- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's marking policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. A double-stack PDA

[8] Let M be an ordinary Turing machine (i.e. M is a single-tape, deterministic TM). Suppose that M accepts (or does not accept) its input strings (i.e. M does **not** compute a function). Let P be a PDA (accepting by final state), having two independent stacks, that both use the same stack alphabet, Γ. P's transition function, δ_P, is therefore of the form

$$\delta_P : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times \Gamma \rightarrow \text{ finite subsets of } Q \times \Gamma^* \times \Gamma^*.$$

Describe how to construct such a P, so that $L(P) = L(M)$ (i.e. prove that every recursively enumerable language can be accepted by a double-stack PDA).

i. Describe how to define the states (including the final states) of P.

ii. Describe how to initialize both of P's stacks, at the start of P's execution.

iii. Then describe how to define δ_P to simulate any single transition from M, inside of P.

2. Reductions

   Let $\Sigma = \{0, 1\}$.

[4] (a) Let L be a language over Σ such that $L \neq \emptyset$ and $L \neq \Sigma^*$. Let L_R be **any** recursive language over Σ. Prove that membership in L_R can be reduced to membership in L.

[4] (b) Let L_{RE} be **any** recursively enumerable language over Σ. Let L_u be the **universal language** as defined in the lecture slides. (In detail, L_u is the set of pairs (e, w) such e is the identifier of a Turing machine, M, which accepts the input word, w.) Prove that membership in L_{RE} can be reduced to membership in L_u.

3. Decidable languages
   Because of the **Church-Turing Thesis**, you only need to describe the required algorithm (e.g. in English or pseudocode) to decide the decision problem, in each part of this question. You do **not** need to provide implementation details for a TM to decide each decision problem.

[4] (a) Consider the language

$$L_a = \{\langle D_1, D_2 \rangle \mid D_1, D_2 \text{ are DFAs and } L(D_1) \subseteq L(D_2)\}.$$

Prove that (membership in) L_a is decidable.

[4] (b) Consider the language

$$L_b = \{\langle D, G\rangle \mid D \text{ is a DFA, } G \text{ is a CFG, and } L(G) \subseteq L(D)\}.$$

Prove that (membership in) L_b is decidable.

4. An undecidable language

   Let Σ be a finite alphabet containing the symbol 0.

[3] (a) Give a reduction from membership in the language

$$L_{0+} = \{w \mid w \text{ represents the Turing machine } M \text{ and } 0 \in L(M)\}$$

to the membership in the language

$$L_{\Sigma^*} = \{w \mid w \text{ represents the Turing machine } M \text{ and } L(M) = \Sigma^*\}.$$

[4] (b) Show that the language L_{0+} from part 4a is undecidable. Do <u>not</u> use Rice's theorem.