- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's marking policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. Regular Expressions
[8] Let $\Sigma = \{0, 1\}$. Give a **rigourous** proof for the equality of languages

$$L((10^*)^*0) = L(0 + 1(0 + 1)^*0).$$

2. Closure Properties of Regular Languages

[8] Let B and C be languages over $\Sigma = \{0, 1\}$. Define the binary operation on languages

$$B \stackrel{1}{\leftarrow} C = \{w \in B \mid \text{ for some } y \in C, n_1(w) = n_1(y)\}.$$

Recall that $n_1(w)$ denotes the number of occurrences of the symbol 1 in the string w. For example, if $B = \{010, 101, 111\}$ and $C = \{01, 011, 1111\}$, then $B \stackrel{1}{\leftarrow} C = \{010, 101\}$.

Prove that the class of regular languages is closed under the $\stackrel{1}{\leftarrow}$ operation.

3. Non-regular languages
   Prove that each of the following languages is not regular.

[4] (a) $L_a = \{0^i 1^j \mid \gcd(i, j) = 1\}$ over $\Sigma = \{0, 1\}$

[4] (b) $L_b = \{a^m \mid m \neq n^2 \text{ for any } n \in \mathbb{N}\}$ over $\Sigma = \{a\}$

4. A Non-Regular Language In Which All Long Words Can Be Pumped
   Let $\Sigma = \{a, b, c\}$.

[4] (a) Prove that $L = \{ab^j c^j \mid j \geq 0\}$ is not regular.

[4] (b) Prove that $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$ is not regular.

[4] (c) Exhibit with proof a choice of a positive integer n, such that, for any $z \in F$ with $|z| \geq n$, we may write $z = uvw$ where

- $|uv| \leq n$,
- $|v| \geq 1$ and
- $uv^iw \in F$, for all $i \geq 0$.

[2] (d) Explain briefly why the results of parts 4b and 4c do **not** contradict the Pumping Lemma for regular languages.

5. Testing a Candidate Criterion for Regular Languages

   Let $\Sigma = \{0, 1\}$ be the alphabet for all languages in this problem.

[4] (a) Prove that $L_a = \{w \mid n_0(w) = n_1(w)\}$ is not regular.

[4] (b) Prove that $L_b = \{w \mid n_0(w) \neq n_1(w)\}$ is not regular.

[4] (c) Suppose that L is a language over Σ and that there is a fixed integer $k \geq 0$ such that, for every $x \in \Sigma^*$, $xz \in L$, for some string z with $|z| \leq k$. Does it follow that L is regular? Prove your answer.