- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's grading policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. **Definition 1** *Let X and Y be sets. Then the* ***intersection of*** X ***and*** Y***, denoted*** $X \cap Y$*, is the set of elements of both X and Y:*

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}.$$

Definition 2 *Let X and Y be sets. Then the* ***union of*** X ***and*** Y***, denoted*** $X \cup Y$*, is the set of elements of either X or Y (or both):*

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y, \text{ or both}\}.$$

Definition 3 *Let X and Y be sets. Then Y* ***is a subset of*** X***, denoted*** $Y \subseteq X$*, if and only if every element of Y is also an element of X.*

[6] Let X, Y and Z be sets. Rigourously prove this **set distributivity law**:

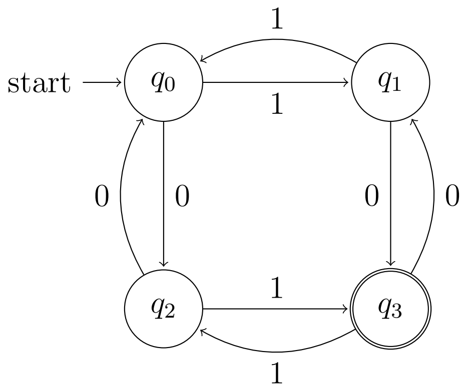
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

2. Let Σ be a non-empty finite alphabet. Let $x, y \in \Sigma^*$.

[4] (a) Prove that, for all integers $i \geq 0$, we have $(xy)^i x = x(yx)^i$.

[6] (b) Prove that $xy = yx$ if and only if there exists a word $z \in \Sigma^*$ such that $x^2y^2 = z^2$.

3. Consider the DFA, M, having alphabet $\Sigma = \{0, 1\}$ and defined by the following diagram.



[3] (a) Determine whether or not $w_a = 0110 \in L(M)$. Briefly justify your answer.

[3] (b) Determine whether or not $w_b = 10 \in L(M)$. Briefly justify your answer.

[3] (c) Determine whether or not $w_c = 010 \in L(M)$. Briefly justify your answer.

[3] (d) Determine whether or not $w_d = 101 \in L(M)$. Briefly justify your answer.

[1] (e) Give a brief description of $L(M)$. No justification is required for the correctness of the description.

4. Draw the diagram of a DFA, NFA or ε-NFA which accepts each of the following languages over $\Sigma = \{0, 1\}$, and argue informally why your automaton accepts exactly the language given.

[4] (a) $L_a = \{w \mid n_0(w) \equiv 0 \bmod 2\}$ (Recall that $n_0(w)$ denotes the number of occurrences of the symbol 0 in the string w.)

[6] (b) $L_b = \{w \mid n_0(w) \equiv 0 \bmod 2 \text{ and each 0 in } w \text{ is followed by at least one 1}\}$

5. Let $M = (\Sigma, Q, q_0, F, \delta)$ be a DFA.

   Let $\hat{\delta}$ denote the extended transition function of M, as defined in the lecture slides.

[4] (a) Prove that, for any $x, y \in \Sigma^*$, and any $q \in Q$, we have

$$\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y).$$

[4] (b) Assume that for some state $q \in Q$, and for every $a \in \Sigma$, we have $\delta(q, a) = q$. Prove that $\hat{\delta}(q, x) = q$ holds for every $x \in \Sigma^*$.

[4] (c) Assume that for some state $q \in Q$, and some string $x \in \Sigma^*$, we have $\hat{\delta}(q, x) = q$. Prove that, for every $n \geq 0$, we have $\hat{\delta}(q, x^n) = q$.