- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's marking policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. Non-context-free languages

[4] (a) Let $\Sigma = \{0, 1\}$. Prove that the language

$$L_a = \left\{ 0^i 1^j \mid i = j^2 \right\}$$

is not context-free.

[4] (b) Let $\Sigma = \{a\}$. Prove that the language

$$L_b = \left\{ a^{n^2} \mid n \geq 0 \right\}$$

is not context-free.

2. Closure rules for CFLs

   Recall that the **set difference** of two sets S and T, denoted $S \setminus T$, is defined as

$$S \setminus T = \{x \in S \mid x \notin T\}.$$

[2] (a) Prove that the class of context-free languages is **not** closed under set difference.

[2] (b) Let L_1 be a context-free language and let L_2 be a regular language. Prove that $L_1 \setminus L_2$ is a context-free language.

[6] (c) Consider two alphabets Σ_1 and Σ_2. Let *Sub* be any function $\Sigma_1 \rightarrow \Sigma_2^*$ (i.e. *Sub* replaces every $a \in \Sigma_1$ with some $w_a \in \Sigma_2^*$). Then *LetSub* is a **Letter Substitution** $L_1 \rightarrow L_2$ if and only if

$$\begin{aligned} L_2 &= LetSub(L_1) \\ &\underbrace{=}_{\text{Definition}} \{w \in \Sigma_2^* \mid \exists y \in L_1, \text{ such that } w = y, \text{ except that} \\ &\qquad \text{every character } c \text{ of } y \text{ has been replaced by } Sub(c)\}. \end{aligned}$$

Prove that the class of context-free languages is closed under Letter Substitution.

3. An algorithm for PDAs

[6] Let P be an arbitrary PDA, with alphabet Σ. Let $n > 0$ be an arbitrary positive integer. Describe an algorithm to decide whether P accepts any words $w \in \Sigma^*$ such that $|w| \leq n$.

4. Computations in a Turing machine

Let M be a Turing Machine over the alphabet $\Sigma = \{0\}$. Let M's tape alphabet be $\{0, X, B\}$. Let M's states be $\{q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$, with q_6 being the sole final state. Let the transition function, δ, for M, be defined by the following table.

q	a	$\delta(q,a)$	q	a	$\delta(q,a)$	q	a	$\delta(q,a)$
q_1	0	(q_2, B, R)	q_2	B	(q_6, B, R)	q_4	X	(q_4, X, R)
q_1	X	(q_7, X, R)	q_3	X	(q_3, X, R)	q_4	B	(q_7, B, R)
q_1	B	(q_7, B, R)	q_3	0	$(q_4, 0, R)$	q_5	B	(q_2, B, R)
q_2	X	(q_2, X, R)	q_3	B	(q_5, B, L)	q_5	0	$(q_5, 0, L)$
q_2	0	(q_3, X, R)	q_4	0	(q_3, X, R)	q_5	X	(q_5, X, L)

Let M begin processing in the configuration $(q_1, \underline{w})$, where $w \in \Sigma^*$ is the input word.

[4] (a) Draw a diagram for M.

[4] (b) Give the sequence of instantaneous descriptions of M as it processes the input word $w = 0000$.

[4] (c) Give the sequence of instantaneous descriptions of M as it processes the input word $w = 00000$.

[2] (d) Briefly describe the algorithm which M performs, given any input word $w \in \Sigma^*$.

5. Constructing A Turing machine

[10] Let $\Sigma = \{0, 1\}$. Let $L_1 = \{0^i 1^j \mid 0 \leq j \leq i\}$. Let $L_2 = \{0^i 1^i \mid 0 \leq i\}$. Construct a Turing machine, M, which takes as input an arbitrary $w_1 \in L_1$, and returns $w_2 \in L_2$, such that $n_0(w_1) = n_0(w_2)$. Recall that $n_0(w)$ denotes the number of occurrences of the symbol 0 in the string w. In other words, M appends $i - j$ copies of 1 to the end of w_1, to produce the correct w_2. For example, for input $w_1 = 0001$, M produces $w_2 = 000111$. Provide a diagram for your machine M.

6. Every context-free language is recursive

[6] Let Σ be a non-empty finite alphabet. Let G be an arbitrary context-free grammar over Σ, and let $L = L(G)$. Give an algorithm for a Turing machine which decides membership in the language L. Argue informally why your algorithm is correct.