Lecture 20: Reductions II

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Overview

- **More Reductions**
  - Hamiltonian Cycle and Traveling Salesman Problem (TSP)
  - SAT, 3SAT & Independent Set
  - Graph Coloring & 3SAT
  - Subset Sum & Vertex Cover

- **Web of Reductions**

- **Acknowledgements**
Claim 1: hamiltonian cycle $\leq_m$ TSP

Reduction: different edge weights (for edges in graph vs edges not in graph)
More Reductions

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Web of Reductions

Acknowledgements
- $x_1, \ldots, x_n$ are boolean variables (i.e., take values in $\{0, 1\}$)
- a *literal* is a variable, or its negation (i.e., $x_i, \overline{x_i}$)
x_1, \ldots, x_n \text{ are boolean variables (i.e., take values in } \{0, 1\}\text{)}

a \textit{ literal} is a variable, or its negation (i.e., } x_i, \overline{x_i} \text{)

\textbf{CNF:} a boolean formula is in conjunctive normal form (CNF) if:
- it is the (conjunction) AND of a number of \textit{clauses},
- each clause being an (disjunction) OR of some \textit{literals}

Example:

\[(x_1 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor x_2 \lor \overline{x_5} \lor x_6) \land (x_2 \lor x_4)\]
SAT

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Example:

$$(x_1 \lor x_3 \lor \overline{x_4}) \land (\overline{x_1} \lor x_2 \lor \overline{x_5} \lor x_6) \land (\overline{x_2} \lor x_4)$$

- **SAT** problem
  - **Input**: a CNF formula
  - **Output**: YES, if it has a satisfying assignment; NO otherwise
3SAT

- a 3CNF formula is a CNF formula with exactly 3 literals per clause

Example:

\[(x_1 \lor x_3 \lor \overline{x_4}) \land (\overline{x_1} \lor x_2 \lor x_5) \land (\overline{x}_2 \lor x_4 \lor x_5)\]
3SAT

- a **3CNF** formula is a CNF formula with *exactly 3 literals* per clause

  Example:

  \[(x_1 \lor x_3 \lor \overline{x_4}) \land (\overline{x_1} \lor x_2 \lor \overline{x_5}) \land (x_2 \lor x_4 \lor x_5)\]

- **3SAT** problem
  - **Input:** a 3CNF formula
  - **Output:** YES, if it has a satisfying assignment; NO otherwise
3SAT

- a 3CNF formula is a CNF formula with *exactly 3 literals* per clause
  Example:

\[
(x_1 \lor x_3 \lor \overline{x_4}) \land (\overline{x_1} \lor x_2 \lor \overline{x_5}) \land (\overline{x_2} \lor x_4 \lor x_5)
\]

- 3SAT problem
  - Input: a 3CNF formula
  - Output: YES, if it has a satisfying assignment; NO otherwise

Why are we talking about this problem?

Exercise: prove that SAT $\leq_m$ 3SAT.
Claim 2: $\text{3SAT} \leq_m \text{IS}$
Claim 2: $3\text{SAT} \leq_m \text{IS}$

Proof: construct “conflict graph.”

Example: $(x \land y \land \bar{z}) \land (\bar{x} \land y \land z) \land (x \land \bar{y} \land z) \land (x \land y \land z) \land (\bar{x} \land y \land z) \land (x \land \bar{y} \land z)$
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Web of Reductions

Acknowledgements
Given graph $G(V, E)$ and $k \in \mathbb{N}$, a proper $k$-coloring of $G$ is a function $C : V \rightarrow \{1, 2, \ldots, k\}$ such that

For all $\{u, v\} \in E$ we have $C(u) \neq C(v)$. 
Graph Coloring

- Given graph $G(V, E)$ and $k \in \mathbb{N}$, a proper $k$-coloring of $G$ is a function $C : V \rightarrow \{1, 2, \ldots, k\}$ such that
  
  For all $\{u, v\} \in E$ we have $C(u) \neq C(v)$.

- **Graph Coloring** problem
  - **Input:** graph $G(V, E)$, $k \in \mathbb{N}$
  - **Output:** does $G$ admit a proper $k$-coloring?
Graph Coloring

- Given graph $G(V, E)$ and $k \in \mathbb{N}$, a **proper $k$-coloring** of $G$ is a function $C : V \rightarrow \{1, 2, \ldots, k\}$ such that
  
  For all $\{u, v\} \in E$ we have $C(u) \neq C(v)$.

- **Graph Coloring** problem
  - **Input**: graph $G(V, E)$, $k \in \mathbb{N}$
  - **Output**: does $G$ admit a proper $k$-coloring?

- **3 Coloring** (3COLOR) problem
  - **Input**: graph $G(V, E)$
  - **Output**: does $G$ admit a proper 3-coloring?
3SAT & 3COLOR

- **Claim 3:** \(3SAT \leq_m 3COLOR\)
3SAT & 3COLOR

- **Claim 3:** \(3\text{SAT} \leq_m 3\text{COLOR}\)
- **Proof:** let \(\varphi = C_1 \land \cdots \land C_m\) be a 3CNF
Claim 3: $3\text{SAT} \leq_m 3\text{COLOR}$

Proof: let $\varphi = C_1 \land \cdots \land C_m$ be a 3CNF

we use gadgets - subgraphs enforcing semantics of input formula $\varphi$

- Truth gadget: triangle with 3 vertices $T, F, X$ (standing for True, False, Other)

  Enforces that $T$ will be assigned color True, $F$ will be assigned color False and $X$ will be assigned the third color
Claim 3: 3SAT $\leq_m$ 3COLOR

Proof: let $\varphi = C_1 \land \cdots \land C_m$ be a 3CNF

we use gadgets - subgraphs enforcing semantics of input formula $\varphi$

- **Truth gadget**: triangle with 3 vertices $T, F, X$ (standing for True, False, Other)
  
  Enforces that $T$ will be assigned color True, $F$ will be assigned color False and $X$ will be assigned the third color

- **Literal gadget**: for each $x_i, \overline{x_i}$, we have a triangle with vertices $X, x_i, \overline{x_i}$
  
  Enforces $x_i$ and $\overline{x_i}$ get a proper assignment.


**3SAT & 3COLOR**

- **Claim 3:** \(3\text{SAT} \leq_m 3\text{COLOR}\)
- **Proof:** let \(\varphi = C_1 \land \cdots \land C_m\) be a 3CNF
  - we use gadgets - subgraphs enforcing semantics of input formula \(\varphi\)
    - **Truth gadget:** triangle with 3 vertices \(T, F, X\) (standing for True, False, Other)
      - Enforces that \(T\) will be assigned color True, \(F\) will be assigned color False and \(X\) will be assigned the third color
    - **Literal gadget:** for each \(x_i, \overline{x}_i\), we have a triangle with vertices \(X, x_i, \overline{x}_i\)
      - Enforces \(x_i\) and \(\overline{x}_i\) get a proper assignment.
    - **Clause gadget:** enforces each clause that becomes true under assignment will have a 3 coloring (iff) \(\text{Clause: } a \lor \overline{b} \lor \overline{c}\)
3SAT & 3COLOR - correctness

Need to prove following claims:

- Literal gadget enforces every variable is properly assigned in a coloring
- Clause gadget enforces that every valid 3-coloring of the graph corresponds to a variable assignment which makes corresponding clause true
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Web of Reductions

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Subset Sum & Vertex Cover

- **Claim 4:** Vertex Cover $\leq_m$ Subset Sum
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Proof: given \( G(V, E) \) and \( k \), need to construct (in poly-time) a (multi)set \( X \) of integers and \( T \) such that:

\[ X \text{ has a subset that sums to } T \iff G \text{ has a vertex cover of size } k. \]
Subset Sum & Vertex Cover

- **Claim 4:** Vertex Cover \( \leq_m \) Subset Sum

- **Proof:** given \( G(V, E) \) and \( k \), need to construct (in poly-time) a (multi)set \( X \) of integers and \( T \) such that:
  
  \( X \) has a subset that sums to \( T \) \( \iff \) \( G \) has a vertex cover of size \( k \).

- **Reduction:**
  - Number edges (arbitrarily) from 0 to \( m - 1 \). Edge \( i \) will correspond to integer \( b_i := 4^i \)
  - For each vertex \( u \in V \) assign number
    
    \[ a_u := 4^m + \sum_{i \in \delta(u)} 4^i \]
    
    where \( \delta(u) \) is the set of edges with \( u \) as one endpoint.
  - Let
    
    \[ T := k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i \]
  - Let \( X = \{a_u, b_i\}_{u \in V, 0 \leq i < m} \)
Subset Sum & Vertex Cover - proof of reduction

- Need to prove that $\langle G, k \rangle$ has a vertex cover of size $k$ iff $X$ has a subset of elements with sum $T$
Subset Sum & Vertex Cover - proof of reduction

- Need to prove that \( \langle G, k \rangle \) has a vertex cover of size \( k \) iff \( X \) has a subset of elements with sum \( T \)

- \( (\Rightarrow) \) Let \( C \subseteq V \) be a vertex cover of size \( k \). Consider the subset

\[
Y := \{a_u \mid u \in C\} \cup \{b_i \mid \text{edge } i \text{ has exactly one endpoint in } C\}
\]

easy to check it has sum \( T \)
Subset Sum & Vertex Cover - proof of reduction

- Need to prove that \( \langle G, k \rangle \) has a vertex cover of size \( k \) iff \( X \) has a subset of elements with sum \( T \)

\((\Leftarrow)\) Let \( \{a_u\}_{u \in C} \cup \{b_i\}_{i \in F} =: Y \subset X \) be a subset with sum \( T \). Must have:

\[
\sum_{u \in C} a_u + \sum_{i \in F} b_i = T = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i
\]

since there are no carries from lower order base-4 digits (i.e., the \( b_i \)'s), it must be the case that \( |C| = k \). moreover, to each \( 4^i \), there is at most one \( b_i \) on LHS that contributes with \( 4^i \), so \( C \) must be a vertex cover.
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Current algorithmic world view

- Wait, why haven’t we proved the missing arrows? Do they even hold?
Is there a way to organize our world view?

- Is there some property that unifies the problems we have seen so far?
- Why would any of these be considered “hard”?
- Can we “classify” problems according to their “difficulty”? How can we measure this?
Acknowledgement

Based on

- [Kleinberg Tardos 2006, Chapter 8]
- [Erickson 2019, Chapter 12]
References

Cormen, Thomas and Leiserson, Charles and Rivest, Ronald and Stein, Clifford (2009)
MIT Press

Dasgupta, Sanjay and Papadimitriou, Christos and Vazirani, Umesh (2006)
Algorithms

Erickson, Jeff (2019)
Algorithms
https://jeffe.cs.illinois.edu/teaching/algorithms/

Kleinberg, Jon and Tardos, Eva (2006)
Algorithm Design.
Addison Wesley