CS 341: Algorithms

Lecture 22: NP-completeness, continued

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based on lecture notes by many other CS341 instructors

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the Traveling Salesman Problem

Traveling salesman is NP-complete

Definition: TSP

- **input:** n, nonnegative integer "distances" $d_{u,v}$, $1 \leq u < v \leq n$, integer K
- **output:** is there a Hamiltonian cycle C in the complete graph G_n on n vertices with $\sum_{\{u,v\} \in C} d_{u,v} \leq K$
- **NP** (certificate? certifier? runtime? correctness?)

Claim

HamiltonianCycle $\leq_P$ TSP

Proof: given $G = (V, E)$, set $n = |V|$

- reduction: $d_{u,v} = 1$ if $\{u, v\}$ in E, $d_{u,v} = 2$ otherwise, and $K = n$
- Hamiltonian cycle in G_n with $\sum_{\{u,v\} \in C} d_{u,v} \leq n \iff d_{u,v} = 1$ for all $\{u, v\}$ on C $\iff$ all $\{u, v\}$ on C are in E

Remark: Euclidean traveling salesman

Definition: EUCLIDEANTSP

- **input:** $(x_u, y_u)_{1 \leq u \leq n}$ integers
- **output:** same as above, with $d_{u,v} = \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2}$ (not necessarily integers)

Unknown if NP

- **open:** how to test efficiently if a sum of square roots of integers is $\leq K$

NP-hard

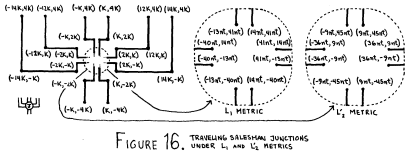


FIGURE 16. TRAVELING SALESMAN JUNCTIONS UNDER L_1 AND L'_1 METRICS

Colorability

kColorable

Definition:

- **input:** symmetric graph G, integer k
- **output:** can we color all vertices of G with $\boldsymbol{k}$ colors, with adjacent vertices having **different** colors?
- **NP**

Remark

- $\boldsymbol{k = 1}$: G 1-colorable iff no edge
- $\boldsymbol{k = 2}$: G 2-colorable iff bipartite

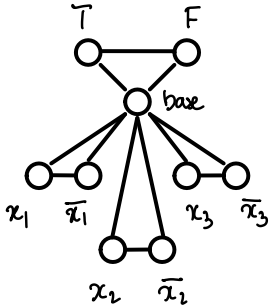
kColorable is NP-complete

Claim

$3\text{SAT} \leq_P \text{kColorable}$

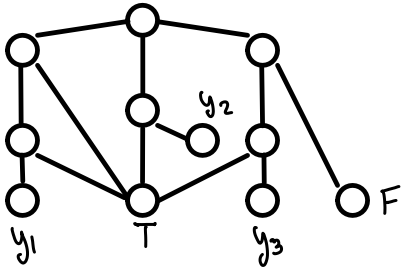
Starting the construction

- given a formula in 3CNF, e.g. $(x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2)$
- build a graph with vertices x_i and $\overline{x_i}$, T, F, and a base
- 3-coloring $\iff$ variable assigments (x_i true if the same color as T)



3-colorability gadget

For any clause $\boldsymbol{y_1} \vee \boldsymbol{y_2} \vee \boldsymbol{y_3}$, attach graph below to the previous construction

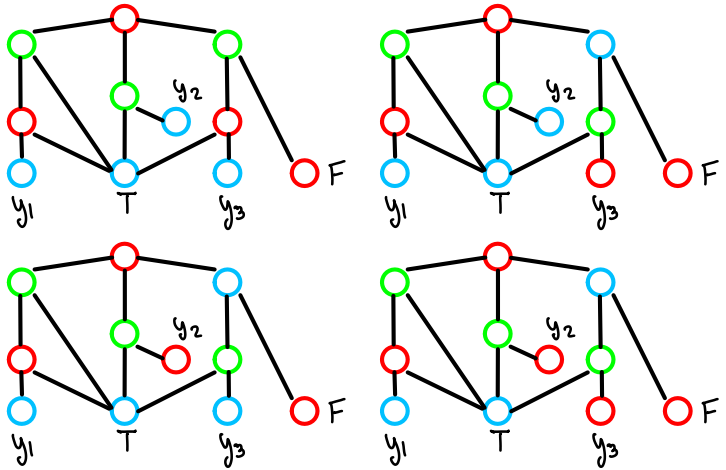


Claim

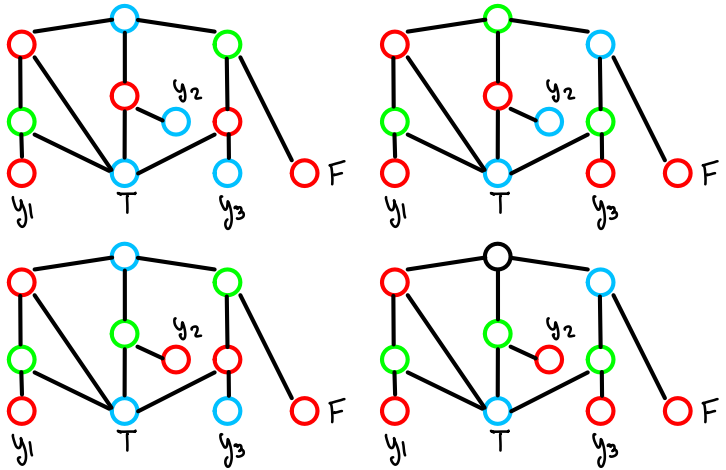
Assuming we can use colors T or F for y_1, y_2, y_3, this graph is 3-colorable **iff** at least one of y_1, y_2, y_3 is colored T

$\Longrightarrow$ new graph 3-coloriable **iff** all clauses can simultaneously be satisfied

Case discussion 1/2



Case discussion 2/2



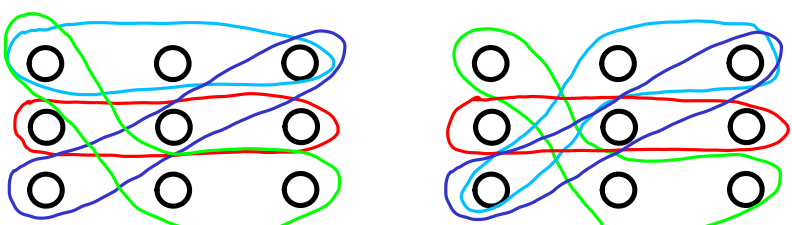
Perfect 3D matchings

Definition

3D matching

- **input:** 3 sets X, Y, Z of size n and a family of **hyperedges** $E \subset X \times Y \times Z$ (don't require $|E| = n$)
- **output:** is there a **perfect matching** (n hyperedges that cover X, Y and Z)? each x_i, and each y_j, and each z_k, is in a unique hyperedge
- **NP**

Examples



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Remark: 2D version

- **input:** 2 sets X, Y of size n and a family of **edges** $E \subset X \times Y$
- **output:** is there a **perfect matching** (n edges that cover X, Y)?
- this is testing if a bipartite graph has a perfect matching
- in **P** via max-flow

3DMatchings is NP-complete

(this is yet another question where 2 is easy and 3 looks hard)

Claim

3SAT $\leq_P$ 3DMatching

- **given:** a formula F in 3CNF, with s clauses $C_1, \ldots, C_s$
- **want:** an instance $H = (X, Y, Z, E)$ of 3DMatching such that F satisfiable iff H admits a perfect 3D matching
- reduction must be polynomial time

The variable gadget

build **one fidget-spinner-thing per variable** x_i, $i = 1, \ldots, n$.

Vertices (not done)

- $2s$ **core vertices** $v_{i,1}, \ldots, v_{i,2s}$ — only used in the gadget
- $2s$ **tip vertices** $z^T_{i,1}, z^F_{i,1}, \ldots, z^T_{i,s}, z^F_{i,s}$ — will connect to clause vertices

Partition

- $\boldsymbol{v_{i,1}, v_{i,3}, v_{i,5}, \ldots}$ in $\boldsymbol{X}$ — $\boldsymbol{ns}$ so far
- $\boldsymbol{v_{i,2}, v_{i,4}, v_{i,6}, \ldots}$ in $\boldsymbol{Y}$ — $\boldsymbol{ns}$ so far
- all tips in $\boldsymbol{Z}$ — $\boldsymbol{2ns}$ (done)

Hyperedges: for $i = 1, \ldots, n,$, $j = 1, \ldots, s$

- $\{v_{i,\mathbf{2j-1}},\ v_{i,\mathbf{2j}},\ z^T_{i,j}\}$
- $\{v_{i,\mathbf{2j+1}},\ v_{i,\mathbf{2j}},\ z^F_{i,j}\}$

(not done)

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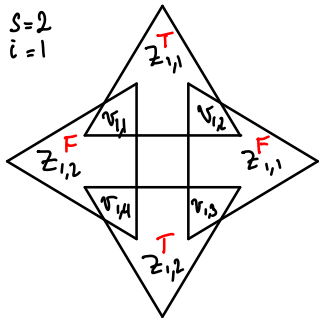
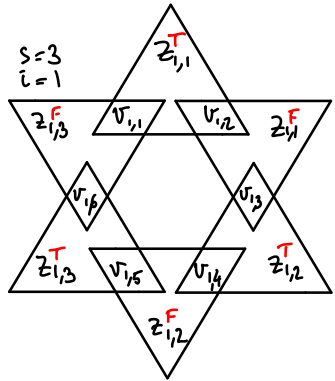
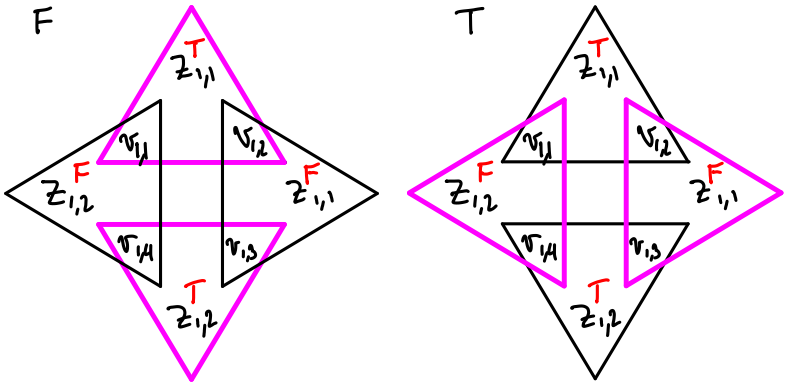




Covering the core vertices

- there will be no other hyperedge covering the core vertices $v_{i,j}$
- so only **two** ways to cover a gadget

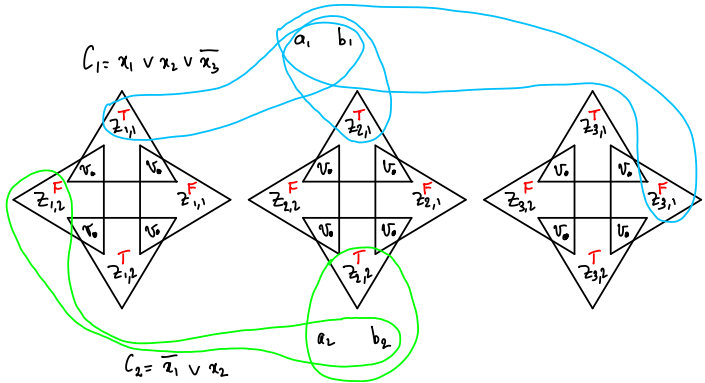


2^n coverings (2 possibilities per gadget) $\iff$ boolean values for the x_i's

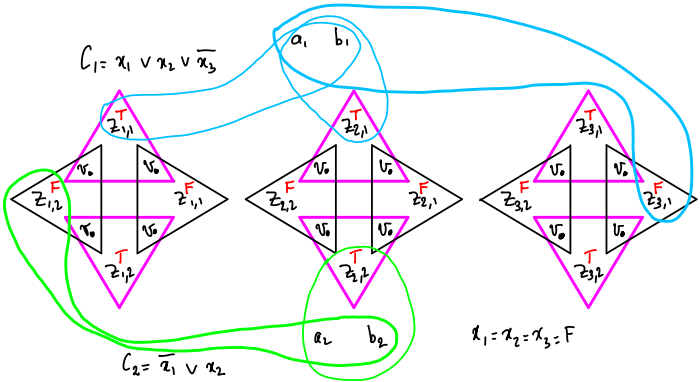
Using the clauses to (almost) finish the graph

For each clause C_j

- add two new vertices $\boldsymbol{a_j} \in \boldsymbol{X}$ and $\boldsymbol{b_j} \in \boldsymbol{Y}$
- for any literal $\boldsymbol{x_i}$ in C_j, add hyperedge $\{\boldsymbol{a_j}, \boldsymbol{b_j}, \boldsymbol{z^T_{i,j}}\}$
- for any literal $\overline{\boldsymbol{x_i}}$ in C_j, add hyperedge $\{\boldsymbol{a_j}, \boldsymbol{b_j}, \boldsymbol{z^F_{i,j}}\}$



Final adjustments



- we have $2ns = 12$ tips, with ns matched in pink and ns left
- in a perfect matching, each clause covers a tip, so $ns - s = 4$ tips left
- add $ns - s$ dummy vertices $\boldsymbol{d_k} \in \boldsymbol{X}$, $\boldsymbol{e_k} \in \boldsymbol{Y}$
- add **all** hyperedges $\{d_k,\ e_k,\ z^T_{i,j}\}$ and $\{d_k,\ e_k,\ z^F_{i,j}\}$ (that's $(ns - s)(2ns)$ of them)

F satisfiable iff perfect 3D matching

If F is satisfiable

- cover gadgets for $x_1, \ldots, x_n$ according to their truth value
- pick **exactly** one true literal per clause C_j
  - if x_i, take hyperedge $\{a_j, b_j, z^T_{i,j}\}$ $\quad z^T_{i,j}$ still free
  - if $\overline{x_i}$, take hyperedge $\{a_j, b_j, z^F_{i,j}\}$ $\quad z^F_{i,j}$ still free
- match all remaining tips with pairs of dummy vertices

If perfect 3D matching

- matching gives truth values
- for each clause C_j, we picked a hyperedge $\{a_j, b_j, z^T_{i,j}\}$, resp. $\{a_j, b_j, z^F_{i,j}\}$
- the corresponding x_i is T, resp. F
- this makes C_j satisfied either way

Subset sum and knapsack

Subset sum and Knapsack are NP-complete

Subset sum

- **given:** positive integers $a_1, \ldots, a_n$ and K
- **want:** is there a subset S of $\{1, \ldots, n\}$ with $\sum_{i \in S} a_i = K$
- **NP**

Claim

3DMatching $\leq_P$ SubsetSum $\leq_P$ Knapsack (this is already known)

- **given:** sets X, Y, Z of size n, m hyperedges $E \subset X \times Y \times Z$
- **want:** integers $a_1, \ldots, a_s, K$ s.t. perfect 3D matching iff $\sum_{i \in S} a_i = K$ for some $S \in \{1, \ldots, n\}$
- reduction must be polynomial time

From 3D matchings to vectors

we define $\boldsymbol{m}$ 0/1 vectors (one per hyperedge) of size $3n$.

- jth hyperedge $= \{x_{\boldsymbol{u}}, y_{\boldsymbol{v}}, z_{\boldsymbol{w}}\}$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\{1, \ldots, n\}$
- jth vector given by

$$v_j = \Big[0 \cdots 0\,1\,0 \cdots 0 \;\; 0 \cdots 0\,1\,0 \cdots 0 \;\; 0 \cdots 0\,1\,0 \cdots 0\Big]$$

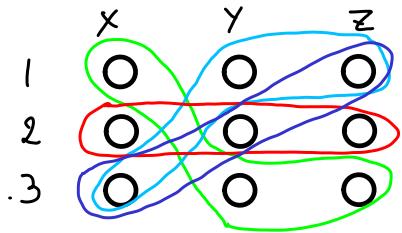
$$\uparrow_{u} \qquad \uparrow_{v+n} \qquad \uparrow_{w+2n}$$

Observation

there is a perfect 3D matching iff there is a subset of $\{v_1, \ldots, v_m\}$ that adds up to

$$\Big[1\,1\,1\,1\,1\,1 \cdots \cdots \cdots 1\,1\,1\,1\,1\,1\Big]$$

Example



$$v_1 = [\,1\,0\,0 \quad 0\,0\,1 \quad 0\,0\,1\,]$$
$$v_2 = [\,0\,1\,0 \quad 0\,1\,0 \quad 0\,1\,0\,]$$
$$v_3 = [\,0\,0\,1 \quad 1\,0\,0 \quad 1\,0\,0\,]$$
$$v_4 = [\,0\,0\,1 \quad 0\,1\,0 \quad 1\,0\,0\,]$$

From vectors to numbers

we define $\boldsymbol{m}$ integers (one per hyperedge) of bit-size polynomial in n, m

- set $\boldsymbol{b = m + 1}$ (large enough)
- given

$$v_j = \Big[0 \cdots 0\ 1\ 0 \cdots 0 \;\; 0 \cdots 0\ 1\ 0 \cdots 0 \;\; 0 \cdots 0\ 1\ 0 \cdots 0\Big]$$

$$\uparrow_{u} \qquad \uparrow_{v+n} \qquad \uparrow_{w+2n}$$

  define

$$\boldsymbol{a_j = b^u + b^{v+n} + b^{w+2n} = v_j \cdot [\ b\ b^2\ \cdots\ b^{3n}\ ]}$$

  (vector $v_j \leftrightarrow$ digits of a_j in base b)
- $a_j \le (m+1)^{3n+1}$ so $\log_2(a_j) \in (mn)^{O(1)}$

End of the proof

set $K = b + b^2 + \cdots + b^{3n}$ (so $\log(K) \in (nm)^{O(1)}$)

Claim

for S subset of $\{1, \ldots, m\}$, $\sum_{i \in S} v_i = [1 \ \cdots \ 1] \iff \sum_{i \in S} a_i = K$

1. we always have

$$\sum_{i \in S} a_i = \left(\sum_{i \in S} v_i \right) \cdot [\ b\ b^2\ \cdots\ b^{3n}\ ] = [\ c_1\ c_2\ \cdots\ c_{3n}\ ] \cdot [\ b\ b^2\ \cdots\ b^{3n}\ ], \quad 0 \leq c_j \leq m$$

2. if $\sum_{i \in S} v_i = [1 \ \cdots \ 1]$, $\sum_{i \in S} a_i = \sum_{j=1}^{3n} b^j = K$
3. if $\sum_{i \in S} a_i = K$,

$$\sum_{j=1}^{3n} b^j = \sum_{j=1}^{3n} c_j b^j$$

so $c_j = 1$ for all j **because $c_j < b$**, and $\sum_{i \in S} v_i = [1 \ \cdots \ 1]$