Lecture 24: Review Session & AMA

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Overview

- Review Session
  - Divide-and-Conquer
  - Greedy
  - Dynamic Programming
  - Graph Algorithms
  - Max-Flow Min-Cut
  - Reductions
  - Intractability

- Ask me Anything

- Acknowledgements
Divide-and-Conquer

Structure of divide-and-conquer:

1. **Divide:** given instance \( I \), construct smaller instances \( I_1, \ldots, I_a \) (subproblems)
   - Ideally want \( |I_j| \) small compared to \( |I| \) (say constant fraction)
Divide-and-Conquer

- **Structure of divide-and-conquer:**
  1. **Divide:** given instance $I$, construct smaller instances $I_1, \ldots, I_a$ (subproblems)
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- “Recursion for running time:”

$$T(I) = T(I_1) + \cdots + T(I_a) + \text{time to combine}$$
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Greedy Algorithms

- Greedy strategy based on following principles:
  1. choose a “progress measure”
  2. preprocess input accordingly
  3. make next decision based on what is best given current partial solution
  4. **Main idea:** must show that the greedy solution is always no worse than any other optimal solution!

     Usually can prove this by begin able to “transform” any optimal solution into the greedy one without losing anything.

**Exchange Argument**

5. **Optimal Substructure:** a problem has optimal substructure if any optimal solution contains optimal solutions to subproblems.
Dynamic Programming

- Sometimes, when trying a divide and conquer approach, we are only able to divide in a way which makes us perform "exhaustive search"
  
  Looks like it is going to be a bad divide and conquer
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  1. solve them once
  2. save them to memory
  3. and if we need them again, we already precomputed them!  (savings)
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**DP template**

1. identify small set of subproblems
2. devise proper recursion
3. show how bottom-up approach correctly compute the subproblems
Graph Search & Connectivity

Undirected graphs:

- **BFS**
  1. Finds shortest paths
  2. Can be used to detect graph is bipartite
  3. Shortest paths encoded in the BFS tree
  4. Non-tree edges in adjacent layers

- **DFS**
  1. Parenthesis lemma: start and finish time intervals are either disjoint, or one contains the other
  2. Non-tree edges (in DFS tree) must be back edges
  3. Checks for cut vertices or cut edges

BFS still gives you shortest paths from source

BFS and DFS trees have less structure, but parenthesis lemma still holds for DFS tree

DAGs (directed acyclic graphs) (topological sort)

Any directed graph is a DAG of SCCs (strongly connected components)

Linear time algorithm to find all SCCs!
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Minimum Spanning Trees (MSTs)

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- Boruvka’s algorithm:
  1. pick cheapest edge from a vertex, and contract it
  2. recurse on contracted graph

Cut property: given any cut \((S, \overline{S})\), there is an MST containing edge with smallest weight across cut.

Prim’s algorithm:
1. start from arbitrary vertex and grow connected component one vertex at a time

Kruskal’s algorithm:
1. consider edges from cheapest to most expensive, add edge to solution so long as it does not create a cycle
2. needs UNION-FIND for that last step

All of the above can be assumed to run in \(O(m \log n)\) time.
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Shortest Paths

- Single-source, all weights non-negative: Dijkstra
  - Similar to Prim’s algorithm (greedy)
  - Start from source, and build shortest paths by adding one vertex at a time
  - efficiently simulates the “water down the pipes” idea
  - Runtime $O((m + n) \log n)$

- Single-source, arbitrary weights: Bellman-Ford
  - Now cannot do the greedy approach, because of negative weights
  - DP for the rescue!
  - Subproblems
    - $D[v, i] :=$ captures shortest $s \rightarrow v$ distance using at most $i$ edges
  - Runtime $O(mn)$

- All-pairs shortest-paths (arbitrary weights, no negative cycles): Floyd-Warshall
  - Subproblems:
    - $D[u, v, k] :=$ shortest $u \rightarrow v$ path using only $\{1, \ldots, k\}$ as intermediate vertices
  - Runtime $O(n^3)$
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Max-Flow & Min-Cut

Let $G(V, E, c)$ be an undirected graph, with capacity (weight) function $C : E \rightarrow \mathbb{R}_{\geq 0}$, two special vertices $s, t \in V$

- **Flows**: $f : E \rightarrow \mathbb{R}_{\geq 0}$ satisfying:
  1. Capacity: $f(e) \leq c(e)$ for all $e \in E$
  2. Conservation: $f_{in}(u) = f_{out}(u)$ for all $u \in V \setminus \{s, t\}$
  3. Value: $f_{out}(s) - f_{in}(s)$

Flow decomposition theorem: any integral flow $f : E \rightarrow \mathbb{N}$ of value $r$ can be decomposed into paths $P_1, \ldots, P_r$ and cycles $C_1, \ldots, C_m$ such that each $e \in E$ appears in exactly $f(e)$ of the paths and cycles.

Cuts: a cut is a partition of the vertices into two sets $(S, V \setminus S)$.

Capacity of a cut: $C_{out}(S)$ is the total capacity coming out of $S$.

Max-Flow Min-Cut Theorem: the value of the maximum flow equals the minimum capacity of a cut.

Ford-Fulkerson algorithm: keep finding $s \rightarrow t$ paths in residual graph, when there is none, found a max-flow and a min-cut.
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Reductions

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- *Turing reductions* (a.k.a. Cook reductions)
  
  $A \leq_T^P B \iff$ there is a poly-time algorithm $M^B$ with oracle access to $B$
  such that $M^B$ solves $A$.
  
  - Oracle access: algorithm $M^B$ can query the oracle on inputs to problem $B$, and each query is counted as 1 unit of time
How do we prove problem \( A \) is “easier than” another problem \( B \)?

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\[ A \leq_T B \iff \text{there is a poly-time algorithm } M^B \text{ with oracle access to } B \text{ such that } M^B \text{ solves } A. \]

Oracle access: algorithm \( M^B \) can query the oracle on inputs to problem \( B \), and each query is counted as 1 unit of time.

**Karp reductions** (a.k.a. polynomial transformations)

\[ A \leq_p B \iff \text{there is a poly-time computable function } f : A \to B \text{ such that} \]

\[ x \text{ is a YES instance of } A \iff f(x) \text{ is YES instance of } B \]
NP-completeness

- NP is a class of *decision problems*
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- To prove a problem $B$ is NP-complete, need to prove
  1. $B \in \text{NP}$
  2. there is an NP-complete problem $A$ such that
     \[ A \leq_p B \]

*Notation:* $A \leq_p B$ is the same as I have used as $A \leq_m B$.
These are polynomial transformations (a.k.a. Karp reductions).
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Example:

- **HittingSet**
  - **Input:** collection of sets $S_1, \ldots, S_m \subseteq [n]$, integer $k \in \mathbb{N}$
  - Is there a collection of $k$ sets $S_i$ which contain all elements of $[n]$?
NP-hardness

A problem $B$ is NP-hard if there is an NP-complete problem $A$ such that

$$A \leq_T B$$

**Notation:** $A \leq_T B$ is the same as I have used as $A \leq_T B$.

These are Cook reductions (which I have denoted as Turing reductions).

**Bonus:** a bogus video on super mario bros and NP-hardness

https://www.youtube.com/watch?v=HhGI-GqAK9c

**Disclaimer:** if you are still confused about the complexity part, please do not watch it!

A real proof can be found here
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References

Cormen, Thomas and Leiserson, Charles and Rivest, Ronald and Stein, Clifford (2009)
MIT Press

Dasgupta, Sanjay and Papadimitriou, Christos and Vazirani, Umesh (2006)
Algorithms

Erickson, Jeff (2019)
Algorithms
https://jeffe.cs.illinois.edu/teaching/algorithms/

Kleinberg, Jon and Tardos, Eva (2006)
Algorithm Design.
Addison Wesley