CS 341: Algorithms

Lecture 5: Greedy algorithms

Slides due to Éric Schost and based on lecture notes by many other CS341 instructors

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Master theorem – correction

Suppose that $\boldsymbol{a \geq 1}$ and $\boldsymbol{b > 1}$. Consider the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + \boldsymbol{f(n)}, \qquad T(n) = d \quad (n \leq 1), \qquad \inf_n f(n) > 0$$

Let $\boldsymbol{x = \log_b a}$ (so $a = b^x$).

$$\text{Then } T(n) \in \begin{cases} \Theta(\boldsymbol{f(n)}) & \text{if } f(n) \in \Omega(n^{x+\varepsilon}), \text{ for some } \varepsilon > 0 \\ & \text{and } \textbf{regularity condition} \\ \Theta(\boldsymbol{n^x \log n}) & \text{if } f(n) \in \Theta(n^x) \\ \Theta(\boldsymbol{n^x}) & \text{if } f(n) \in O(n^{x-\varepsilon}), \text{ for some } \varepsilon > 0 \end{cases}$$

Regularity: $af(n/b) \leq cf(n)$ for all sufficiently large n and some $c < 1$.

Regularity

Theorem

If $f(n)/n^{x+\varepsilon}$ is non-decreasing for some $\varepsilon > 0$ then the regularity condition holds.

Proof

$$\frac{f(n/b)}{(n/b)^{x+\varepsilon}} \leq \frac{f(n)}{n^{x+\varepsilon}}$$

$$b^x f(n/b) \leq b^{-\varepsilon} f(n)$$

$$a f(n/b) \leq c f(n)$$

where we recall $a = b^x$ and let $c = b^{-\varepsilon} < 1$.

Closest Pair

ClosestPair(A)
A: array of points of size n sorted by y
$x_{median} \leftarrow$ **Median**(A)
$L, R \leftarrow$ **Partition**(A, x_{median})
$\delta_L \leftarrow$ **ClosestPair**(L)
$\delta_R \leftarrow$ **ClosestPair**(R)
$\delta \leftarrow \min(\delta_L, \delta_R)$
return **TransversePairs**(A, x_{median}, δ)

TransversePairs(A, x_{median}, δ)
A: array of points of size n sorted by y
opt $\leftarrow \delta$
$A \leftarrow [(x, y) \in A \mid -\delta < x - x_{median} < \delta]$
for $P \in A$:
    for $Q \in A$ such that $y_P \leq y_Q < y_P + \delta$
        compute $d(P, Q)$, replace opt if better
return opt

Greedy Algorithms

Goals

This chapter: the greedy paradigm through examples

- job scheduling
- interval scheduling
- more scheduling
- fractional knapsack
- and so on

Computational model:

- all input quantities we work with (weights, capacities, deadlines, ...) fit in a word
- unit cost

Greedy algorithms

Context: we are trying to solve a **combinatorial optimization** problem:

- have a **large, but finite,** set $\mathcal{S}$ (orderings of tasks, sets of possible tasks, trees, ...)
- want to find an element E in $\mathcal{S}$ that **minimizes / maximizes** a cost function

Greedy strategy:

- build E step-by-step
- don't think ahead, just try to improve as much as you can at every step
- simple algorithms, but it is often **hard** to prove correctness

A recurrent proof pattern

- let $\boldsymbol{E}_{\mathbf{greedy}}$ be the greedy solution
- let $\boldsymbol{E}$ be any other solution
- transform $\boldsymbol{E}$ into $\boldsymbol{E}_{\mathbf{greedy}}$ progressively, making sure the cost never increases

Example: Huffman

Review from CS240: the **Huffman tree**

- we are given **"frequencies"** $f_1, \dots, f_n$ **for characters** $c_1, \dots, c_n$
- want a code (character $c_i \mapsto$ word w_i in $\{0, 1\}$)
- want prefix-free: build a **binary tree**
- minimize expected codeword length, $\mathbb{E}[\mathrm{length}(w_i)] = \sum_i f_i \mathrm{length}(w_i)$

Greedy strategy: we build the tree **bottom up**.

- create n single-letter trees
- define the **frequency** of a tree as the sum of the frequencies of the letters in it
- build the final tree by joining smaller trees
- greedy choice: **join the two trees with the least frequencies**

Claim

this minimizes $\sum_i f_i \times \{\text{length of } w_i\}$

Proof: takes some work. Progressively transform any other tree into the greedy one.

Minimizing completion time

The problem

Input:

- n jobs, with processing times $[t(1), \ldots, t(n)]$

Output:

- an ordering of the jobs that minimizes the **sum T of the completion times**
- **completion time:** how long it took **(since the beginning)** to complete a job

Example:

- $n = 5$, processing times $[2, 8, 1, 10, 5]$
- in this order,
  $T = 2 \ + \ (8+2) \ + \ (1+8+2) \ + \ (10+1+8+2) \ + \ (5+10+1+8+2) = 70$
- in the order $[1, 2, 8, 5, 10]$,
  $T = 1 \ + \ (2+1) \ + \ (8+2+1) \ + \ (5+8+2+1) \ + \ (10+5+8+2+1) = 57$
- in the order $[1, 2, 5, 8, 10]$,
  $T = 1 \ + \ (2+1) \ + \ (5+2+1) \ + \ (8+5+2+1) \ + \ (10+8+5+2+1) = 54$

Greedy algorithm

Lemma

Let $L = [e_1, \ldots, e_i, e_{i+1}, \ldots, e_n]$ and $L' = [e_1, \ldots, e_{i+1}, e_i, \ldots, e_n]$ be permutations of $[1, \ldots, n]$ that differ by a swap of e_i and e_{i+1}.

The cost difference is $\textbf{cost}(L') - \textbf{cost}(L) = t(e_{i+1}) - t(e_i)$.

Proof

$$\begin{aligned} \textbf{cost}(L') - \textbf{cost}(L) = \quad & (n-i+1)t(e_{i+1}) + (n-i)t(e_i) \\ & - (n-i-1)t(e_i) - (n-i)t(e_{i+1}) \\ = \; & t(e_{i+1}) - t(e_i). \end{aligned}$$

Greedy algorithm

Algorithm: order the jobs in **non-decreasing** processing times

To prove correctness

- let $L = [e_1, \ldots, e_n]$ be a permutation of $[1, \ldots, n]$
- suppose that **we don't** have $t(e_1) \leq t(e_2) \cdots \leq t(e_n)$
- then we can find a better permutation L' by removing an inversion

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1. by assumption there exists i such that $\boldsymbol{t(e_i) > t(e_{i+1})}$
2. use the lemma: $\textbf{cost}(L') - \textbf{cost}(L) = t(e_{i+1}) - t(e_i) < 0$
3. cost is improved

Greedy algorithm

Algorithm: order the jobs in **non-decreasing** processing times

Review from CS240

- optimal static order for linked list implementation of dictionaries
- same result (up to reverse), same proof

$$\textbf{cost}(L) = \sum_{i=1}^{n} i f(e_i)$$

Interval scheduling

The problem

Input:

- n intervals $I_1 = [s_1, f_1], \ldots, I_n = [s_n, f_n]$ start time, finish time
- also write $s_j = \textbf{start}(I_j)$, $f_j = \textbf{finish}(I_j)$

Output:

- a choice T of intervals that **do not overlap** and that has **maximal cardinality**
- $\textbf{finish}(I_j) = \textbf{start}(I_k)$ not an overlap

Example: A car rental company has the following requests for a given day:

I_1: 2pm to 8pm

I_2: 3pm to 4pm

I_3: 5pm to 6pm

Optimum is $T = [I_2, I_3]$.

A few attempts

Attempt 1:

- pick the interval with the **earliest starting time** that creates no conflict

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Attempt 2:

- pick the **shortest interval** that creates no conflict

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Attempt 3:

- pick the interval with the **fewest overlaps** that creates no conflict

A few attempts

Attempt 1:

- pick the interval with the **earliest starting time** that creates no conflict
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- pick the **shortest interval** that creates no conflict
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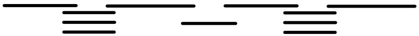

A few attempts

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- **no**, previous example

Attempt 2:

- pick the **shortest interval** that creates no conflict
- **no**, for example

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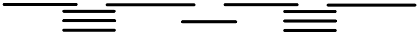

Attempt 4:

- pick the interval with the **earliest finish time**, that creates no conflict

A $\Theta(n \log(n))$ implementation

Greedy($\boldsymbol{I} = [I_1, \ldots, I_n]$)

1. $T \leftarrow [\,]$
2. sort $\boldsymbol{I}$ by non-decreasing finish time
3. **for** $k = 1, \ldots, n$ **do**
4.   if I_k does not overlap the last entry in T
5.   append I_k to T

Correctness: greedy is optimal

To prove correctness

- $T = [\boldsymbol{x_1}, \ldots, \boldsymbol{x_p}]$ be the intervals chosen by algorithm
- $S = [\boldsymbol{y_1}, \ldots, \boldsymbol{y_q}]$ be any feasible choice (sorted by increasing finish time)
- want to prove $\boldsymbol{p} \geq \boldsymbol{q}$

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Proof by induction: **for** $\boldsymbol{k = 0, \ldots, q}$, $\boldsymbol{p \geq k}$, $[\boldsymbol{x_1, \ldots, x_k, y_{k+1}, \ldots, y_q}]$ **feasible and sorted by increasing finish time**

Correctness: greedy is optimal

To prove correctness

- $T = [\boldsymbol{x_1, \ldots, x_p}]$ be the intervals chosen by algorithm
- $S = [\boldsymbol{y_1, \ldots, y_q}]$ be any feasible choice (sorted by increasing finish time)
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Proof by induction: for $\boldsymbol{k = 0, \ldots, q}$, $\boldsymbol{p \geq k}$, $[\boldsymbol{x_1, \ldots, x_k, y_{k+1}, \ldots, y_q}]$ feasible and sorted by increasing finish time

- OK for $k = 0$, so we suppose true for some $k < q$, and prove for $k + 1$
- **finish$(\boldsymbol{x_k}) \leq$ finish$(\boldsymbol{y_{k+1}})$** and $[\boldsymbol{x_1, \ldots, x_k, y_{k+1}}]$ **feasible**, so the algorithm didn't stop at x_k. So $\boldsymbol{p \geq k + 1}$.
- by definition, **finish$(\boldsymbol{x_{k+1}}) \leq$ finish$(\boldsymbol{y_{k+1}})$**. So we can replace y_{k+1} by x_{k+1} and we get $[\boldsymbol{x_1, \ldots, x_{k+1}, y_{k+2}, \ldots, y_q}]$, which is still feasible and sorted by increasing finish time

Interval coloring

The problem

Input:

- n intervals $I_1 = [s_1, f_1], \ldots, I_n = [s_n, f_n]$ start time, finish time
- also write $s_j = \textsf{start}(I_j)$, $f_j = \textsf{finish}(I_j)$

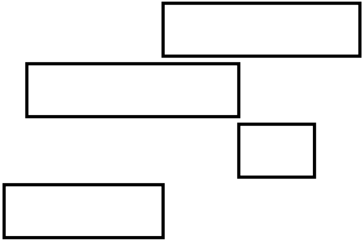
Output:

- assignment of **colors** to each interval
- overlapping intervals get **different colors**
- **minimize** the number of colors used overall

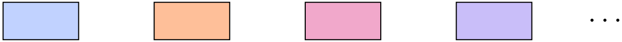
Remarks:

- another version: finding classrooms for lectures
- colors $\leftrightarrow$ indices $1, 2, 3, \ldots$
- $\textsf{finish}(I_j) = \textsf{start}(I_k)$ not an overlap

An example



Available colors:



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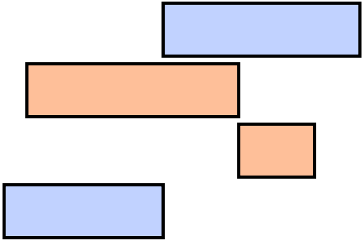

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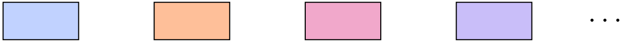

A few attempts

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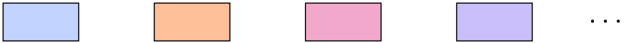
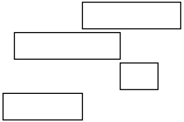


Attempt 1:

- sort intervals by **non-decreasing finish times**
- for $j = 1, \ldots, n$, use for I_j **the smallest existing color** (smallest index) that creates no conflict, or **a new color** if needed



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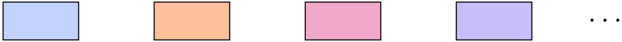

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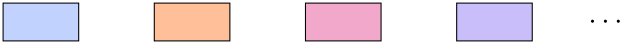

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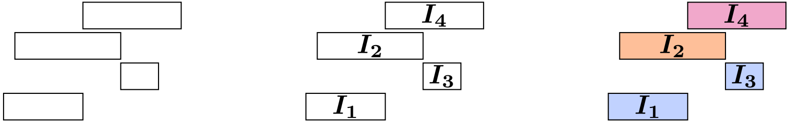

- does not give the optimal

A few attempts

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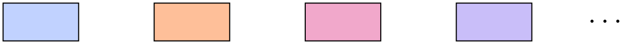

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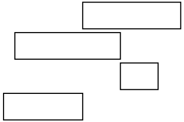

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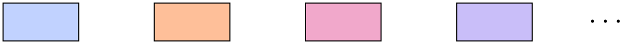

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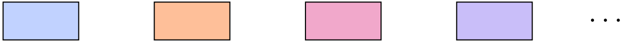

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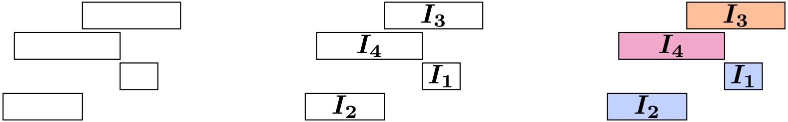

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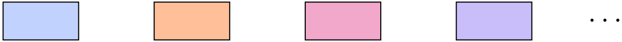

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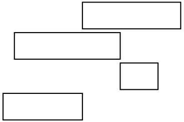

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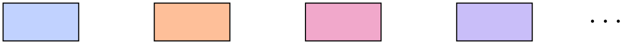

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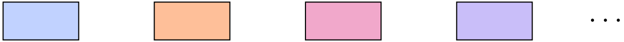

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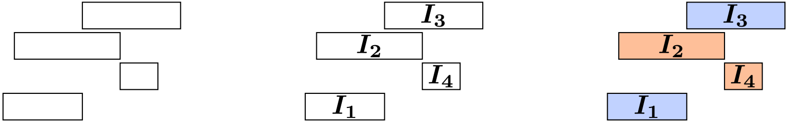

- maybe, needs proof

Correctness of the third attempt

Claim

Suppose the output uses $\boldsymbol{k}$ colors. Then, **we cannot use fewer than $\boldsymbol{k}$**.

Proof

- suppose we color I_t with color k
- so I_t overlaps with $k-1$ intervals, say $I_{\alpha_1}, \ldots, I_{\alpha_{k-1}}$ seen previously
- **because we sorted by start time**, for all $j = 1, \ldots, k-1$, $\boldsymbol{s_{\alpha_j} \leq s_t < f_{\alpha_j}}$
- so at time s_t, we can't do with less than k colors

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Remark: could also write a proof closer in spirit to the previous ones:

- $T = [\boldsymbol{c_1, \ldots, c_n}]$ are the colors chosen by algorithm
- $S = [\boldsymbol{d_1, \ldots, d_n}]$ is any other feasible choice
- prove that S uses more (or same) number of colors by transforming it into T progressively

A $\Theta(n \log n)$ implementation

Color

1. sort the array $A = [[s_i, \text{“start”}, i]]_{1..n}$ **cat** $[[f_i, \text{“finish”}, i]]_{1..n}$ by time
   (to break ties: **finish** comes before **start**)
2. $C[1..n] \leftarrow$ array (of color indices)
3. $H \leftarrow$ **min-heap** of available color indices, initially empty
4. $k \leftarrow 0$
5. **for all entries of** A (in increasing order)
   if interval i starts, set $C[i]$ = min element in H (if not empty) or $++k$ (if empty)
   if interval i ends, insert $C[i]$ in H
6. **return** C

Remark: picking **any** available color would work too.