CS 341: Algorithms

Lecture 18: Applications of flows and cuts

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based on lecture notes by many other CS341 instructors

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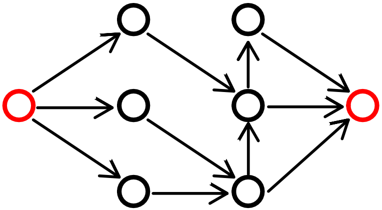
Fall 2024

Edge disjoint paths

Edge disjoint paths

Problem:

- **input:** a directed graph, with source $\boldsymbol{s}$ and sink $\boldsymbol{t}$
- **output:** the maximum number of **edge-disjoint paths** $\boldsymbol{s} \to \boldsymbol{t}$



Algorithm

- all edges get capacity 1
- Ford-Fulkerson outputs a 0/1 flow
  (0/1 flow = flow which only takes values 0 and 1)
  proof: flow values are integers ≤ 1

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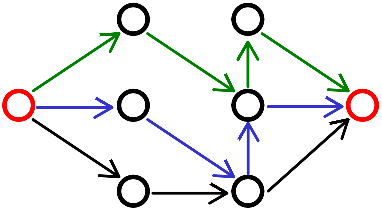

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Edge disjoint paths

Claim

There is a $0/1$ flow with value k **iff** there are k edge-disjoint paths $s \to t$ in G (using the edges with flow 1)

Paths $\to$ flow: set flow to 1 on edges covered by paths

Flow $\to$ paths by induction: given a $0/1$ flow f with N **flow-1 edges**, can find $\mathsf{Val}(f)$ edge-disjoint paths $s \to t$ in G using only flow-1 edges

- OK for $N = 0$
- induction assumption: true for $0, 1, \ldots, N-1$
- start from s, follow edges with flow 1
- **if we loop**, set flow to 0 on the cycle $\to$ flow with same value and smaller N
- **if we get to t**, set flow to 0 on the path $\to$ flow with value $\mathsf{Val}(f) - 1$ and smaller N

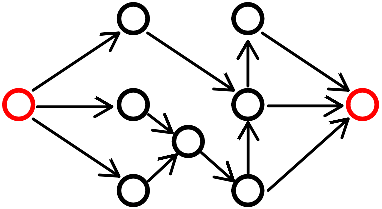
Remarks

1. Runtime:

- value of the flow is at most $\boldsymbol{n}$ so runtime $\boldsymbol{O(mn)}$ if no isolated vertex, $\boldsymbol{O((m+n)n)}$ otherwise

2. Edge version of Menger's theorem (from max flow = min cut)

- max number of edge-disjoint paths $s \to t$ = min number of edges to remove if we want to ensure there is no path $s \to t$
- exercise: fill in the details



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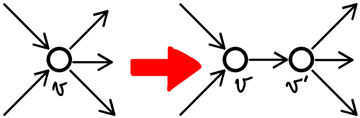
3. Vertex-disjoint paths

- transform the graph and find edge-disjoint paths



- $n' \le 2n$, $m' \le m + n$

Bipartite matching

Example

Context: students and professors:

- each professor offers **one** internship, but not all students may be eligible;
- each student wants **one** internship, and is ready to go for any of them.

Example: 3 students and 3 professors.

$$
\begin{aligned}
&P_1 \text{ will only consider } S_1 \text{ and } S_2\\
&P_2 \text{ will only consider } S_2\\
&P_3 \text{ will only consider } S_1 \text{ and } S_3
\end{aligned}
$$

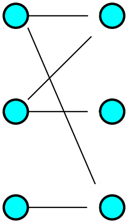
How to find the best matching?

Bipartite graphs

Definition

a (symmetric) graph G whose vertices are split into two groups S_i and P_j, with no edge between S_i's, or between P_j's.

Can be turned into a weighted directed graph G', adding a source and a sink



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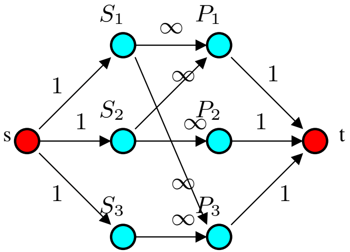


In general

Bipartite matching:

- a set of r edges (between the S_i and the P_j) with no common vertex, *i.e.*,
- vertices $S_{m_1}, \ldots, S_{m_r}$ and $P_{\ell_1}, \ldots, P_{\ell_r}$ such that S_{m_i} is connected to P_{ℓ_i}

How to find one: set up a flow problem as before

- a matching of size $\boldsymbol{r}$ gives a $0/1$ flow of value $\boldsymbol{r}$
- a $0/1$ flow of value $\boldsymbol{r}$ gives a matching of size $\boldsymbol{r}$ as in edge-disjoint paths (using 1s instead of ∞s, this is edge-disjoint paths)
- Ford-Fulkerson's algorithm returns a $0/1$ flow

Runtime: $n + 2$ vertices, $m + n$ edges so $O((m + n)n)$

Minimum vertex cover

Definition

- $G = (V, E)$ is a **symmetric graph**
- a **vertex cover** C is a subset of **vertices** that contains an extremity of **every edge**
- C vertex cover iff $V - C$ **independent set** (no vertices in it are directly connected)
- want C as small as possible

Complexity

- **trees:** dynamic programming (for independent sets)
- **bipartite graphs:** min cut
- **general graphs:** NP-hard

König's theorem

Thm

in a **bipartite graph**, maximum size of a matching $=$ minimum size of a vertex cover

take a matching of maximum size $\boldsymbol{r}$

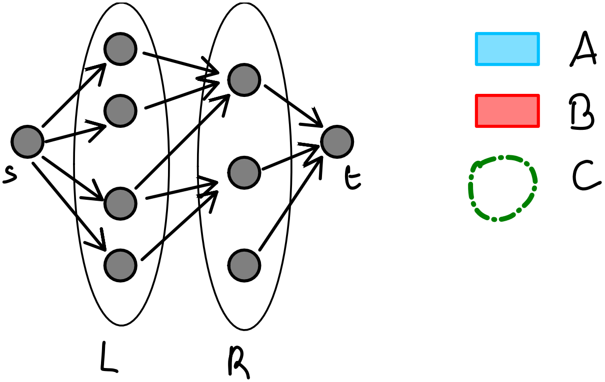
All vertex covers have size at least r

- need at least $\boldsymbol{r}$ vertices to cover these edges

Proof of $=$

- max flow has value $\boldsymbol{r}$ in G'
- max flow $=$ min cut: there is a cut $A, B = V - A$ of capacity $\boldsymbol{r}$ in G'
- use it to find a vertex cover of size $\boldsymbol{r}$

König's theorem



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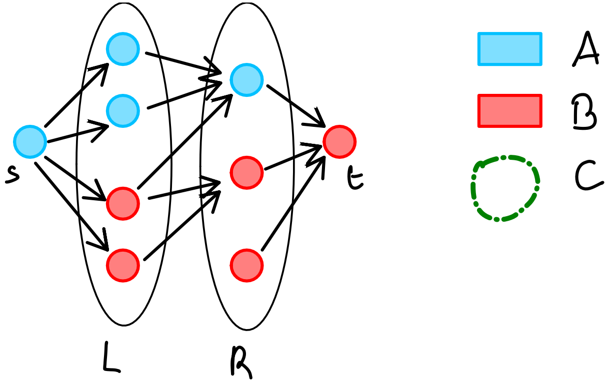

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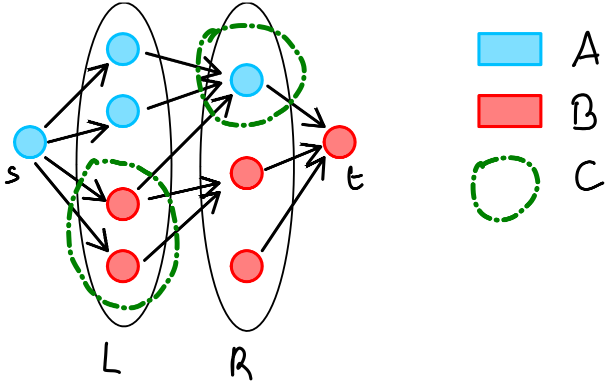

Define $\boldsymbol{C} = (\boldsymbol{L} \cap \boldsymbol{B}) \ \cup \ (\boldsymbol{R} \cap \boldsymbol{A})$

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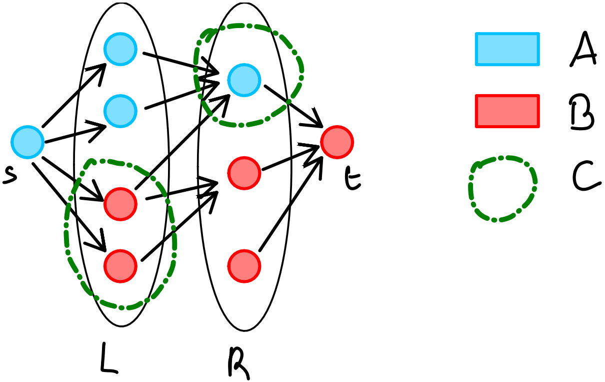


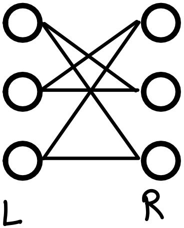
Define $\boldsymbol{C = (L \cap B) \ \cup \ (R \cap A)}$

- there are no edges **blue in $\boldsymbol{L}$** $\to$ **red in $\boldsymbol{R}$** (they have infinite capacity)
- so C is a vertex cover in G
- and $r = c(A) = |(\text{edges } s \to C)| + |(\text{edges } C \to t)| = |C|$

Special case: d-regular bipartite graphs (bonus)

d-regular:

- all vertices have d incident edges
- if G is also **bipartite**, then $\boldsymbol{m = d|L| = d|R|}$ and so $\boldsymbol{|L| = |R|}$



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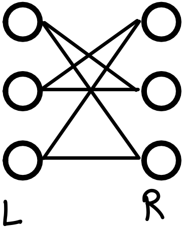

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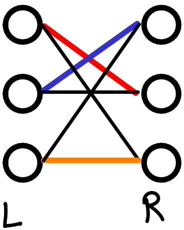


Claim

In a d-regular bipartite graph, there is a **perfect matching** (all vertices are matched)

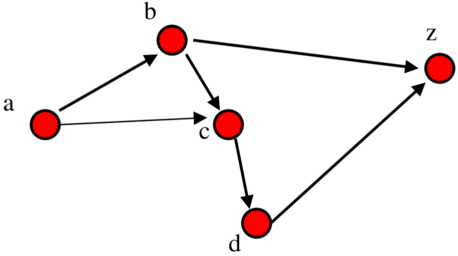
Proof

König's theorem: maximum size of a matching = minimum size of a vertex cover

- call $r = |L| = |R|$
  - 1 vertex covers d edges
  - 2 vertices cover at most $2d$ edges
  - $\ldots$
  - $r - 1$ vertices cover at most $(r-1)d$ edges
- we have **rd** edges, so $r - 1$ vertices are not sufficient
- so r vertices are necessary, and also sufficient (easy), and min vertex cover has size r

A shipping problem

Suppliers $a, b, c, d, \ldots$ want to send stuff to a buyer z, through a network



Initially, there are $s_a, s_b, \ldots$ units of stuff in $a, b, \ldots$.

We take **shipping time** into account: a maximum of stuff should arrive before $t = T$.

- the amount of stuff that can leave a toward b **per time unit** is $c_{(a,b)}$, and the same for $c_{(a,c)}, \ldots$
- the **traversal time** from a to b is $t_{(a,b)}$, and the same for $t_{(a,c)}, \ldots$

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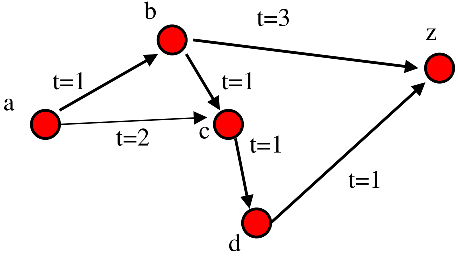

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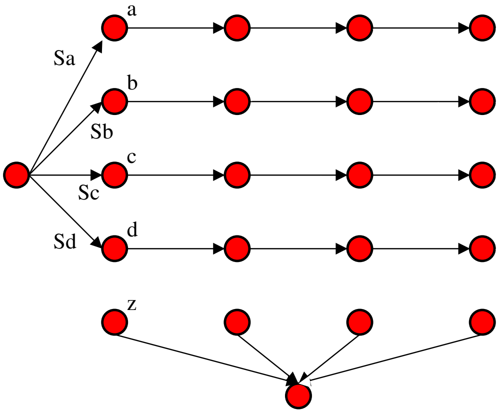
- previous graph is copied $T+1$ times: one copy for each time step
- edges are arranged to match the time contraints
- capacities are the $c_{(u,v)}$
- super-source and super-sink



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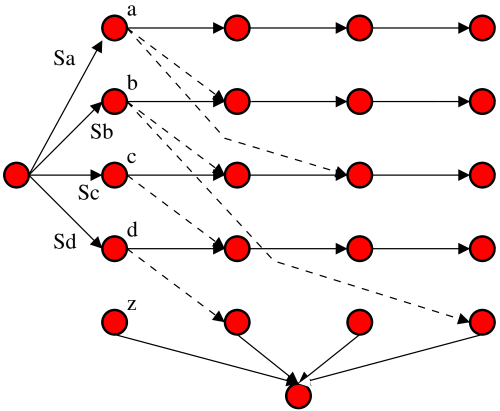

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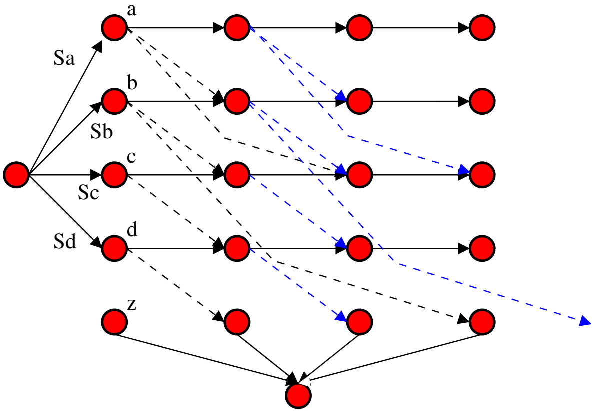

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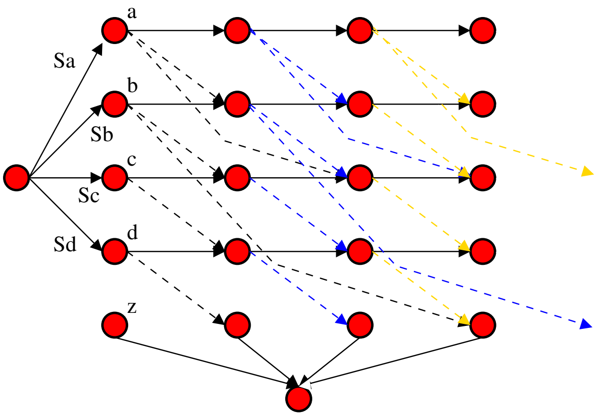

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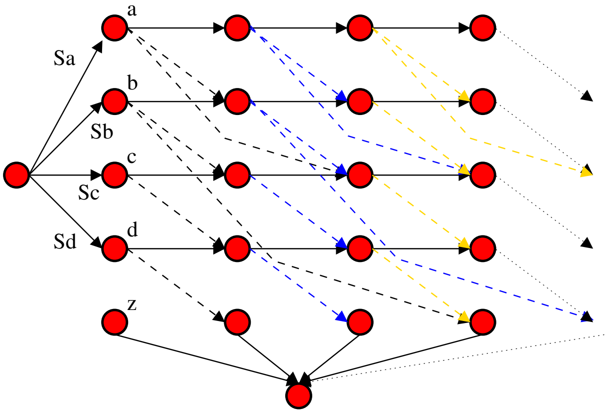


Image segmentation

Segmenting an image: separating a grid P of pixels into **background** and **foreground**.

Constraints:

- **single-pixel:** each pixel v in P comes with integers f_v and b_v
  - f_v large: v should be in the **foreground**
  - b_v large: v should be in the **background**
- **connectivity:** for adjacent pixels v, w, there is a **penalty** $p_{v,w}$ to pay if they are **not** in the same component.

Finding $f_v, b_v, p_{v,w}$ takes some work. Assuming we know them, want to find subset F **(foreground)** and $B = P - F$ **(background)** that maximizes

$$W(F) = \sum_{v \in F} f_v + \sum_{v \in B} b_v - \sum_{v \in F,\ w \in B,\ (v,w)\ \text{connected}} p_{v,w}.$$

Making it a min-cut problem

Step 1: turning a max into a min.

Let $K = \sum_v f_v + \sum_v b_v$ (independent of the choice of F and B).

Then

$$K = \sum_{v \in F} f_v \; + \sum_{v \in B} f_v + \sum_{v \in F} b_v \; + \sum_{v \in B} b_v .$$

So

$$W(F,B) \; = \; K \; - \; \sum_{v \in B} f_v \; - \; \sum_{v \in F} b_v \; - \sum_{v \in F,\ w \in B,\ (v,w) \text{ connected}} p_{v,w} .$$

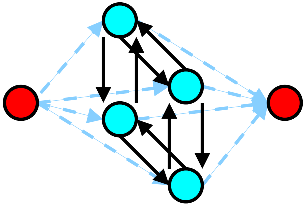
Since K does not depend on F and B, **maximizing** W is the same thing as **minimizing**

$$\sum_{v \in B} f_v \; + \; \sum_{v \in F} b_v \; + \sum_{v \in F,\ w \in B,\ (v,w) \text{ connected}} p_{v,w} .$$

Making it a min-cut problem

Step 2: setting up the graph G:

- vertices of G are **all pixels**, plus a source s and a sink t;
- edges of G:
  - for all v, an edge (s, v), with capacity b_v;
  - for all v, an edge (v, t), with capacity f_v;
  - for all neighbours v, w, edges (v, w) and (w, v), with capacities $p_{v,w}$.



Making it a min-cut problem

Step 3: understanding the cuts.

A **cut** $(U, V = P - U)$ in G gives a **partition** of the pixels:

- the background B is $U - \{s\}$,
- the foreground F is $V - \{t\}$.

What is its capacity? The edges $U \to V$ come into 3 categories:

- edges (v, t), for v in B, **contributes f_v to the capacity**,
- edges (s, v), for v in F, **contributes b_v to the capacity**,
- edges (v, w), for v in B and w in F, **contributes $p_{v,w}$ to the capacity**.

The capacity of the cut is

$$\sum_{v \in B} f_v + \sum_{v \in F} b_v + \sum_{v \in F,\ w \in B,\ (v,w) \text{ connected}} p_{v,w},$$

so solving min-cut solves the problem.