CS 341: Algorithms

Lecture 18: Applications of flows and cuts

Slides due to Éric Schost and based on lecture notes by many other CS341 instructors

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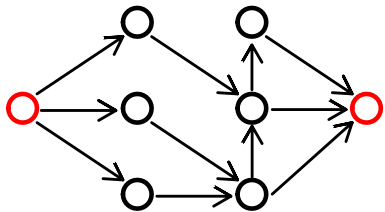
Fall 2025

Edge disjoint paths

Edge disjoint paths

Problem:

- **input:** a directed graph, with source s and sink t
- **output:** the maximum number of **edge-disjoint paths** $s \to t$



Algorithm

- all edges get capacity 1
- Ford-Fulkerson outputs a 0/1 flow
  (0/1 flow = flow which only takes values 0 and 1)
  proof: flow values are integers ≤ 1

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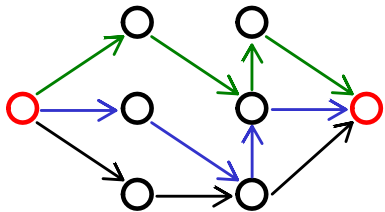

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Edge disjoint paths

Claim

There is a 0/1 flow with value k **iff** there are k edge-disjoint paths $s \to t$ in G (using the edges with flow 1)

Paths → flow: set flow to 1 on edges covered by paths

Flow → paths by induction: given a 0/1 flow f with N **flow-1 edges**, can find $\mathsf{Val}(f)$ edge-disjoint paths $s \to t$ in G using only flow-1 edges

- OK for $N = 0$
- induction assumption: true for $0, 1, \ldots, N-1$
- start from s, follow edges with flow 1
- **if we loop**, set flow to 0 on the cycle → flow with same value and smaller N
- **if we get to t**, set flow to 0 on the path → flow with value $\mathsf{Val}(f) - 1$ and smaller N

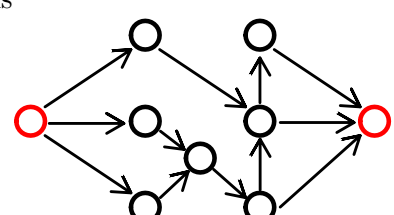
Remarks

1. Runtime:

- value of the flow is at most n so runtime $O(mn)$ if no isolated vertex, $O((m+n)n)$ otherwise

2. Edge version of Menger's theorem (from max flow = min cut)

- max number of edge-disjoint paths $s \to t$ = min number of edges to remove if we want to ensure there is no path $s \to t$
- exercise: fill in the details



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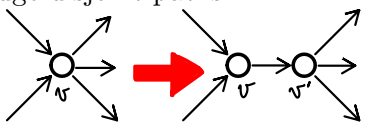
3. Vertex-disjoint paths

- transform the graph and find edge-disjoint paths



- $n' \leq 2n$, $m' \leq m+n$

Bipartite matching

Example

Sharing candy: n_1 children, n_2 pieces of candy

- children may have allergies
- candy can't be shared

Example: 3 kids, 3 pieces of candy

S_1 can only have P_1 and P_3
S_2 can only have P_1 and P_2
S_3 can only have P_3

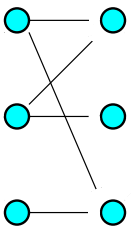
How to find the best matching?

Bipartite graphs

Definition

a (symmetric) graph G whose vertices are split into two groups S_i and P_j, with no edge between S_i's, or between P_j's.

Can be turned into a weighted directed graph G', adding a source and a sink



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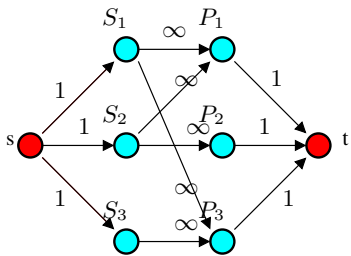


Bipartite matching

Output:

- a set of r edges (between the S_i and the P_j) with no common vertex, *i.e.*,
- want to maximize r

Algorithm: set up a flow problem as before

- a matching of size $\boldsymbol{r}$ gives a 0/1 flow of value $\boldsymbol{r}$
- a 0/1 flow of value $\boldsymbol{r}$ gives a matching of size $\boldsymbol{r}$
- Ford-Fulkerson's algorithm returns a 0/1 flow

Runtime: $n + 2$ vertices, $m + n$ edges so $O((m + n)n)$

Remark:

- could also have use capacity 1 on middle edges
- then, particular case of edge-disjoint paths

Minimum vertex cover

Definition

- $G = (V, E)$ is a **symmetric graph**
- a **vertex cover** C is a subset of **vertices** that contains an extremity of **every edge**
- C vertex cover iff $V - C$ **independent set** (no vertices in it are directly connected)
- want C as small as possible

Sometimes easy, usually hard

- **trees:** dynamic programming (for independent sets)
- **bipartite graphs:** min cut
  (a tree is bipartite, by even/odd levels)
- **general graphs:** NP-hard

König's theorem

Thm

in a **bipartite graph**, maximum size of a matching = minimum size of a vertex cover

take a matching of maximum size r

All vertex covers have size at least r

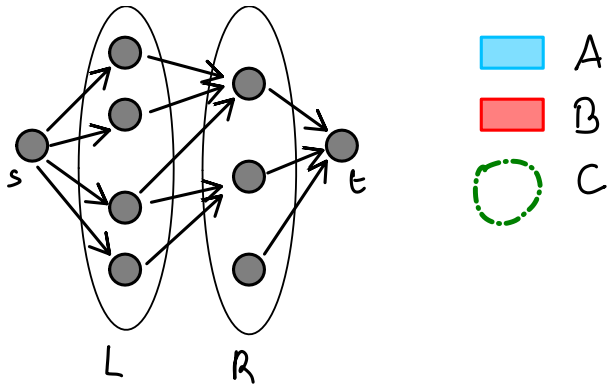
- need at least r vertices to cover these edges

Proof of $=$

- max flow has value r in G'
- max flow = min cut: there is a cut A, B of capacity r in G'
- use it to find a vertex cover of size r

In general: cut of capacity $r < \infty \implies$ vertex cover of size r

König's theorem



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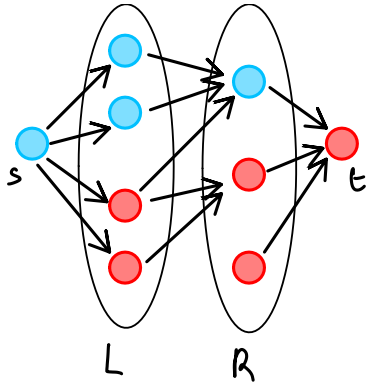

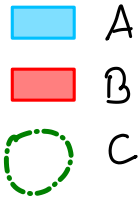

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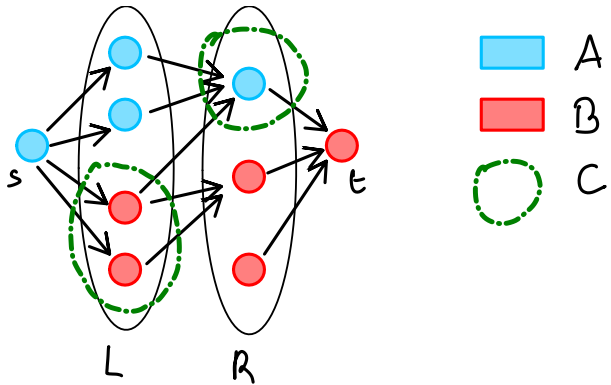

Define $\boldsymbol{C} = (\boldsymbol{L} \cap \boldsymbol{B}) \cup (\boldsymbol{R} \cap \boldsymbol{A})$

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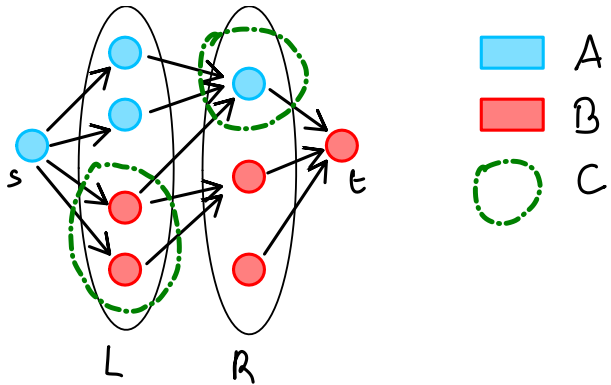


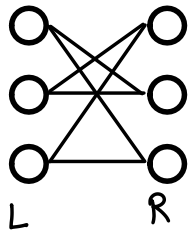
Define $\boldsymbol{C = (L \cap B) \ \cup \ (R \cap A)}$

- there are no edges **blue in L** $\rightarrow$ **red in R** (they have infinite capacity)
- so C is a vertex cover in G
- and $c(A) = |(\text{edges } s \rightarrow C)| + |(\text{edges } C \rightarrow t)| = |C|$

Special case: d-regular bipartite graphs (bonus)

d-regular:

- all vertices have d incident edges
- if G is also **bipartite**, then $\boldsymbol{m = d|L| = d|R|}$ and so $\boldsymbol{|L| = |R|}$



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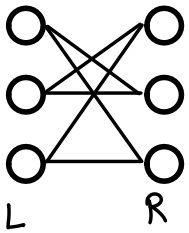

Claim

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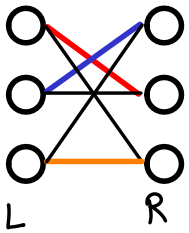

Claim

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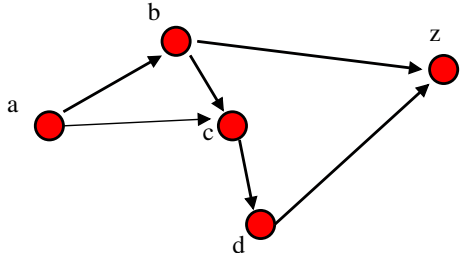
Proof

König's theorem: **maximum size of a matching = minimum size of a vertex cover**

- call $r = |L| = |R|$
  - 1 vertex covers d edges
  - 2 vertices cover at most $2d$ edges
  - ...
  - $r - 1$ vertices cover at most $(r-1)d$ edges
- we have **rd** edges, so $r - 1$ vertices are not sufficient
- so r vertices are necessary, and also sufficient (easy), and min vertex cover has size r

A shipping problem

Suppliers $a, b, c, d, \ldots$ want to send stuff to a buyer z, through a network



Initially, there are $s_a, s_b, \ldots$ units of stuff in $a, b, \ldots$.

We take **shipping time** into account: a maximum of stuff should arrive before $t = T$.

- the amount of stuff that can leave a toward b **per time unit** is $c_{(a,b)}$, and the same for $c_{(a,c)}, \ldots$
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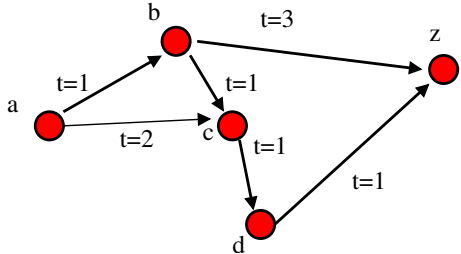

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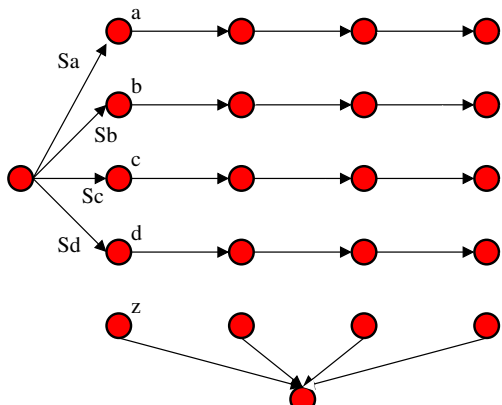
- previous graph is copied $T + 1$ times: one copy for each time step
- edges are arranged to match the time contraints
- capacities are the $c_{(u,v)}$
- super-source and super-sink



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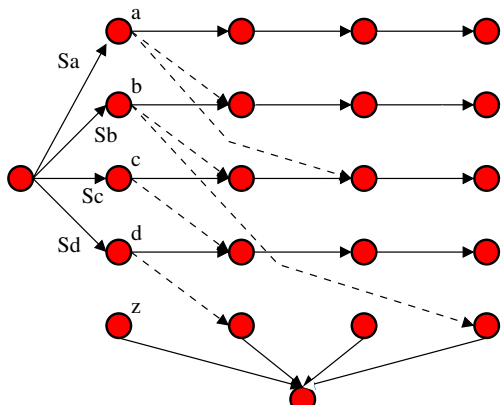

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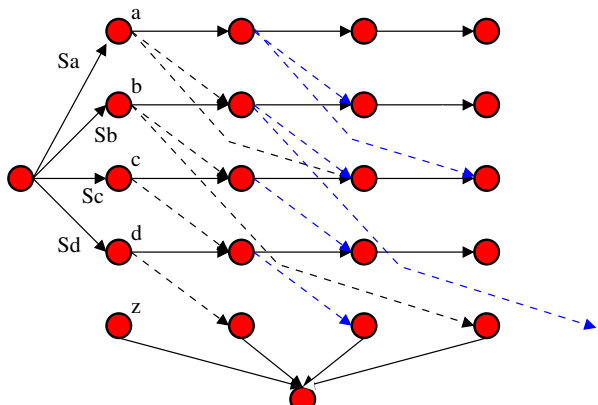

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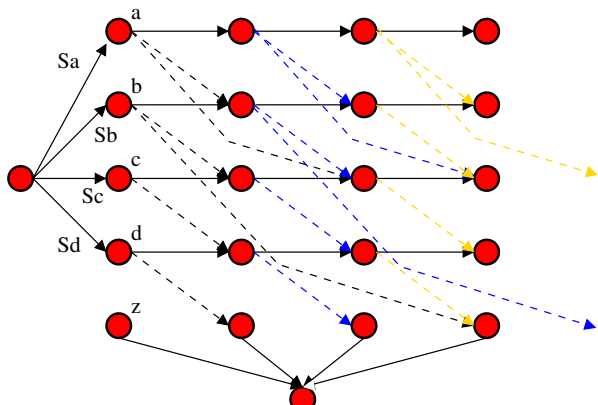

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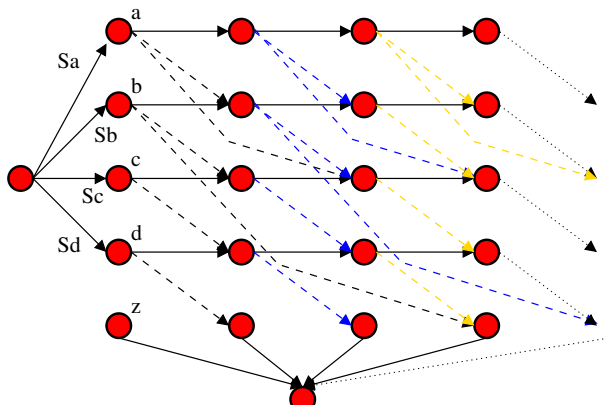


Image segmentation

Segmenting an image: separating a grid P of pixels into **background** and **foreground**.

Constraints:

- **single-pixel:** each pixel v in P comes with integers f_v and b_v
  - f_v large: v should be in the **foreground**
  - b_v large: v should be in the **background**
- **connectivity:** for adjacent pixels v, w, there is a **penalty** $p_{v,w}$ to pay if they are **not** in the same component.

Finding $f_v, b_v, p_{v,w}$ takes some work. Assuming we know them, want to find subset F **(foreground)** and $B = P - F$ **(background)** that maximizes

$$W(F) = \sum_{v \in F} f_v + \sum_{v \in B} b_v - \sum_{v \in F,\ w \in B,\ (v,w) \text{ connected}} p_{v,w}.$$

Making it a min-cut problem

Step 1: turning a max into a min.

Let $K = \sum_v f_v + \sum_v b_v$ (independent of the choice of F and B).

Then

$$K = \sum_{v \in F} f_v \; + \sum_{v \in B} f_v + \sum_{v \in F} b_v \; + \sum_{v \in B} b_v.$$

So

$$W(F,B) \; = \; K \; - \sum_{v \in B} f_v \; - \sum_{v \in F} b_v \; - \sum_{v \in F,\; w \in B,\; (v,w) \text{ connected}} p_{v,w}.$$

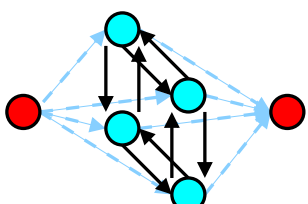
Since K does not depend on F and B, **maximizing** W is the same thing as **minimizing**

$$\sum_{v \in B} f_v \; + \sum_{v \in F} b_v \; + \sum_{v \in F,\; w \in B,\; (v,w) \text{ connected}} p_{v,w}.$$

Making it a min-cut problem

Step 2: setting up the graph G:

- vertices of G are **all pixels**, plus a source s and a sink t;
- edges of G:
  - for all v, an edge (s, v), with capacity b_v;
  - for all v, an edge (v, t), with capacity f_v;
  - for all neighbours v, w, edges (v, w) and (w, v), with capacities $p_{v,w}$.



Making it a min-cut problem

Step 3: understanding the cuts.

A **cut** $(U, V = P - U)$ in G gives a **partition** of the pixels:

- the background B is $U - \{s\}$,
- the foreground F is $V - \{t\}$.

What is its capacity? The edges $U \to V$ come into 3 categories:

- edges (v, t), for v in B, **contributes f_v to the capacity**,
- edges (s, v), for v in F, **contributes b_v to the capacity**,
- edges (v, w), for v in B and w in F, **contributes $p_{v,w}$ to the capacity**.

The capacity of the cut is

$$\sum_{v \in B} f_v \; + \sum_{v \in F} b_v \; + \sum_{v \in F,\; w \in B,\; (v,w) \text{ connected}} p_{v,w},$$

so solving min-cut solves the problem.