Proving Languages Non-Regular via Closure Properties

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Closure properties

Recall that we proved that the class of regular languages is closed under

- union
- concatenation
- Kleene closure
- intersection
- complement
- relative complement
- symmetric difference and
- reversal.

Sometimes closure properties can be used to prove languages non-regular.
The basic idea

Here’s the basic idea.

Start with a language $L$ that you wish to prove non-regular.

Assume, to get a contradiction, that $L$ is regular.

Then try to use the closure properties to “massage” $L$ into another language $L'$ that you already know is non-regular.

Since the closure properties preserve regularity, if you can do this, you get a contradiction.

Thus $L$ is not regular.
Here’s a really simple example.

Suppose we have the language $L = \{a^i b^{i+1} : i \geq 0\}$.

What can we do to $L$ to turn it into a language we already know to be non-regular?

We could concatenate the language $\{a\}$ on the front:

$\{a\}L = \{a^{i+1}b^{i+1} : i \geq 0\}$. This is almost a language we studied already.

So union this with $\{\epsilon\}$, and we do get a language we already studied:

$\{\epsilon\} \cup \{a\}L = \{a^i b^i : i \geq 0\}$.

This is a language we already saw in Week 3 was non-regular.

Since the regular languages are closed under union and concatenation, we know $L$ is non-regular. ■
Another example

That was a pretty simple example, because we could have used the pumping lemma directly on \( L \) instead.

Now let’s look at an example where a proof using the pumping lemma is rather hard, but using closure properties is easy.

Let \( L = \{ a^p b^q : p \neq q \} \).

What can we do to “massage” \( L \) into a non-regular language we’ve already seen?

What’s a good candidate to massage \( L \) into?
Another example

Let’s try to massage $L$ into $\{a^i b^i : i \geq 0\}$.

First, take the complement of $L = \{a^p b^q : p \neq q\}$. What is it?

You might be tempted to think it is $\{a^p b^q : p = q\}$, or in other words, $\{a^i b^i : i \geq 0\}$. But not quite...

Actually

$$\overline{L} = \{a^i b^i : i \geq 0\} \cup \{x : x \text{ is not of the form } a^* b^* \}.$$ 

So $\overline{L} \cap a^* b^* = \{a^i b^i : i \geq 0\}$.

Since the regular languages are closed under complement and intersection, we’ve proven that $L$ is not regular. ■
Things to keep in mind

Although one can use any closure property for this trick, generally speaking, the most useful operations are complement and intersection.

The reason why intersection is useful is because it makes a language “smaller” and lets you concentrate on the essence of $L$ that makes it non-regular.

But remember that a single string is never “non-regular”. Regularity is a property of a language, not a string.

Also remember that neither regularity nor non-regularity are properties that are necessarily “inherited” by subsets or supersets: if $L_1 \subseteq L_2$, then each of the four cases $L_1$ regular or nonregular, $L_2$ regular or nonregular, are possible.

Exercise: give examples of each of the four cases.
A hard example using the pumping lemma

Now let’s go back to the language

\[ L = \{ a^p b^q : p \neq q \} \].

Let’s try to prove \( L \) non-regular directly, using the pumping lemma.

The way to “win” here is to choose \( z \) appropriately.

Choose \( z = a^n b^{n+n!} \).

Clearly \( z \in L \) because \( n \neq n + n! \).
A hard example using the pumping lemma

The adversary writes $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$. So $u = a^j$, $v = a^k$, and $w = a^{n-j-k} b^{n+n!}$. Note that $1 \leq k \leq n$.

Now we want to choose $i$ such that $uv^i w \not\in L$. But $uv^i w = a^{n+(i-1)k} b^{n+n!}$. We need to choose $i$ such that $n + (i-1)k = n + n!$. In other words, such that $(i-1)k = n!$.

Now $k$ divides $n!$ exactly, since $1 \leq k \leq n$. So $n!/k$ is a natural number, and we can take $i = n!/k + 1$.

This completes the proof.

That was a lot harder than using the closure properties!

Moral of the story: when confronted with a language you want to prove non-regular, try to use closure properties to turn it into a non-regular language you already studied.