CS 341: Algorithms

Lecture 15: Single-source and all pairs shortest paths

Slides due to Éric Schost and based on lecture notes by many other CS341 instructors

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The Bellman-Ford algorithm

Outlook

Bellman-Ford

- given a **directed** graph $G = (V, E)$ with **weights** $w(e)$ on the edges
- assuming **no negative cycles**, want the shortest (=minimal weight) path / walk between a **source s** and **all vertices**

  (write $\delta(s, v)$ for the length of a minimal path = **distance** from s to v)
- can **detect** the existence of negative cycles
- very simple pseudo-code, but slower than Dijkstra's algorithm
- no isolated vertex, so $m \geq n/2$

Dynamic programming for shortest paths

Definition:

- for $i = 0, \ldots, n$, set

$$\boldsymbol{\delta_i(s, v)} = \text{length of the shortest } \textbf{walk } \boldsymbol{s \rightsquigarrow v} \text{ with at most } \boldsymbol{i} \text{ edges}$$

  if no walk, $\delta_i(s, v) = \infty$

Easy observations:

- $\delta_0(s, s) = 0$ and $\delta_0(s, v) = \infty$ for $v \neq s$
- $\boldsymbol{\delta_{n-1}(s, v) < \infty}$ iff v reachable from s
- if **no negative cycle reachable from** s, $\boldsymbol{\delta_{n-1}(s, v) = \delta(s, v)}$ for all v

Recurrence

Claim

$$\boldsymbol{\delta_i(s,v) = \min(\delta_{i-1}(s,v), \min_{(u,v)\in E} \delta_{i-1}(s,u) + w(u,v))}$$

Proof: call M the right-hand side.

1. $\delta_i(s,v) \leq \delta_{i-1}(s,v)$ (take more walks into consideration on the left)
   for all u, $\delta_i(s,v) \leq \delta_{i-1}(s,u) + w(u,v)$ (same)
   so $\boldsymbol{\delta_i(s,v) \leq M}$
2. Take a path $\gamma : s \to v$, at most i edges
   if at most $i-1$ edges, $w(\gamma) \geq \delta_{i-1}(s,v) \geq M$
   if i edges, let u be the last stop before v, so $w(\gamma) \geq \delta_{i-1}(s,u) + w(u,v) \geq M$
   so $\boldsymbol{\delta_i(s,v) \geq M}$

Remark: if $s \neq v$, you don't need the term $\delta_{i-1}(s,v)$

Pseudo-code

BellmanFord(G, s)

```
d_0 ← [0, ∞, ..., ∞] (s is the first index)
parent ← [s, •, ..., •] (s is the first index)
for i = 1, ..., n − 1 do
    for all v in V do
        d_i[v] ← d_{i−1}[v]
        for all (u, v) in E do
            if d_{i−1}[u] + w(u, v) < d_i[v] then
                d_i[v] ← d_{i−1}[u] + w(u, v)
                parent[v] ← u
```

Correctness: $d_i[v] = \delta_i(s, v)$, so **if no negative cycle** $d_{n-1}[v] = \delta(s, v)$ for all v

Runtime: $\boldsymbol{\Theta(mn)}$

Remark: need to loop over edges directed **toward** v

Saving a bit of space

Idea: use a single array d

BellmanFord2.0(G, s)

1. $d \leftarrow [0, \infty, \ldots, \infty]$ (s is the first index)
2. parent $\leftarrow [s, \bullet, \ldots, \bullet]$ (s is the first index)
3. **for** $i = 1, \ldots, n-1$ **do**
4.   **for all** (u, v) in E **do**
5.   **if** $d[u] + w(u, v) < d[v]$ **then**
6. $d[v] \leftarrow d[u] + w(u, v)$
7.   parent$[v] \leftarrow u$

Runtime: $\boldsymbol{\Theta(mn)}$

Correctness, part 1

Claim

For all i, after iteration i, we have $\boldsymbol{d[v] \leq d_i[v]}$ for all v

Idea: all quantities can only go down in version 2.0

Proof: by induction

- true for $i = 0$, so we suppose true at index $i - 1$ and prove true at i
- at the beginning of the loop, for all v, $\boldsymbol{d[v] \leq d_{i-1}[v]}$
- $d[v]$ can only decrease, so this stays true throughout the loop
- $d[v]$ is replaced by $\min(d[v], \min_{(u,v)\in E}(\square + w(u,v)))$, where $\square \leq d_{i-1}[u]$
- so at the end of iteration i, $\boldsymbol{d[v] \leq d_i[v]}$

Relaxations

The operation $\boldsymbol{d[v] \leftarrow \min(d[v], d[u] + w(u, v))}$ is called a **relaxation**

Claim

if $\boldsymbol{\delta(s, u) \leq d[u]}$ and $\boldsymbol{\delta(s, v) \leq d[v]}$ before relaxation, then $\boldsymbol{\delta(s, v) \leq d[v]}$ post-relaxation.

Proof

- $\delta(s, v) \leq \delta(s, u) + w(u, v)$ **(triangle inequality)**, so $\delta(s, v) \leq d[u] + w(u, v)$
- but also $\delta(s, v) \leq d[v]$

Consequence: if **all** $d[v]$ satisfy $\delta(s, v) \leq d[v]$, and we apply **any number** of relaxations, all inequalities stay true

Correctness, part 2

Claim

For $i = 0, \ldots, n-1$, after iteration i, $\boldsymbol{\delta(s, v) \leq d[v] \leq \delta_i(s, v)}$ for all v.

Proof:

- correctness part 1 gives $\boldsymbol{d[v] \leq \delta_i(s, v)}$
- previous slide gives $\boldsymbol{\delta(s, v) \leq d[v]}$

Summary

If there is no negative cycle reachable from s

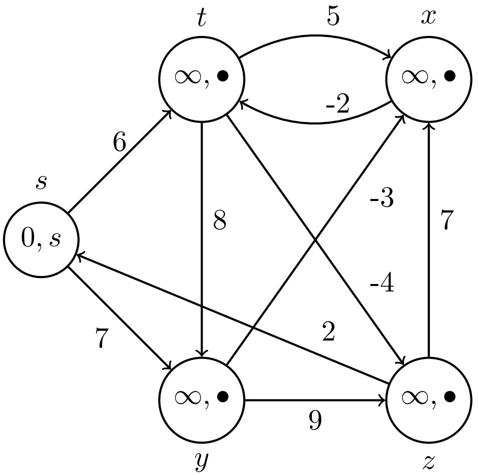
- at the end with $i = n-1$, $\boldsymbol{d[v] = \delta(s,v)}$ for all v
- in particular, for any edge (u,v), $\boldsymbol{d[v] \le d[u] + w(u,v)}$ (triangle inequality)

If there is a negative cycle reachable from s

- say $v_1, v_2, \ldots, v_k$, with $\boldsymbol{v_k = v_1}$
- at the end, all $d[v_i]$ are $< \infty$
- **claim:** there must be an edge (v_i, v_{i+1}) with $d[v_{i+1}] > d[v_i] + w(v_i, v_{i+1})$
- **else:** $d[v_{i+1}] \le d[v_i] + w(v_i, v_{i+1})$ for all i; sum and derive a contradiction

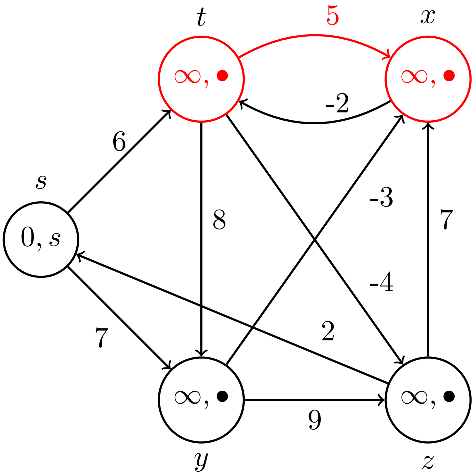
Conclusion: for extra $\boldsymbol{\Theta(m)}$, can check the presence of a negative cycle reachable from s

Example: Bellman-Ford



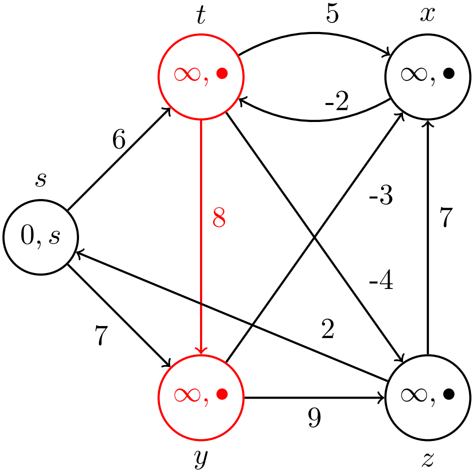
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)

Example: Bellman-Ford



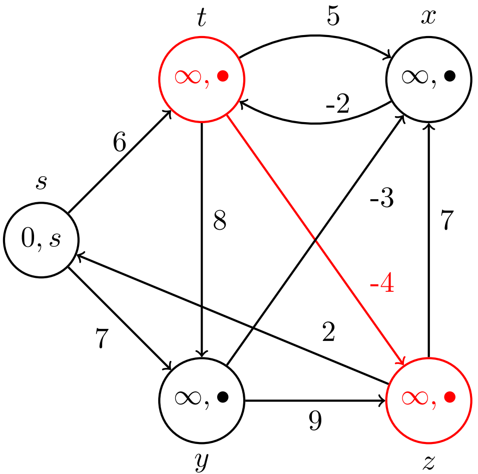
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$									

Example: Bellman-Ford



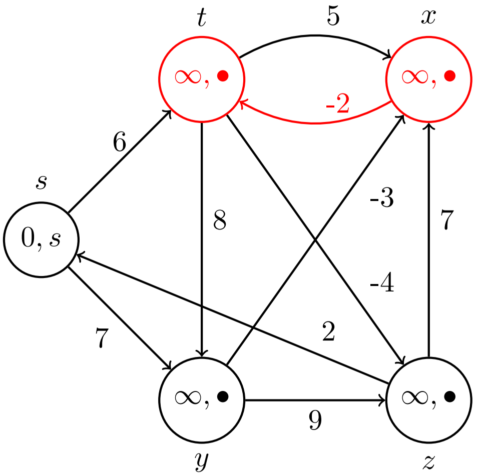
$i = 1$	(t, x)	$\boldsymbol{(t, y)}$	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×								

Example: Bellman-Ford



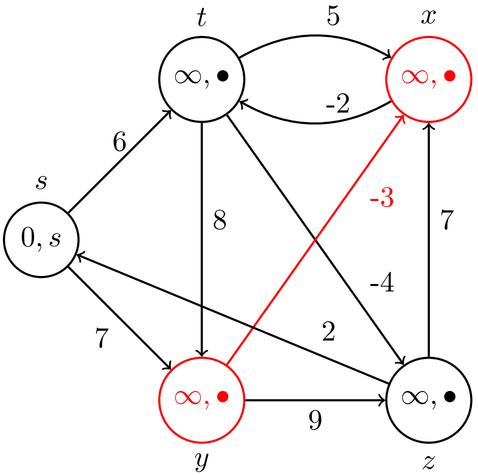
$i = 1$	(t, x)	(t, y)	$\mathbf{(t, z)}$	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$	$\times$	$\times$							

Example: Bellman-Ford



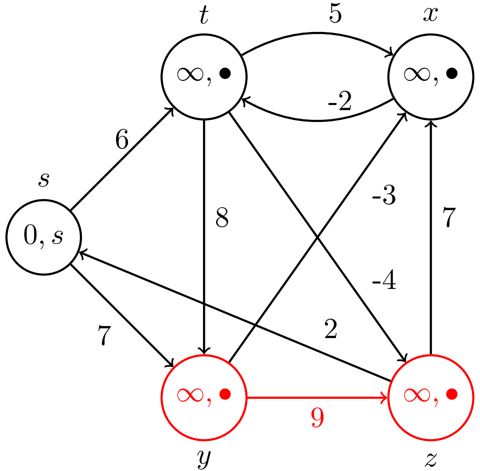
$i = 1$	(t, x)	(t, y)	(t, z)	$\boldsymbol{(x, t)}$	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	×						

Example: Bellman-Ford



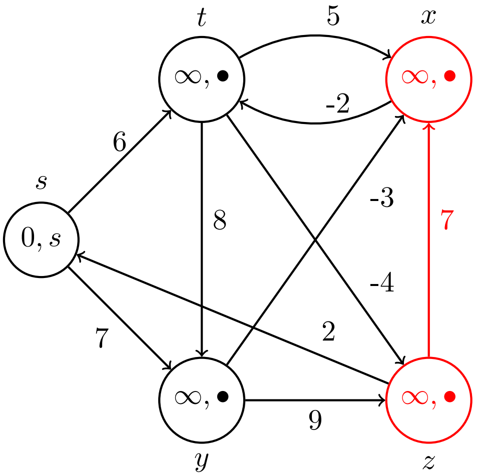
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	$\boldsymbol{(y, x)}$	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	×	×					

Example: Bellman-Ford



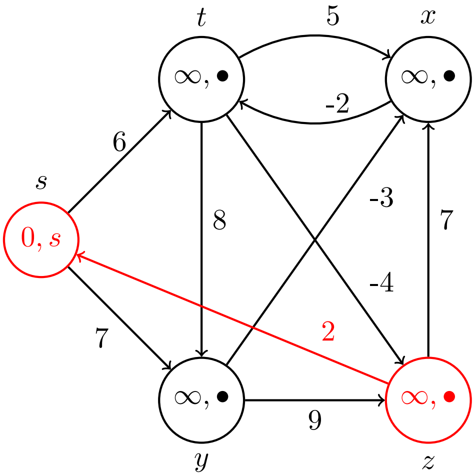
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	$\boldsymbol{(y, z)}$	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	×	×	×				

Example: Bellman-Ford



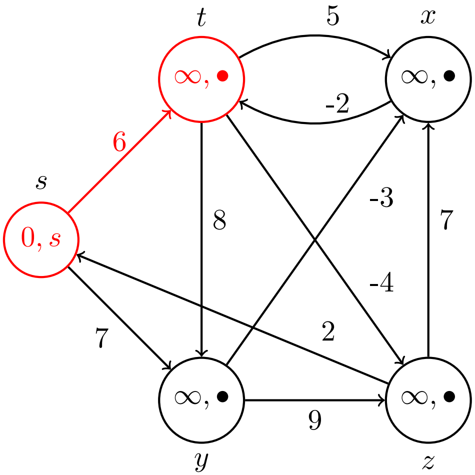
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	$\boldsymbol{(z, x)}$	(z, s)	(s, t)	(s, y)
	×	×	×	×	×	×	×			

Example: Bellman-Ford



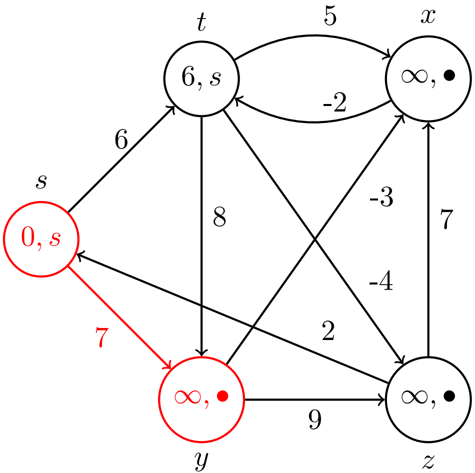
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	×	×	×	×	×		

Example: Bellman-Ford



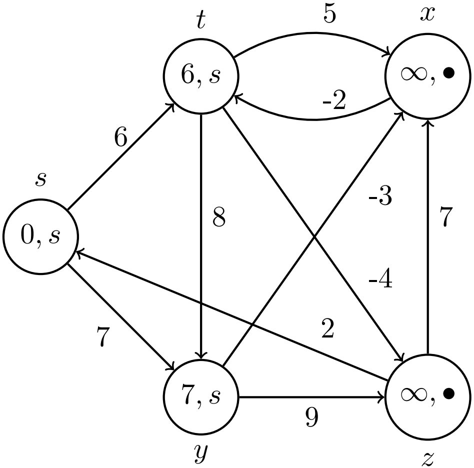
$i = 1$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	$(\boldsymbol{s},\boldsymbol{t})$	(s,y)
	×	×	×	×	×	×	×	×	✓	

Example: Bellman-Ford



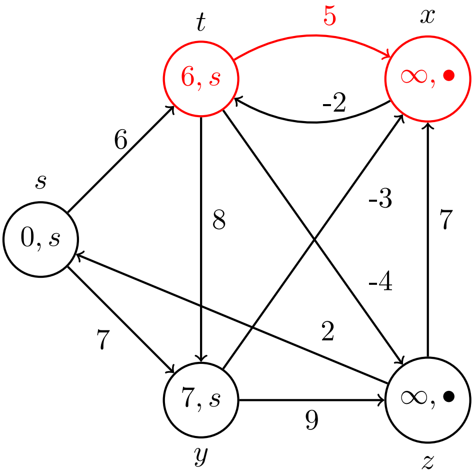
$i = 1$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	$(\boldsymbol{s}, \boldsymbol{y})$
	×	×	×	×	×	×	×	×	✓	✓

Example: Bellman-Ford



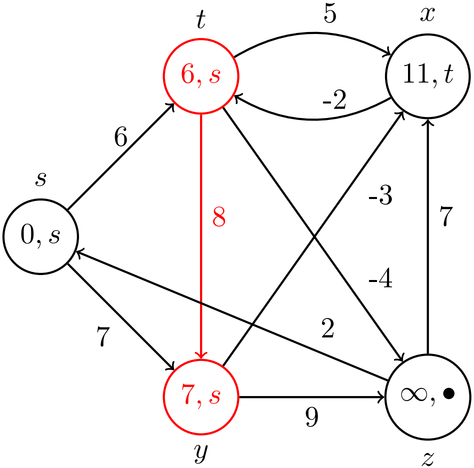
$i = 1$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	✓	✓

Example: Bellman-Ford



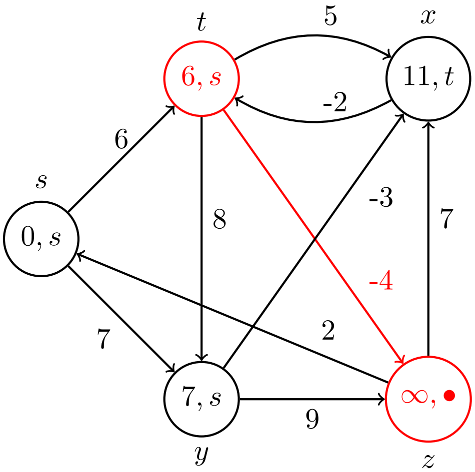
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	✓									

Example: Bellman-Ford



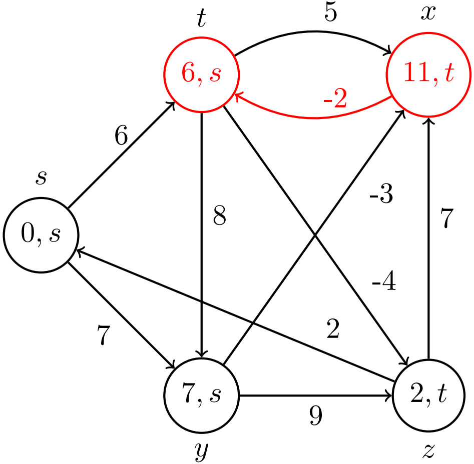
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	✓	×								

Example: Bellman-Ford



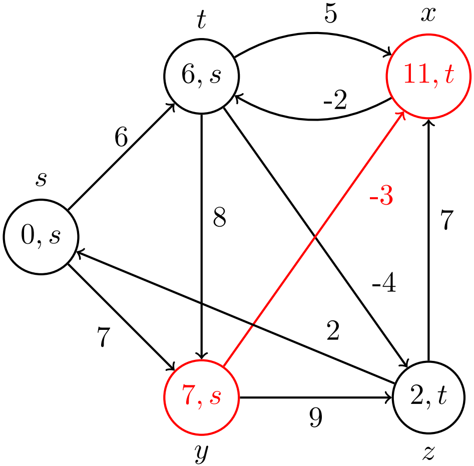
$i = 2$	(t,x)	(t,y)	$\mathbf{(t,z)}$	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	✓	×	✓							

Example: Bellman-Ford



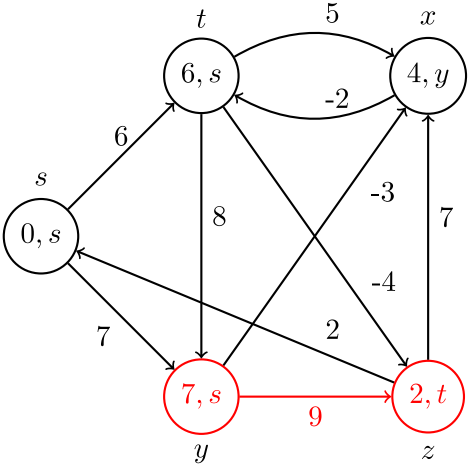
$i = 2$	(t, x)	(t, y)	(t, z)	$\boldsymbol{(x, t)}$	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	✓	×	✓	×						

Example: Bellman-Ford



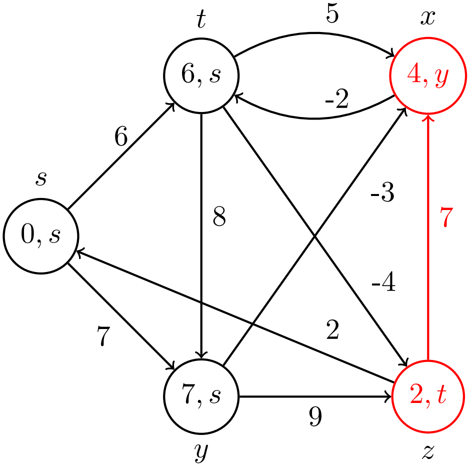
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	$\boldsymbol{(y, x)}$	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	✓	×	✓	×	✓					

Example: Bellman-Ford



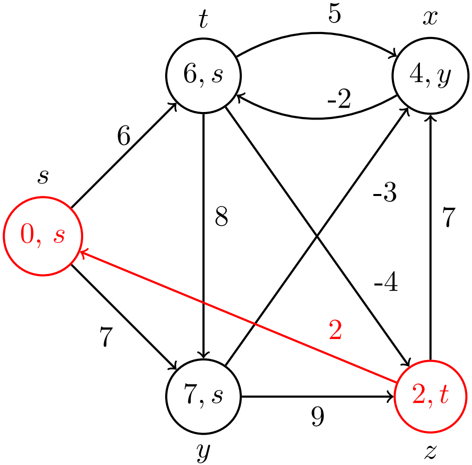
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	$\boldsymbol{(y, z)}$	(z, x)	(z, s)	(s, t)	(s, y)
	✓	×	✓	×	✓	×				

Example: Bellman-Ford



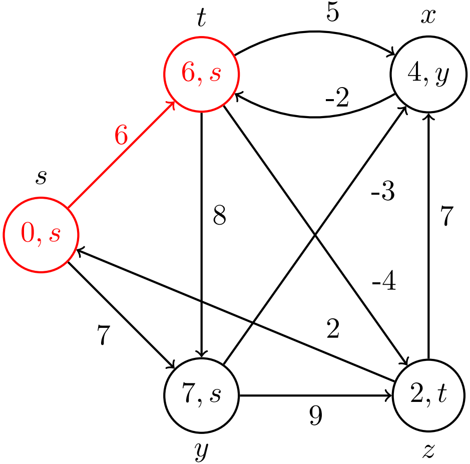
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	$\boldsymbol{(z, x)}$	(z, s)	(s, t)	(s, y)
	✓	×	✓	×	✓	×	×			

Example: Bellman-Ford



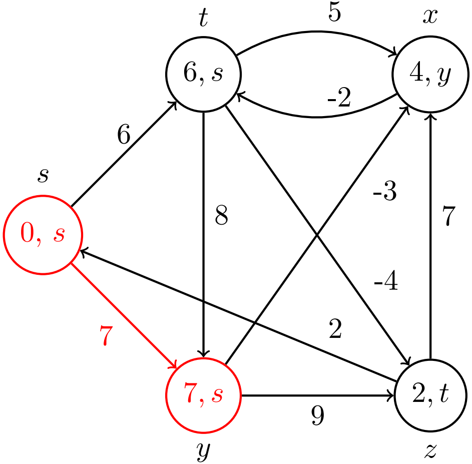
$i = 2$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	✓	×	✓	×	✓	×	×	×		

Example: Bellman-Ford



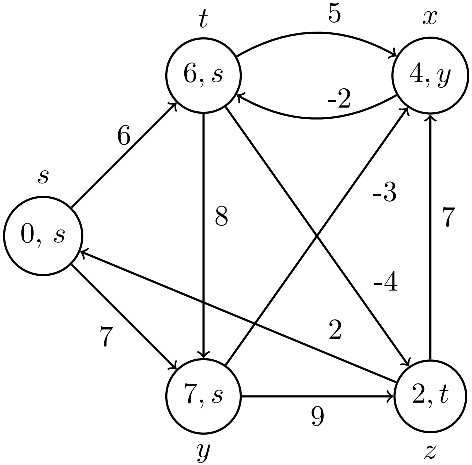
$i = 2$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	$(\boldsymbol{s},\boldsymbol{t})$	(s,y)
	✓	×	✓	×	✓	×	×	×	×	

Example: Bellman-Ford



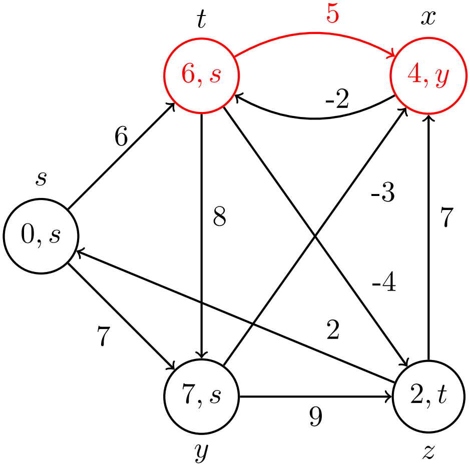
$i = 2$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	$\boldsymbol{(s,y)}$
	✓	×	✓	×	✓	×	×	×	×	×

Example: Bellman-Ford



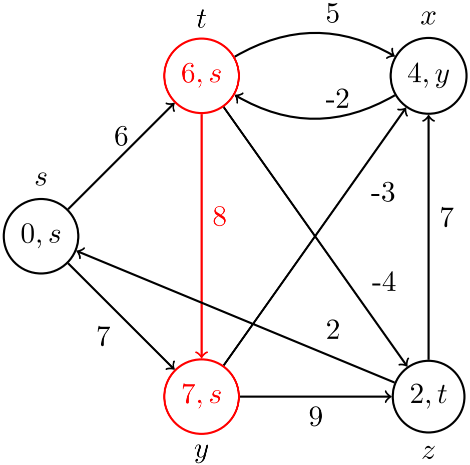
$i = 2$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	✓	×	✓	×	✓	×	×	×	×	×

Example: Bellman-Ford



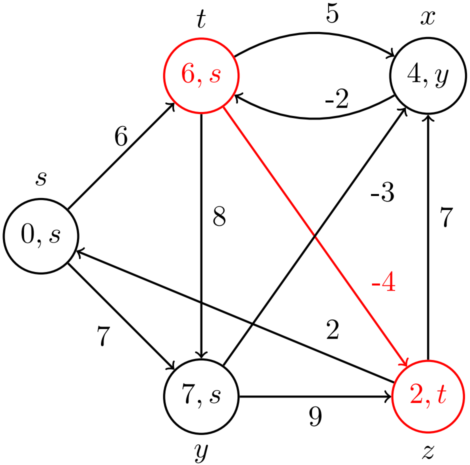
$i = 3$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	$\times$									

Example: Bellman-Ford



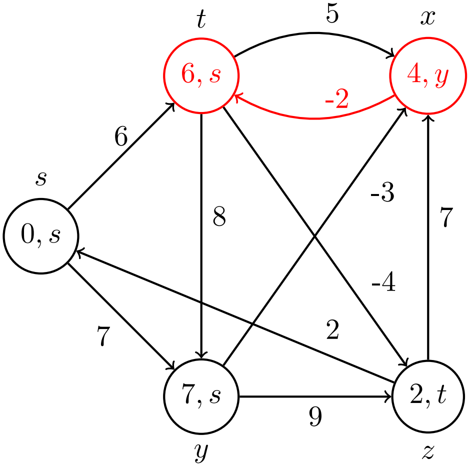
$i = 3$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$	$\times$								

Example: Bellman-Ford



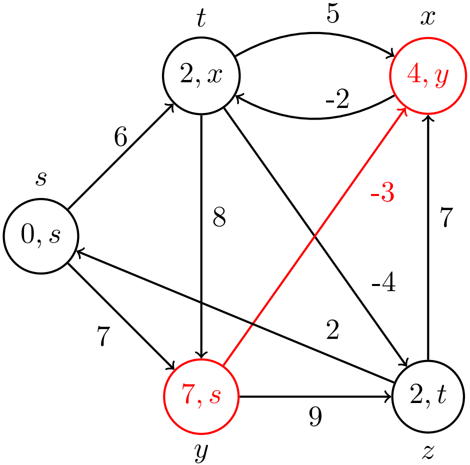
$i = 3$	(t, x)	(t, y)	$\mathbf{(t, z)}$	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$	$\times$	$\times$							

Example: Bellman-Ford



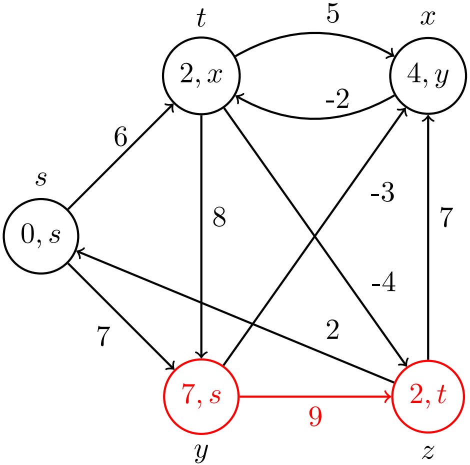
$i = 3$	(t, x)	(t, y)	(t, z)	$(\boldsymbol{x}, \boldsymbol{t})$	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	✓						

Example: Bellman-Ford



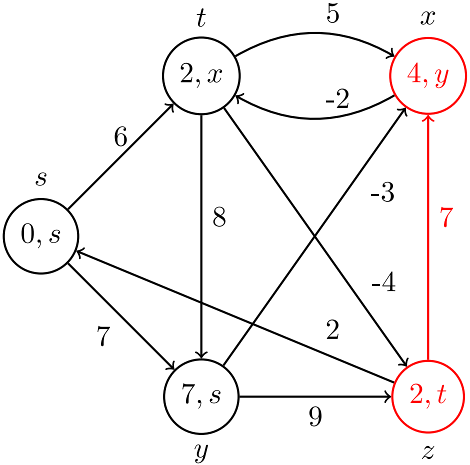
$i = 3$	(t,x)	(t,y)	(t,z)	(x,t)	$\boldsymbol{(y,x)}$	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	$\times$	$\times$	$\times$	$\checkmark$	$\times$					

Example: Bellman-Ford



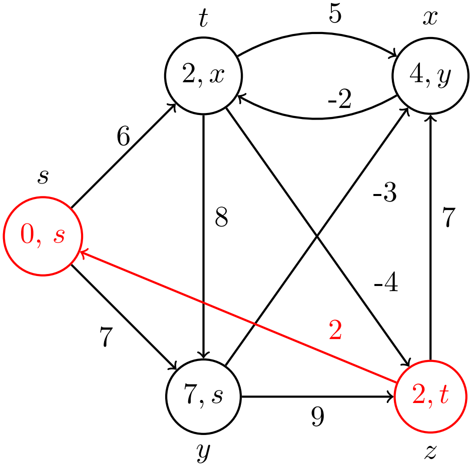
$i = 3$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	$\boldsymbol{(y, z)}$	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	✓	×	×				

Example: Bellman-Ford



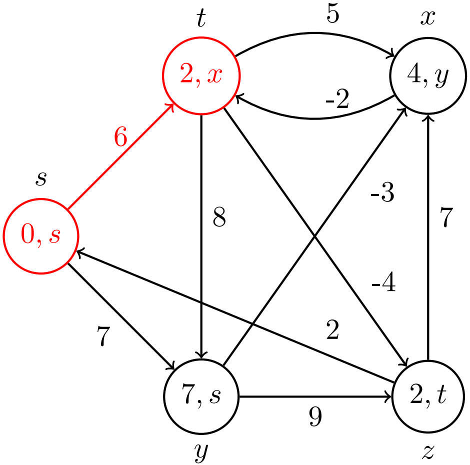
$i = 3$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	×	✓	×	×	×			

Example: Bellman-Ford



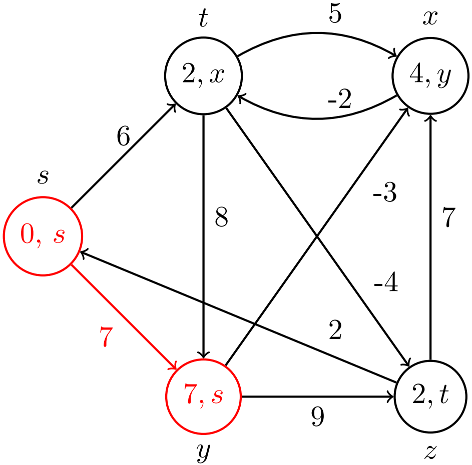
$i=3$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$		

Example: Bellman-Ford



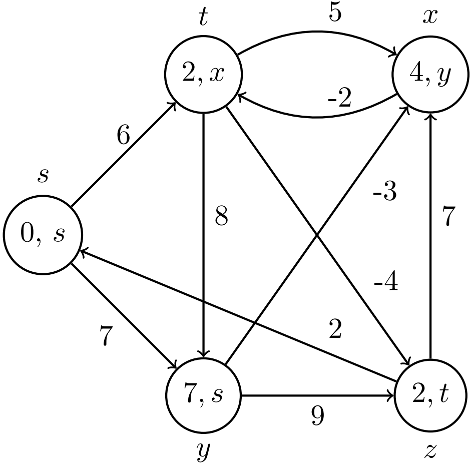
$i = 3$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	$\boldsymbol{(s, t)}$	(s, y)
	×	×	×	✓	×	×	×	×	×	

Example: Bellman-Ford



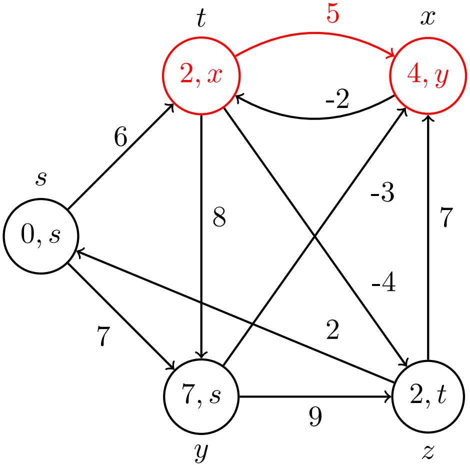
$i=3$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	×	×	×	✓	×	×	×	×	×	×

Example: Bellman-Ford



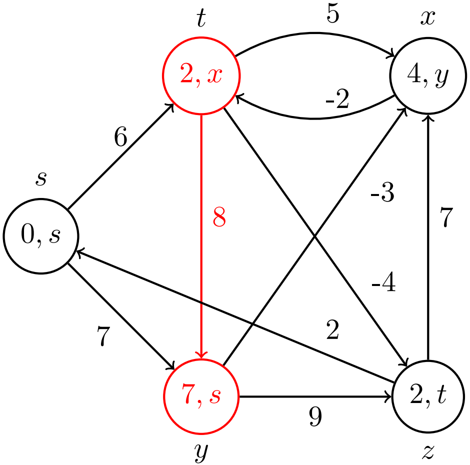
$i = 3$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	×	×	×	✓	×	×	×	×	×	×

Example: Bellman-Ford



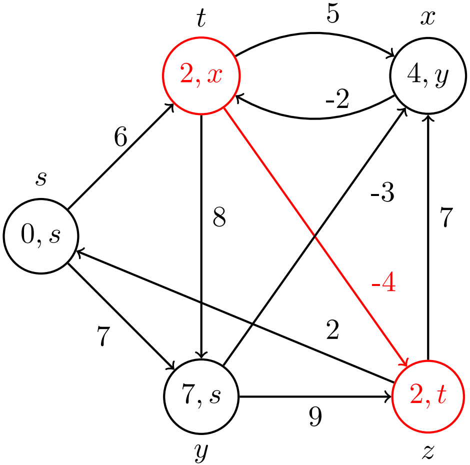
$i = 4$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$									

Example: Bellman-Ford



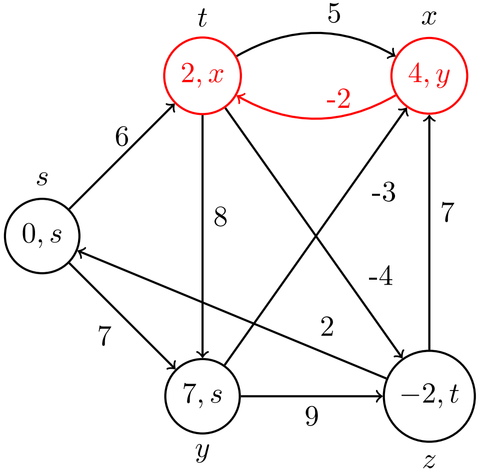
$i = 4$	(t, x)	$\mathbf{(t, y)}$	(t, z)	(x, t)	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	$\times$	$\times$								

Example: Bellman-Ford



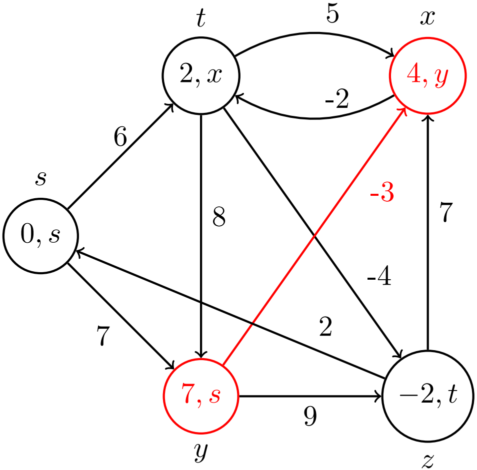
$i = 4$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	×	×	✓							

Example: Bellman-Ford



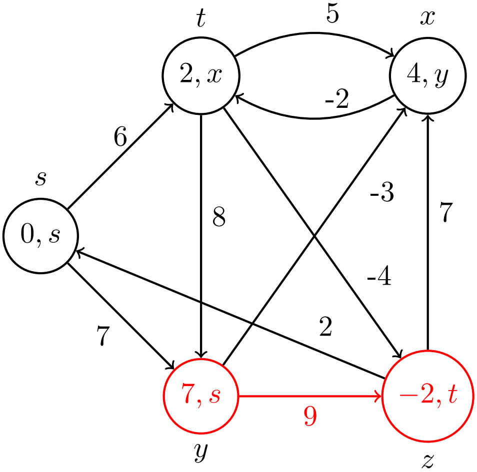
$i = 4$	(t, x)	(t, y)	(t, z)	$\boldsymbol{(x, t)}$	(y, x)	(y, z)	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	✓	×						

Example: Bellman-Ford



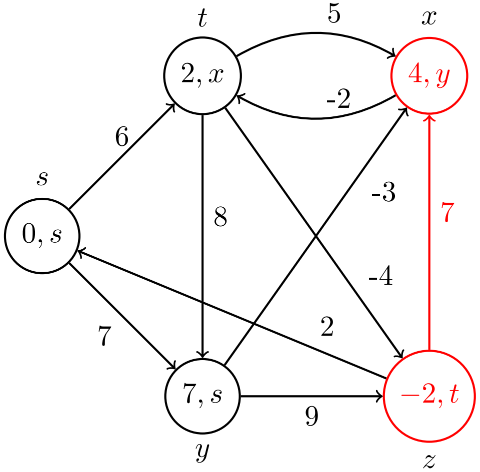
$i=4$	(t,x)	(t,y)	(t,z)	(x,t)	$\boldsymbol{(y,x)}$	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	$\times$	$\times$	$\checkmark$	$\times$	$\times$					

Example: Bellman-Ford



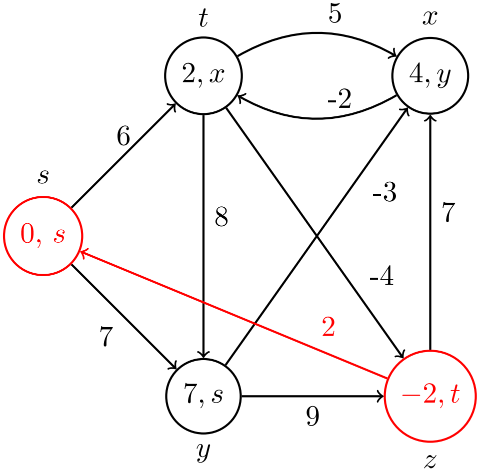
$i = 4$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	**(y, z)**	(z, x)	(z, s)	(s, t)	(s, y)
	×	×	✓	×	×	×				

Example: Bellman-Ford



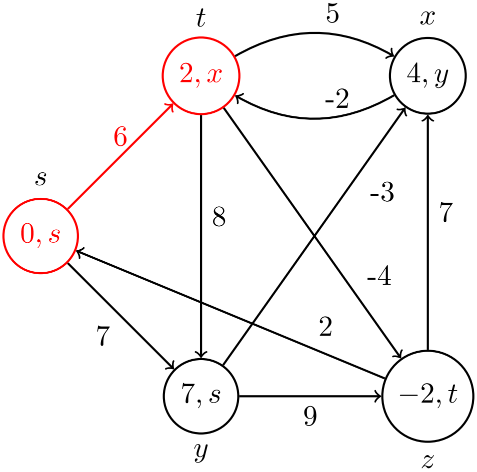
$i = 4$	(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y, z)	$\boldsymbol{(z, x)}$	(z, s)	(s, t)	(s, y)
	×	×	✓	×	×	×	×			

Example: Bellman-Ford



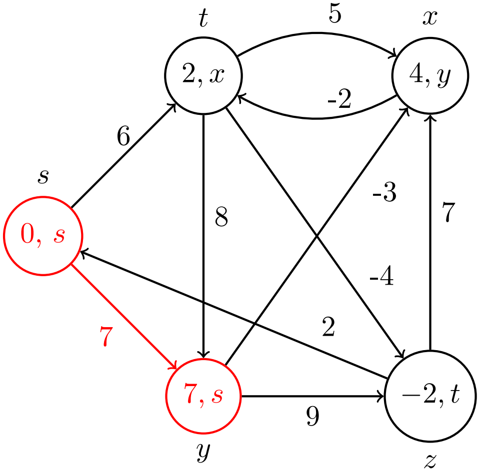
$i=4$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	$\mathbf{(z,s)}$	(s,t)	(s,y)
	×	×	✓	×	×	×	×	×		

Example: Bellman-Ford



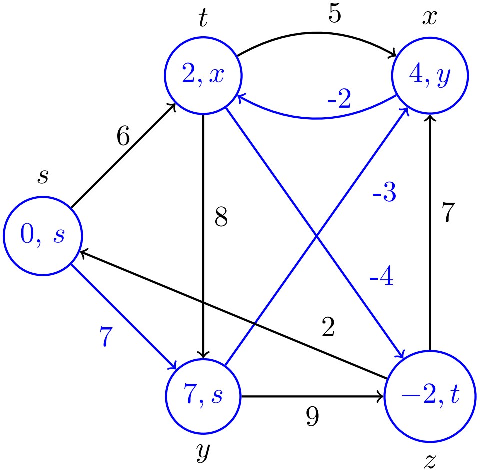
$i = 4$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	$(\boldsymbol{s,t})$	(s,y)
	×	×	✓	×	×	×	×	×	×	

Example: Bellman-Ford



$i = 4$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	$\boldsymbol{(s,y)}$
	×	×	✓	×	×	×	×	×	×	×

Example: Bellman-Ford



$i = 4$	(t,x)	(t,y)	(t,z)	(x,t)	(y,x)	(y,z)	(z,x)	(z,s)	(s,t)	(s,y)
	×	×	×	✓	×	×	×	×	×	×

The Floyd-Warshall algorithm

Outlook

Floyd-Warshall

- **no fixed source**: computes all distances $\delta(u, v)$
- negative weights OK but **no negative cycle**
  (can be tested in $O(mn)$ with Bellman-Ford in each SCC of G)
- very simple pseudo-code, but slower than other algorithms: $\boldsymbol{\Theta(n^3)}$
- another application of dynamic programming

Remark: doing Bellman-Ford from all u takes $\boldsymbol{\Theta(mn^2)}$

Looking at subsets of vertices

Subproblems for dynamic programming

- Bellman-Ford uses paths with **fixed numbers of steps**
- Floyd-Warshall restricts which **vertices** can be used

Definition:

- for $i = 0, \ldots, n$, set $\boldsymbol{D_i(v_j, v_k)}$ = length of the shortest path $v_j \rightsquigarrow v_k$ with all intermediate vertices in $v_1, \ldots, v_i$
- for $i = 0$, we get
  - $D_0(v_j, v_j) = 0$
  - $D_0(v_j, v_k) = w(v_j, v_k)$ if there is an edge (v_j, v_k)
  - $D_0(v_j, v_k) = \infty$ otherwise
- $D_n(v_j, v_k) = \delta(v_j, v_k)$

Pseudo-code

Claim

$$D_i(v_j, v_k) = \min(\boldsymbol{D_{i-1}(v_j, v_k)}, \boldsymbol{D_{i-1}(v_j, v_i) + D_{i-1}(v_i, v_k)})$$

Proof: either the shortest path does not go through v_i, or it does (if it does, it's only once)

```
FloydWarshall(G)
set up D_0 above
for i = 1, ..., n do
    for j = 1, ..., n do
        for k = 1, ..., n do
            D_i[v_j, v_k] ← min(D_{i-1}[v_j, v_k], D_{i-1}[v_j, v_i] + D_{i-1}[v_i, v_k])
```

Analysis

Runtime and memory: $\Theta(n^3)$

Exercise 1

prove that we can use only a single array $D[v_j, v_k]$, with

$$D[v_j, v_k] \leftarrow \min(D[v_j, v_k], D[v_j, v_i] + D[v_i, v_k])$$

(if no negative cycle, this computes the same values, unlike in Bellman-Ford)

Exercise 2

to find all shortest paths, use an array $P[v_j, v_k]$, which gives the vertex following v_j on the shortest path to v_k

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_0

$$\begin{bmatrix} \mathbf{0} & \mathbf{6} & \boldsymbol{\infty} & \mathbf{7} & \boldsymbol{\infty} \\ \boldsymbol{\infty} & 0 & 5 & 8 & -4 \\ \boldsymbol{\infty} & -2 & 0 & \infty & \infty \\ \boldsymbol{\infty} & \infty & -3 & 0 & 9 \\ \mathbf{2} & \infty & 7 & \infty & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_0

$$\begin{bmatrix} \mathbf{0} & \mathbf{6} & \boldsymbol{\infty} & \mathbf{7} & \boldsymbol{\infty} \\ \boldsymbol{\infty} & 0 & 5 & 8 & -4 \\ \boldsymbol{\infty} & -2 & 0 & \infty & \infty \\ \boldsymbol{\infty} & \infty & -3 & 0 & 9 \\ \mathbf{2} & \infty & 7 & \infty & 0 \end{bmatrix} \qquad \begin{bmatrix} 0+0 & 6+0 & \infty+0 & 7+0 & \infty+0 \\ 0+\infty & 6+\infty & \infty+\infty & 7+\infty & \infty+\infty \\ 0+\infty & 6+\infty & \infty+\infty & 7+\infty & \infty+\infty \\ 0+\infty & 6+\infty & \infty+\infty & 7+\infty & \infty+\infty \\ 0+2 & 6+2 & \infty+2 & 7+2 & \infty+2 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_0

$$\begin{bmatrix} \mathbf{0} & \mathbf{6} & \boldsymbol{\infty} & \mathbf{7} & \boldsymbol{\infty} \\ \boldsymbol{\infty} & 0 & 5 & 8 & -4 \\ \boldsymbol{\infty} & -2 & 0 & \infty & \infty \\ \boldsymbol{\infty} & \infty & -3 & 0 & 9 \\ \mathbf{2} & \infty & 7 & \infty & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 6 & \infty & 7 & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 2 & 8 & \infty & 9 & \infty \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_1

$$
\begin{bmatrix}
0 & \mathbf{6} & \infty & 7 & \infty \\
\boldsymbol{\infty} & \mathbf{0} & \mathbf{5} & \mathbf{8} & \mathbf{-4} \\
\infty & \mathbf{-2} & 0 & \infty & \infty \\
\infty & \boldsymbol{\infty} & -3 & 0 & 9 \\
2 & \mathbf{8} & 7 & 9 & 0
\end{bmatrix}
$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_1

$$\begin{bmatrix} 0 & \mathbf{6} & \infty & 7 & \infty \\ \boldsymbol{\infty} & \mathbf{0} & \mathbf{5} & \mathbf{8} & \mathbf{-4} \\ \infty & \mathbf{-2} & 0 & \infty & \infty \\ \infty & \boldsymbol{\infty} & -3 & 0 & 9 \\ 2 & \mathbf{8} & 7 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} \infty+6 & 0+6 & 5+6 & 8+6 & -4+6 \\ \infty+0 & 0+0 & 5+0 & 8+0 & -4+0 \\ \infty-2 & 0-2 & 5-2 & 8-2 & -4-2 \\ \infty+\infty & 0+\infty & 5+\infty & 8+\infty & -4+\infty \\ \infty+8 & 0+8 & 5+8 & 8+8 & -4+8 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_1

$$\begin{bmatrix} 0 & \mathbf{6} & \infty & 7 & \infty \\ \boldsymbol{\infty} & \mathbf{0} & \mathbf{5} & \mathbf{8} & \mathbf{-4} \\ \infty & \mathbf{-2} & 0 & \infty & \infty \\ \infty & \boldsymbol{\infty} & -3 & 0 & 9 \\ 2 & \mathbf{8} & 7 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} \infty & 6 & 11 & 14 & 2 \\ \infty & 0 & 5 & 8 & -4 \\ \infty & -2 & 3 & 6 & -6 \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 8 & 13 & 16 & 4 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_2

$$\begin{bmatrix} 0 & 6 & \mathbf{11} & 7 & 2 \\ \infty & 0 & \mathbf{5} & 8 & -4 \\ \boldsymbol{\infty} & \mathbf{-2} & \mathbf{0} & \mathbf{6} & \mathbf{-6} \\ \infty & \infty & \mathbf{-3} & 0 & 9 \\ 2 & 8 & \mathbf{7} & 9 & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_2

$$\begin{bmatrix} 0 & 6 & \mathbf{11} & 7 & 2 \\ \infty & 0 & \mathbf{5} & 8 & -4 \\ \boldsymbol{\infty} & \mathbf{-2} & \mathbf{0} & \mathbf{6} & \mathbf{-6} \\ \infty & \infty & \mathbf{-3} & 0 & 9 \\ 2 & 8 & \mathbf{7} & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} \infty + 11 & -2 + 11 & 0 + 11 & 6 + 11 & -6 + 11 \\ \infty + 5 & -2 + 5 & 0 + 5 & 6 + 5 & -6 + 5 \\ \infty + 0 & -2 + 0 & 0 + 0 & 6 + 0 & -6 + 0 \\ \infty - 3 & -2 - 3 & 0 - 3 & 6 - 3 & -6 - 3 \\ \infty + 7 & -2 + 7 & 0 + 7 & 6 + 7 & -6 + 7 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_2

$$\begin{bmatrix} 0 & 6 & \mathbf{11} & 7 & 2 \\ \infty & 0 & \mathbf{5} & 8 & -4 \\ \boldsymbol{\infty} & \mathbf{-2} & \mathbf{0} & \mathbf{6} & \mathbf{-6} \\ \infty & \infty & \mathbf{-3} & 0 & 9 \\ 2 & 8 & \mathbf{7} & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} \infty & 9 & 11 & 17 & 5 \\ \infty & 3 & 5 & 11 & -1 \\ \infty & -2 & 0 & 6 & -6 \\ \infty & -5 & -3 & 3 & -9 \\ \infty & 5 & 7 & 13 & 1 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_3

$$\begin{bmatrix} 0 & 6 & 11 & \mathbf{7} & 2 \\ \infty & 0 & 5 & \mathbf{8} & -4 \\ \infty & -2 & 0 & \mathbf{6} & -6 \\ \boldsymbol{\infty} & \mathbf{-5} & \mathbf{-3} & \mathbf{0} & \mathbf{-9} \\ 2 & 5 & 7 & \mathbf{9} & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_3

$$\begin{bmatrix} 0 & 6 & 11 & \mathbf{7} & 2 \\ \infty & 0 & 5 & \mathbf{8} & -4 \\ \infty & -2 & 0 & \mathbf{6} & -6 \\ \boldsymbol{\infty} & \mathbf{-5} & \mathbf{-3} & \mathbf{0} & \mathbf{-9} \\ 2 & 5 & 7 & \mathbf{9} & 0 \end{bmatrix} \qquad \begin{bmatrix} \infty+7 & -5+7 & -3+7 & 0+7 & -9+7 \\ \infty+8 & -5+8 & -3+8 & 0+8 & -9+8 \\ \infty+6 & -5+6 & -3+6 & 0+6 & -9+6 \\ \infty+0 & -5+0 & -3+0 & 0+0 & -9+0 \\ \infty+9 & -5+9 & -3+9 & 0+9 & -9+9 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_3

$$\begin{bmatrix} 0 & 6 & 11 & \mathbf{7} & 2 \\ \infty & 0 & 5 & \mathbf{8} & -4 \\ \infty & -2 & 0 & \mathbf{6} & -6 \\ \boldsymbol{\infty} & \mathbf{-5} & \mathbf{-3} & \mathbf{0} & \mathbf{-9} \\ 2 & 5 & 7 & \mathbf{9} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \infty & 2 & 4 & 7 & -2 \\ \infty & 3 & 5 & 8 & -1 \\ \infty & 1 & 3 & 6 & -3 \\ \infty & -5 & -3 & 0 & -9 \\ \infty & 4 & 6 & 9 & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_4

$$\begin{bmatrix} 0 & 2 & 4 & 7 & \mathbf{-2} \\ \infty & 0 & 5 & 8 & \mathbf{-4} \\ \infty & -2 & 0 & 6 & \mathbf{-6} \\ \infty & -5 & -3 & 0 & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{9} & \mathbf{0} \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_4

$$\begin{bmatrix} 0 & 2 & 4 & 7 & \mathbf{-2} \\ \infty & 0 & 5 & 8 & \mathbf{-4} \\ \infty & -2 & 0 & 6 & \mathbf{-6} \\ \infty & -5 & -3 & 0 & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{9} & \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} 2-2 & 4-2 & 6-2 & 9-2 & 0-2 \\ 2-4 & 4-4 & 6-4 & 9-4 & 0-4 \\ 2-6 & 4-6 & 6-6 & 9-6 & 0-6 \\ 2-9 & 4-9 & 6-9 & 9-9 & 0-9 \\ 2+0 & 4+0 & 6+0 & 9+0 & 0+0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_4

$$\begin{bmatrix} 0 & 2 & 4 & 7 & \mathbf{-2} \\ \infty & 0 & 5 & 8 & \mathbf{-4} \\ \infty & -2 & 0 & 6 & \mathbf{-6} \\ \infty & -5 & -3 & 0 & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{9} & \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 & 4 & 7 & -2 \\ -2 & 0 & 2 & 5 & -4 \\ -4 & -2 & 0 & 3 & -6 \\ -7 & -5 & -3 & 0 & -9 \\ 2 & 4 & 6 & 9 & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

D_5

$$\begin{bmatrix} 0 & 2 & 4 & 7 & -2 \\ -2 & 0 & 2 & 5 & -4 \\ -4 & -2 & 0 & 3 & -6 \\ -7 & -5 & -3 & 0 & -9 \\ 2 & 4 & 6 & 9 & 0 \end{bmatrix}$$

Floyd-Warshall 1.0 Example

Simple code: Take min with ith row + ith column matrix for $i = 1, \ldots, n$.

```
for i in 1:n
        A = min.(A, A[i,:]+A[:,i])
end
```

$$\begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{7} & \mathbf{-2} \\ -2 & 0 & 2 & 5 & -4 \\ -4 & -2 & 0 & 3 & -6 \\ -7 & -5 & -3 & 0 & -9 \\ 2 & 4 & 6 & 9 & 0 \end{bmatrix}$$

Note: First row matches Bellman-Ford solution.