CS 341: Algorithms

Lecture 17: Max flow = Min cut

Slides due to Éric Schost and based on lecture notes by many other CS341 instructors

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Sir Tony Hoare

- Age 92
- Inventor of quicksort/quickselect
- Inventor of “null”
- Turing Award (1980)
- Concurrency and correctness

Cuts

Cuts

Definition

- a **cut** is a partition of the vertices into sets $\boldsymbol{A}$ and $\boldsymbol{B = V - A}$, with $s \in A$ and $t \in B$.
- the **capacity** of the cut is

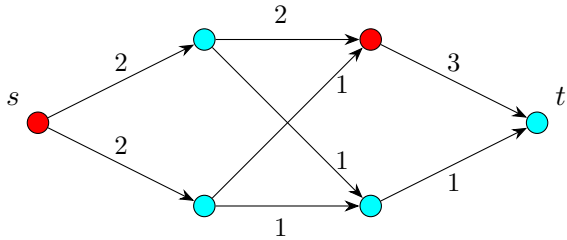
$$c(A) = \sum_{e:A \to B} c(e)$$

  (does not depend on any flow, only on the graph and its capacities)
- if f is a flow, the **outgoing** and **incoming** flows of the cut are

$$v_{\text{out}}(f, A) = \sum_{e:A \to B} f(e), \quad v_{\text{in}}(f, A) = \sum_{e:B \to A} f(e)$$

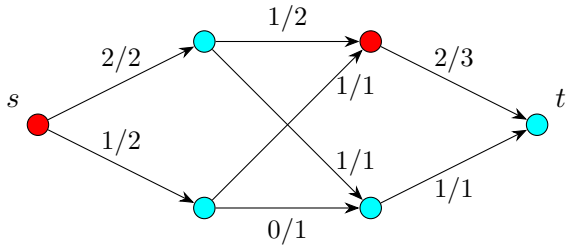
- **net flow** is $v_{\text{net}}(f, A) = v_{\text{out}}(f, A) - v_{\text{in}}(f, A)$

Examples



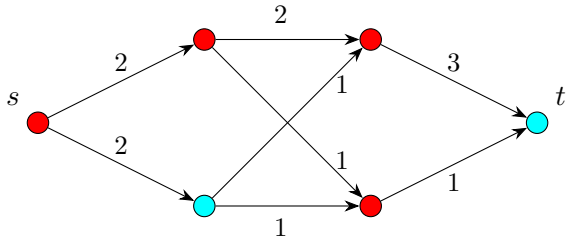
- A is in red and B in light blue,
- capacity is $2 + 2 + 3 = 7$,

Examples



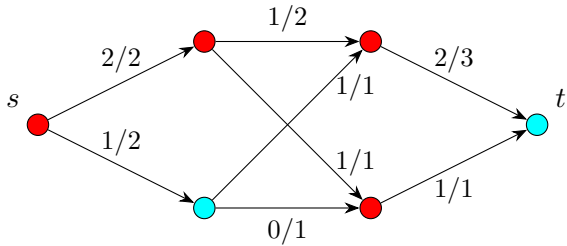
- A is in red and B in light blue,
- capacity is $2 + 2 + 3 = 7$,
- outgoing flow is $2 + 1 + 2 = 5$,
- incoming flow is $1 + 1 = 2$,
- net flow is 3

Examples



- A is in red and B in light blue,
- capacity is $2 + 3 + 1 = 6$,

Examples



- A is in red and B in light blue,
- capacity is $2 + 3 + 1 = 6$,
- outgoing flow is $1 + 2 + 1 = 4$,
- incoming flow is 1,
- net flow is 3

Flows and cuts

Claim

For any flow f and any cut A, we have

$$\mathsf{Val}(f) = v_{\text{out}}(f, A) - v_{\text{in}}(f, A)$$

Proof: induction on A.

- true when $A = \{s\}$, by definition.
- suppose this is true for a cut A, $B = V - A$, we show this is true for the cut $A' = A \cup \{v\}$, $B' = B - \{v\}$, for any vertex $v \in B$ (with $v \neq t$).

What we need to do:

- relate $v_{\text{out}}(f, A)$ to $v_{\text{out}}(f, A')$,
- relate $v_{\text{in}}(f, A)$ to $v_{\text{in}}(f, A')$.

Step 1

$$\begin{aligned} v_{\text{out}}(f, A) &= \sum_{e:A\to B} f(e) \\ &= \sum_{e:A\to v} f(e) \quad + \quad \sum_{e:A\to B'} f(e) \end{aligned}$$

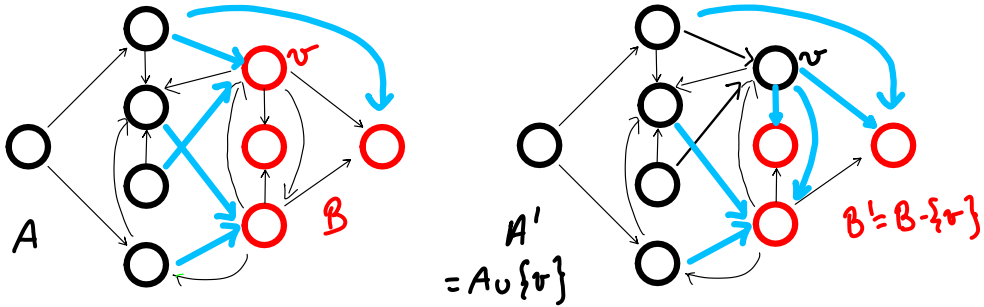
and

$$\begin{aligned} v_{\text{out}}(f, A') &= \sum_{e:A'\to B'} f(e) \\ &= \sum_{e:A\to B'} f(e) \quad + \quad \sum_{e:v\to B'} f(e). \end{aligned}$$

so

$$v_{\text{out}}(f, A') = v_{\text{out}}(f, A) - \sum_{e:A\to v} f(e) + \sum_{e:v\to B'} f(e)$$

Step 1



so

$$v_{\text{out}}(f, A') \quad = \quad v_{\text{out}}(f, A) - \sum_{e:A \to v} f(e) + \sum_{e:v \to B'} f(e)$$

Step 2

$$\begin{aligned} v_{\text{in}}(f, A) &= \sum_{e:B \to A} f(e) \\ &= \sum_{e:v \to A} f(e) \quad + \quad \sum_{e:B' \to A} f(e) \end{aligned}$$

and

$$\begin{aligned} v_{\text{in}}(f, A') &= \sum_{e:B' \to A'} f(e) \\ &= \sum_{e:B' \to A} f(e) \quad + \quad \sum_{e:B' \to v} f(e). \end{aligned}$$

so

$$v_{\text{in}}(f, A') = v_{\text{in}}(f, A) - \sum_{e:v \to A} f(e) + \sum_{e:B' \to v} f(e)$$

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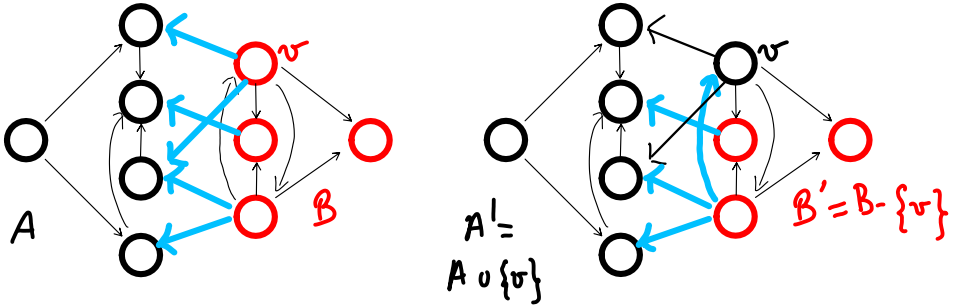


so

$$v_{\text{in}}(f, A') = v_{\text{in}}(f, A) - \sum_{e:v \to A} f(e) + \sum_{e:B' \to v} f(e)$$

Step 3

Because f is a flow, we have

$$\sum_{e:v\to A} f(e) \;+\; \sum_{e:v\to B'} f(e) \;=\; \sum_{e:B'\to v} f(e) \;+\; \sum_{e:A\to v} f(e)$$

so

$$\begin{aligned} v_{\text{out}}(f,A') &= v_{\text{out}}(f,A) - \sum_{e:A\to v} f(e) + \sum_{e:v\to B'} f(e) \\ &= v_{\text{out}}(f,A) - \sum_{e:v\to A} f(e) + \sum_{e:B'\to v} f(e) \end{aligned}$$

and still

$$v_{\text{in}}(f,A') = v_{\text{in}}(f,A) - \sum_{e:v\to A} f(e) + \sum_{e:B'\to v} f(e)$$

This gives

$$\begin{aligned} v_{\text{out}}(f,A') - v_{\text{in}}(f,A') &= v_{\text{out}}(f,A) - v_{\text{in}}(f,A) \\ &= \mathsf{Val}(f). \end{aligned}$$

Maximum flow and minimal cut

Consequence

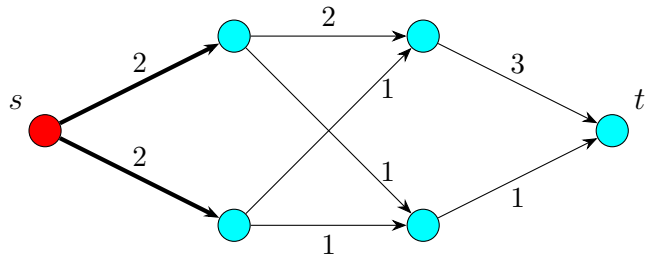
for **any flow** f and **any cut** A, we have $\mathsf{Val}(f) \leq c(A)$.

proof:

$$\begin{aligned} \mathsf{Val}(f) &= v_{\text{out}}(f, A) - v_{\text{in}}(f, A) \\ &\leq v_{\text{out}}(f, A) \\ &\leq c(A) \end{aligned}$$

- so the **maximal value** of a flow $\leq$ **minimal capacity** of a cut
- and if we find **any** flow and cut with equality, they are optimal

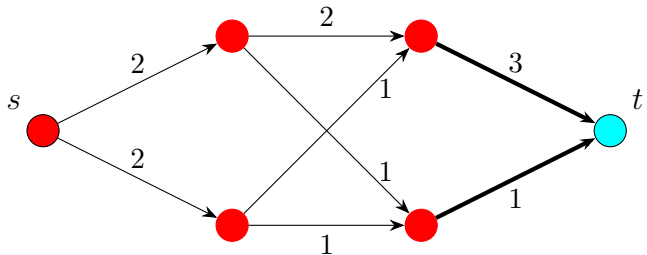
Example 1



Max flow?

- we found 4 in the previous lecture
- with $A = \{s\}$, $c(A) = 4$
- so max flow = min cut = 4

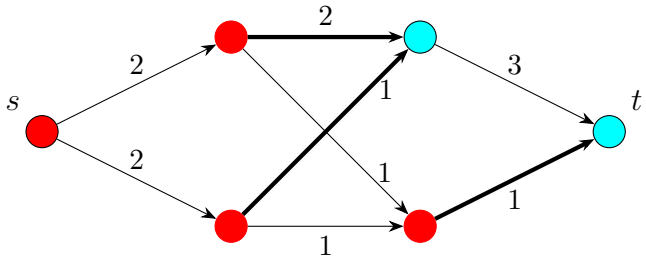
Example 1



Max flow?

- we found 4 in the previous lecture
- with $A = V - \{s\}$, $c(A) = 4$
- so max flow = min cut = 4

Example 1



Max flow?

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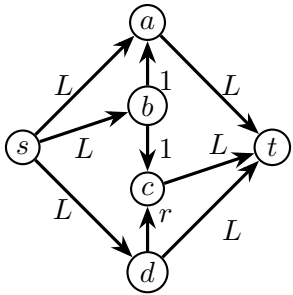
Example 2

A bonus example from last lecture: $r = (\sqrt{5} - 1)/2 \simeq 0.618$, $L \geq 1$



Max flow?

- easy to get $2L + 1$
- with $A = \{s, a, b\}$, $c(A) = 2L + 1$
- so max flow = min cut $= 2L + 1$

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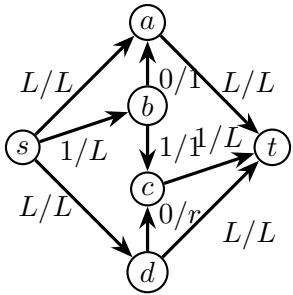

Max flow?

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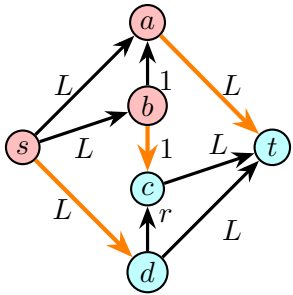


Max flow?

- easy to get $2L + 1$
- with $A = \{s, a, b\}$, $c(A) = 2L + 1$
- so max flow = min cut $= 2L + 1$

Max flow = min cut

Claim

no improving path in G_f $\implies$ can find a cut A such that $\mathsf{Val}(f) = c(A)$

Proof (sketch)

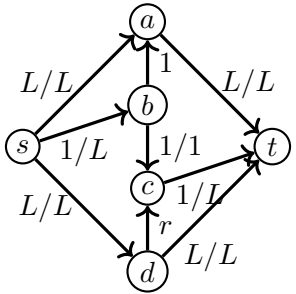
- let A be of vertices **reachable from s** in G_f
- this is a cut (s is in A, no path $s \to t$ in G_f so t is in $B = V - A$)
- needs a bit more work: $\mathsf{Val}(f) = c(A)$

Consequences

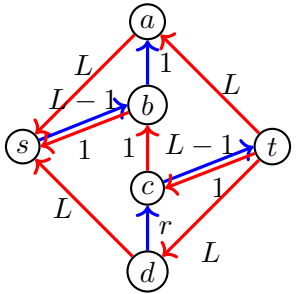
1. max flow = min cut (because no improving path for a max flow)
2. Ford-Fulkerson computes a max flow (because no improving path at the end)
(when/if it terminates)

WHY IS THERE A MAX FLOW?

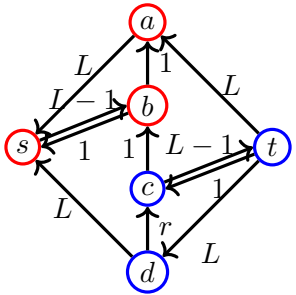
Example (same graph as before, $L > 1$)



Example (same graph as before, $L > 1$)



Example (same graph as before, $L > 1$)



$\boldsymbol{A}$ in red, $\boldsymbol{B} = \boldsymbol{V} - \boldsymbol{A}$ in blue

A few remarks
(time permitting)

Remark 1: Edmonds-Karp

A strategy that refines Ford-Fulkerson: choose a **shortest** path (BFS)

Edmonds-Karp Proof I

Let f be the old flow, γ a shortest augmenting path, and f' the new (augmented) flow.

Theorem

The distances from s do not decrease: for all $v \in V$, $\delta_{f'}(s, v) \geq \delta_f(s, v)$.

Proof:

Suppose to the contrary $\delta_{f'}(s, v) < \delta_f(s, v)$. Choose v with minimal distance $\delta_{f'}(s, v)$, and let (u, v) be the last edge in a shortest path in $G_{f'}$.

- $\delta_{f'}(s, u) \geq \delta_f(s, u)$ because v is minimal.
- (u, v) is a new edge only if (v, u) is on γ.
- Since (u, v) is on a shortest path, $\delta_{f'}(s, v) = \delta_{f'}(s, u) + 1$.
- Since (v, u) is on a shortest path, $\delta_f(s, u) = \delta_f(s, v) + 1$.
- $\implies \delta_{f'}(s, v) \geq \delta_f(s, v) + 2$.

Edmonds-Karp Proof II

Remarks:

- The augmenting paths in Edmonds-Karp get monotonically longer.
- Each improving step deletes at least one edge.
- New edges are created, **but** any shortest path that uses them is at least two steps longer than the current shortest.

Theorem

Edmonds-Karp runs in $O(m^2 n)$ time

Proof

- Each improving step runs in time $O(m)$.
- At distance d, each edge can be critical in at most one augmenting path
- Paths range from length 1 to $n - 1$.

Remark 2: thick paths

A slightly weaker strategy to refine Ford-Fulkerson: choose a path that **maximizes the bottleneck** capacity x.

Key ideas

- finding the thickest path: similar to Dijkstra
  - Dijkstra minimizes $\sum_{e\in\gamma} w(e)$
  - here we maximize $\min_{e\in\gamma} c(e)$
- in G_f, there is a path with $x \geq (M - \mathsf{Val}(f))/2m$, $M =$ max flow so

$$\mathsf{Val}(f') \geq \mathsf{Val}(f) + (M - \mathsf{Val}(f))/2m$$

  (takes some work)
- if capacities are integers, implies we do $O(m \log(M))$ iterations
- total $\boldsymbol{O(m^2 \log(n) \log(M))}$

Remark 3: maximal flow from linear programming

Equations for the max flow problem:

1. create a variable $f_{u,v}$ for each edge (u, v) and the linear constraints

$$f_{u,v} \geq 0, \quad f_{u,v} \leq c(u, v), \quad \sum_{(u,v) \text{ edge}} f_{u,v} = \sum_{(v,w) \text{ edge}} f_{v,w}$$

2. maximize

$$\sum_{(s,v) \text{ edge}} f_{s,v}.$$

- this is an instance of a **linear programming** problem
- max flow / min cut special case of **linear programming duality** (max something = min something else)

  (takes work)