Lecture 9: Dynamic Programming III

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Overview

- Edit Distance
- Graphs & DP on Trees
- Acknowledgements
Edit Distance

- **Input:** two strings $A := a_1 a_2 \cdots a_m$ and $B := b_1 b_2 \cdots b_n$, where $a_i, b_j \in \Sigma$
- **Output:** minimum number of edits to string $A$ (*add, delete, change*) to transform it into string $B$
- **Model:** word RAM
- **Example:**

  SNOWY and SUNNY

  - approach 1
    
    S – N O W Y
    S U N N – Y

  - approach 2
    
    – S N O W Y
    S U N N – Y
**Edit Distance**

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- Looks a bit hard. Can we DP it?
Edit Distance

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- **Output:** minimum number of edits to string $A$ (add, delete, change) to transform it into string $B$

Subproblems: let $A_i := a_1 \cdots a_i$ and $B_j := b_1 \cdots b_j$, and let $D(i,j)$ be edit distance between $A_i$, $B_j$. Base case: $D(0,0) = 0$. 
Edit Distance

- **Input:** two strings \( A := a_1 a_2 \cdots a_m \) and \( B := b_1 b_2 \cdots b_n \), where \( a_i, b_j \in \Sigma \)
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Subproblems: let \( A_i := a_1 \cdots a_i \) and \( B_j := b_1 \cdots b_j \), and let \( D(i, j) \) be edit distance between \( A_i, B_j \). Base case: \( D(0, 0) = 0 \).

Cases (based on allowed operations)

1. **Add:** add \( b_j \) to string \( A_i \).
   Total cost: \( Sol_1 := 1 + D(i, j - 1) \)

2. **Delete:** delete \( a_i \) from \( A_i \).
   Total cost \( Sol_2 := 1 + D(i - 1, j) \)

3. **Change/Match:** can change \( a_i \mapsto b_j \). (if \( a_i = b_j \) we simply match them)
   Total cost: \( Sol_3 := \begin{cases} 1 + D(i - 1, j - 1), & \text{if } a_i \neq b_j \\ D(i - 1, j - 1), & \text{if } a_i = b_j \end{cases} \)
Edit Distance - Recurrence

- Thus, recurrence given by

\[ D(i, j) = \min\{Sol_1, Sol_2, Sol_3\} \]
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• **Correctness:** proof by induction.
  1. True for base case, i.e. \( D(0, 0) = 0 \).
  2. If all subcases are correct, then recurrence tells us all possible ways to handle the last symbols of the strings (thus one must lead to the optimum distance).
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Runtime:
1. \# Subproblems: \( O(mn) \)
2. Time per subproblem (given previous subproblems computed): \( O(1) \)
3. Runtime: \( O(mn) \)
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- Computing the table (bottom up): want to go from \((0, 0)\) to \((m, n)\). Can compute in increasing row order, from left to right.
● Edit Distance

● Graphs & DP on Trees

● Acknowledgements
Wait graphs already?!
Graphs - Definition
Trees
Maximum Independent Set on Trees

- **Input:** A tree $T([n], E)$
- **Output:** A maximum independent set in $T$

**Idea:** Pick a vertex as the root. Traverse the tree downwards as follows:

1. Given vertex $v$, if we include it in our independent set, then don't include its children (look at the grandchildren).
2. If don't include $v$, then pick all its children.

**Recurrence:**

$$\text{MIS}(v) = \max \left\{ \begin{array}{ll}
1 + \sum_{w \text{ grandchild of } v} \text{MIS}(w), \\
\sum_{u \text{ child of } v} \text{MIS}(u) \end{array} \right\}$$

**Running time:**

1. # subproblems: $O(n)$ (# vertices)
2. Time per subproblem (once we have subproblems): $O(|E|) = O(n)$
3. Total runtime: $O(n^2)$
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Acknowledgement

- Based on [DPV 2006]
Cormen, Thomas and Leiserson, Charles and Rivest, Ronald and Stein, Clifford. (2009)
*MIT Press*

Dasgupta, Sanjay and Papadimitriou, Christos and Vazirani, Umesh (2006)
Algorithms

Kleinberg, Jon and Tardos, Eva (2006)
Algorithm Design.
*Addison Wesley*