Module 2

Finite Automata

The simplest computers for the simplest languages

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Topics of Module 2

- Deterministic finite automata
- Nondeterministic finite automata and the equivalence to DFAs
- ε-NFAs, and their equivalence to DFAs.

Deterministic finite automata

Finite automaton: a simple computing machine

- Given a finite input word, it moves from one program state to another.
- Each move is based on one input letter
- At the end of the input, the machine either accepts or rejects the input, depending on the machine state.

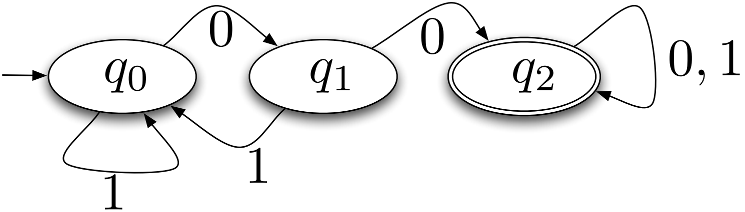
Vital limitation of a finite automaton:

- It cannot look back in its input.
- The only memory is in the state; aside from that, it has forgotten everything.

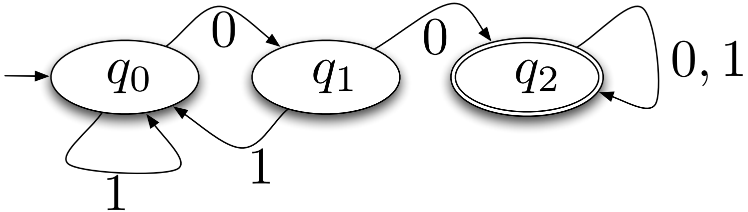
Definitions

Finite automaton described by 5 parameters:

- Q = set of computation states
- Σ = finite input alphabet
- δ = transition function
  - **Important:** In a DFA, there must be a transition defined for every state and for every possible alphabet character.
- q_0 = start state
- F = accept states



This DFA



In this finite automaton:

- $Q = \{q_0, q_1, q_2\}$,
- $\Sigma = \{0, 1\}$,
- δ is a function from $Q \times \Sigma \to Q$.
  It includes $(q_0, 0) \to q_1$
- $q_0 = q_0$,
- $F = \{q_2\}$.

This DFA accepts the language of words with 00 as a substring.
Question: How would you prove that?

Acceptance, extension

Given a DFA M, what does "M accepts w" mean?

- Starting at q_0, follow transition function δ for each letter in w, in order.
- String w accepted by M if at the end of w's transitions, we wind up in a state in F.

A more formal definition of acceptance comes by looking at the extended transition function, $\hat{\delta}$.

- $\hat{\delta}(q, w)$: state we finish in if we start at state q and follow δ for each letter in the word w in turn.

Function $\hat{\delta}$ is a function from $Q \times \Sigma^* \rightarrow Q$. It usually cannot be written down in a closed form, which leads us to the following technique.

Formal definition of extended transition function

Formally, $\hat{\delta}(q, w)$ is defined recursively:

- $\hat{\delta}(q, \varepsilon) = q$, for all states q.
- If $|w| > 0$, then we can write $w = xa$, where $|a| = 1$.
- Then define $\hat{\delta}(q, w) = \delta(\hat{\delta}(q, x), a)$.

Unpack that:

- Let $q_1 = \hat{\delta}(q, x)$.
- In words, q_1 is the state that we reach starting from q after we have read the prefix x.
- Then, process the single letter a: $\delta(q_1, a) = \delta(\hat{\delta}(q, x), a)$.

This can be defined from the other end:

- If $w = ax$, then $\hat{\delta}(q, w) = \hat{\delta}(\delta(q, a), x)$.
- The textbook gives the first definition, so we will stick with that.

Lemma about extended transition function

Lemma: Let $D = (Q, \delta, \Sigma, q_0, F)$ be a DFA, with extended transition function $\hat{\delta}$. Let $q \in Q$ and $a \in \Sigma$ be arbitrary. Then $\hat{\delta}(q, a) = \delta(q, a)$, i.e. $\hat{\delta}$ agrees with δ for any state q and on any single alphabet symbol a.
Proof:

$$\begin{aligned} \hat{\delta}(q, a) &= \hat{\delta}(q, \varepsilon a) \\ &= \delta(\hat{\delta}(q, \varepsilon), a) \\ &= \delta(q, a). \end{aligned}$$

□

Language of an DFA

The language of the DFA M: all words M accepts.
Formally, acceptance of a word:

- $M = (Q, \Sigma, \delta, q_0, F)$ accepts $w \in \Sigma^*$ exactly when $\hat{\delta}(q_0, w) \in F$.

Language of the DFA: all accepted words.

- $L(M) = \{w \in \Sigma^* \text{ where } \hat{\delta}(q_0, w) \in F\}$.
- (or just $\{w \in \Sigma^* \text{ where } M \text{ accepts } w\}$.)

Terminology: $L(M)$ can be called:

- The language of the DFA M
- The language accepted by the DFA M
- The language recognized by the DFA M

Do DFAs compute?

In a certain sense, yes.
Example:

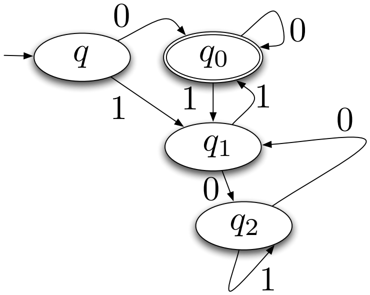
- "Is x a multiple of 3?" can be answered by a DFA.
  - The input word w is the binary representation of x.
  - Then the machine accepts w if x is a multiple of 3.
- This language, $L = \{w|w$ is the binary representation of a number x that is divisible by $3\}$, is accepted by an DFA.
  - L includes $11, 110, 1111, 0$, and does not include $10, \varepsilon$.
- So in that sense, yes, they compute.

Divisibility DFA

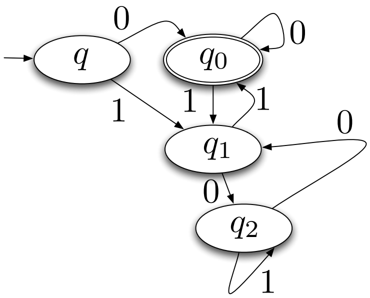
Reminder of conventions:

- start state has an unlabelled arrow
- accept states are double circles
- label arrows with values for δ

We will have a state for each remainder (0, 1, and 2), plus a special start state, so we do not accept ε.
This gives this DFA:



Why does that DFA work?



- Adding a new symbol (0 or 1): double and add the new symbol.
- Double a multiple of 3 and add 0: a new multiple of 3,
- Double a multiple of 3 and add 1: a number with remainder 1,
- Double a number with remainder 1 and add 1: a multiple of 3,
- *etc.*
- We are in state q_i when i is the remainder considering what we have already read.

This machine only accepts multiples of 3.

Enhancements to DFAs

We are going to prove a theorem soon that FAs accept a specific class of languages, called regular languages.

- That should be robust: changes to an FA should not invalidate the property.
- So we will change FAs in a variety of small and large ways.
- The first major change is nondeterminism.

Nondeterminism

How does a DFA work?

- In a given state, for each input letter, there is exactly one choice for what to do, and
- The machine accepts if after all letters have been read, it is in an accept state.

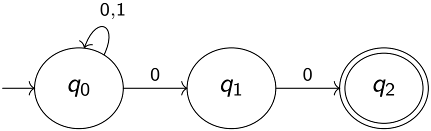
What if there were choices, instead?

- In a given state, for each letter, there may be a choice of what to do.
- The machine accepts if some sequence of choices results in an accept state.

Nondeterministic FAs allow a huge blow-up in computation: there will be lots going on in parallel. As such they may not be very realistic models of any kind of computers.

An example

$L = \{\text{all words with 00 as the last two symbols}\}$



- Nondeterministic FA: Transition from a state, given an alphabet symbol, is to a set of possible states.
- Note: sometimes a state has no outgoing transition for a given symbol; the second state has no output labelled 1.
- If a thread reaches a state which has no outgoing transition for the given input symbol, then that thread crashes; it proceeds no further.

Formal definition

NFA defined by 5 parameters

- Q = set of computation states
- Σ = finite input alphabet
- δ = transition function
  - **Important:** In an NFA, there need not be a transition defined for every state and for every possible alphabet character.
- q_0 = start state
- F = accept states

Differences from DFAs:

- δ: function from $Q \times \Sigma \to \{\text{subsets of } Q\}$.
  - Recall notation: $2^Q = \{\text{subsets of } Q\}$.
  - Based on a state in Q and an input letter from Σ, which states are now active in Q?
  - (Different from DFA, where it is from $Q \times \Sigma \to Q$.)
- Accepts whenever any state path from q_0 is to an accept state.

New form for extended transition function

We must enhance the definition for our extended transition function. We now need to allow the output of the transition function to be a set of states instead of a single state.

- $\hat{\delta}(q, w)$ = all states that we can reach from start state q processing the input word w.
- $\hat{\delta}$: function $Q \times \Sigma^* \to 2^Q$.
- Base case: $\hat{\delta}(q, \varepsilon) = \{q\}$.
- Recursive case: If $|w| > 0$, then we may write $w = xa$, where $|a| = 1$.
- Then can we define $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$?
  - Well, no.
  - The function δ is $Q \times \Sigma \to 2^Q$.
  - But $\hat{\delta}(q, x)$ is in 2^Q instead of in Q, so $\delta(\hat{\delta}(\dots), a)$ is not allowed.
  - We really need to define $\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$ instead.
  - Note: This definition handles threads that crash correctly.
    - A thread that crashes has no outgoing transition from state p for input symbol a.
    - In other words, $\delta(p, a) = \emptyset$.
    - But then, $\delta(p, a)$ contributes nothing to the union of sets of states, reflecting the fact that the thread has crashed and proceeds no further.

Acceptance by an NFA and the language of an NFA

Acceptance of an NFA:

- NFA M accepts a word w if $\hat{\delta}(q_0, w) \cap F$ is nonempty.
- There is a path from q_0, labelled by letters of w, winding up in an accept state from F.

Language of an NFA:

- The language of the NFA $M = (Q, \Sigma, \delta, q_0, F)$ is:

$$L(M) = \{w \in \Sigma^* | \hat{\delta}(q_0, w) \cap F \neq \emptyset\}.$$

- That is, all words accepted by the NFA.

Where are we?

DFAs:

- Model of computation: finite states, follow single transitions
- Acceptance of a word: transitions lead to an accept state from M
- Language: All accepted words

NFAs:

- Model of computation: finite states, possibly many transitions per letter, or possibly none.
- Acceptance: any path of transitions leads to an accept state.
- The extended transition function is more complicated.
- Language: All accepted words.

Are these different from each other in terms of power?
Both are limited in that they only read each input letter once, left-to-right.

NFAs are no more powerful than DFAs

Theorem: Let L be a language that is accepted by an NFA.
Then L is accepted by a DFA.

- Note: The opposite direction is easy: DFAs are NFAs! (Well, must change δ trivially, so that the NFA transitions to the 1-element set corresponding to the transition in the DFA.)
- Need to show: Given an NFA, we can construct a DFA that accepts its language.

Outline of proof

Here is how we will do this:

- Given an NFA N, we will construct a DFA, D.
- Then, we will have to show that $L(D) = L(N)$.
- As this is an equality of sets, we need to show that every word in $L(N)$ is in $L(D)$, and every word in $L(D)$ is in $L(N)$.

You have seen a little of this in CS 241.
First, let's build the DFA D, using the same alphabet Σ that N uses.
Remember: The machine D does not have to have the same states as N, just the same alphabet and language!

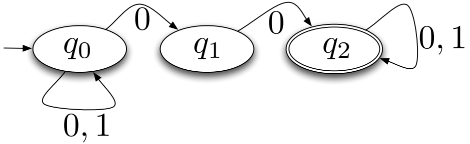
Computation in an NFA

How does the NFA N work?

- If N is in state q, after processing one letter a, N could be in any state from the set $\delta(q, a)$.
- Then, N processes the next letter and winds up in any of another set of states.
  (That is what is built into the extended transition function, $\hat{\delta}_N$.)
- In the DFA D, a single state represents a set of states in the NFA N.
- When D reads a new letter in, we jump from one state in the DFA to another (corresponding to the appropriate sets of states in the NFA N).

Sketch of the process of the proof

This NFA accepts the language $L = \{\text{words with 00 as a substring}\}$. How would you prove that?



- Its only accept state is q_2.
- Suppose we have processed some letters, and the NFA could be in either state q_0 or q_1, and the next input letter is 1.
  - From q_0 we go to q_0.
  - From q_1, no transition labelled 1.
- The new DFA, if it is in the set state $\{q_0, q_1\}$ and reads in a 1, must go to the state corresponding to the set $\{q_0\}$.

Sketch of the process of the proof

Here is idea of the algorithm for the subset construction for the transition function of D:

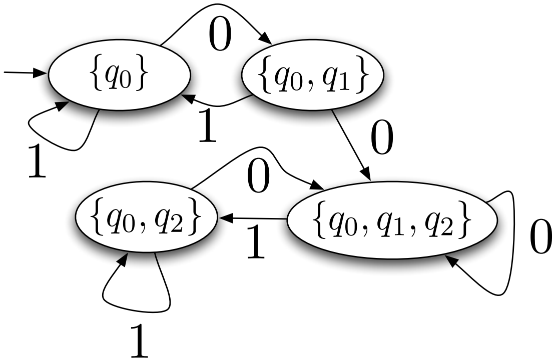
- For each subset $S \subseteq Q$ of states from N:
  - Recall that S corresponds to a single state in D.
  - For each alphabet symbol, a:
    - For each state $p \in S$, consider $\delta(p, a)$ (recall, this is a set of states).
    - Gather all of these together. Let $T = \bigcup_{p \in S} \delta(p, a)$.
    - Then T also corresponds to a single state in D.
    - Add the transition $S \xrightarrow{a} T$ to the transition function δ_D for D.

This procedure may make many new states

This 3-state NFA turns into a 4-state DFA.

- In general, if the NFA N has k states, the DFA D could have 2^k states.

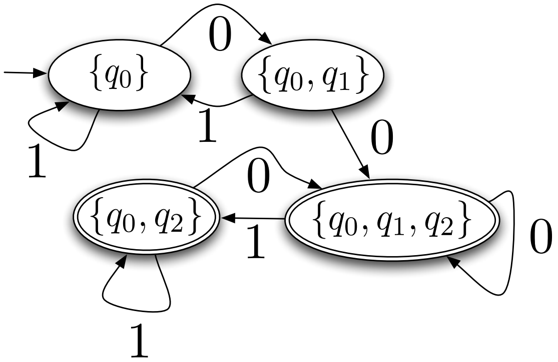
Here is the DFA:



And what are accept states?

- $F_D = \{$States in D that represent sets of states in N that include at least one accept state from $F_N\}$.

The full DFA



- Convince yourself that accepts the same language as the original construction.
- Interestingly, the DFA state $\{q_0, q_2\}$ is not needed, since we only get to it from another accept state.
- We really only needed 3 states.

Now, let's generalize this idea.

Subset construction (formal)

Given NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$, construct a new DFA $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$, with these parameters:

- $Q_D = 2^{Q_N}$.
- Let $S \in Q_D$, i.e. think of S as a set of states from the definition of N. Then
- $F_D = \{S \in Q_D \mid S \cap F_N \neq \emptyset\}$
- That is, each acceptance state in D corresponds to a set of states in N with at least one accept state.
- And a more complicated transition function:

$$\begin{aligned} & \delta_D(S, a) \\ = \; & \{\text{all states reachable in } N \text{ from } S \text{ when we read } a\} \\ = \; & \bigcup_{p \in S} \delta_N(p, a). \end{aligned}$$

- Let D's initial state be $\{q_0\}$, where q_0 is the initial state of N.

This is a DFA, not an NFA. We know which state D is in after reading any alphabet symbol.

The languages are equal

Now we must show that the languages of the NFA N and the DFA D equal. Think about which words are in the languages of N and of D.

- $w \in L(N) \Leftrightarrow \hat{\delta}_N(q_0, w) \cap F_N \neq \emptyset$.
- $w \in L(D) \Leftrightarrow \hat{\delta}_D(\{q_0\}, w) \in F_D$.

By the definition of F_D, the second statement is equivalent to:

- $w \in L(D) \Leftrightarrow \hat{\delta}_D(\{q_0\}, w) \cap F_N \neq \emptyset$.

We must show these are the same: that if $w \in L(D)$, then $w \in L(N)$, and vice versa.

Proof

Must show: $w \in L(N) \Leftrightarrow w \in L(D)$.

- That is:
  $\hat{\delta}_N(q_0, w) \cap F_N \neq \emptyset \Leftrightarrow \hat{\delta}_D(\{q_0\}, w) \cap F_N \neq \emptyset$.
- It suffices to show: $\hat{\delta}_N(q, w) = \hat{\delta}_D(\{q\}, w)$, for any state q from the definition of N.
- (If one of these sets has a non-empty intersection with F_N, then so does the other)

Proof: By induction on $|w|$.

- Base case ($|w| = 0$): Thus $w = \varepsilon$.
  Then $\hat{\delta}_N(q, \varepsilon) = \{q\} = \hat{\delta}_D(\{q\}, \varepsilon)$.
- Inductive case ($|w| > 0$): The inductive hypothesis is that, for every x with $|x| < |w|$, we have $\hat{\delta}_N(q, x) = \hat{\delta}_D(\{q\}, x)$, for any state q from N.
- Since $|w| > 0$, we may write $w = xa$, where $|a| = 1$.
- Then the induction hypothesis applies to x, so
  $\hat{\delta}_N(q, x) = \hat{\delta}_D(\{q\}, x)$.

Inductive case of the proof

- The induction hypothesis is that $\hat{\delta_N}(q, x) = \hat{\delta_D}(\{q\}, x)$.
- We need to show that $\hat{\delta_N}(q, xa) = \hat{\delta_D}(\{q\}, xa)$.
- So now we process the last character, a, on both sides.

$$\begin{aligned}
\hat{\delta}_D(\{q\}, xa) \quad & \underbrace{=}_{\text{Definition of } \hat{\delta}_D \text{ for the DFA } D} \quad \delta_D(\hat{\delta}_D(\{q\}, x), a) \\
& \underbrace{=}_{\text{induction hypothesis}} \quad \delta_D(\hat{\delta_N}(q, x), a) \\
& \underbrace{=}_{\text{Definition of } \delta_D} \quad \bigcup_{p \in \hat{\delta}_N(q,x)} \delta_N(p, a) \\
& \underbrace{=}_{\text{Definition of } \hat{\delta}_N} \quad \hat{\delta_N}(q, xa).
\end{aligned}$$

Reading one more symbol in N, the set of states of N we can be in corresponds with the state we will be in in D (recall that a single state of D represents a set of states of N).

- The NFA N accepts the same language as the DFA D.

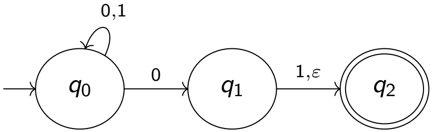
The class of languages accepted by NFAs is the same as for DFAs.

Another expansion to NFA's: ε-transitions

Sometimes in designing an NFA, it is handy to have transitions that happen automatically, without reading any letters of the input.

- Machine models that allow this are ε-NFAs.
- An ε-NFA is a 5-tuple, like an NFA.
- The only difference is transition function:
- δ is now a function: $Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$.
- (Transitions may exist that do not consume input letters.)

Example: The ε-NFA



Accepts binary words ending with 0 or with 01. (We could do this without ε-transitions, by making the second state an accept state.)

Formality about ε-NFAs

How do their extended transition functions look?

- Which states could I be in after processing the word x?
- Any states I could be in after just x, with no ε-transitions, plus
- Any states after just x, and one ε transition, plus
- Any states after just x, and two ε transitions, and so on ...
- ...until we exhaust all possible ε-transitions.

Unroll this to get a recursive definition for $\hat{\delta}$.

Defining the ε-closure of a state

The definitions should be fairly similar:

- $\hat{\delta}(q, y)$: states we can get to after processing y and having ε-transitions.

Let's start with ε-transitions:

- Let $\text{ECLOSE}(p)$ = all states reachable starting from p only using ε-transitions.
- We can define $\text{ECLOSE}(p)$ recursively:
  - $p \in \text{ECLOSE}(p)$
  - If $q \in \text{ECLOSE}(p)$, then so are all of the states in $\delta(q, \varepsilon)$.

The set $\text{ECLOSE}(p)$ is called the ε-closure of the state p.
We will also allow $\text{ECLOSE}(S)$ to be defined for a set S of states via:

$$\text{ECLOSE}(S) = \bigcup_{s \in S} \text{ECLOSE}(s).$$

Defining the ε-closure of a state

Lemma: For an ε-NFA E, subset $S \subseteq Q$ and decomposition $S = \bigcup_i S_i$,

$$\bigcup_i \textsc{Eclose}\,(S_i) = \textsc{Eclose}\left(\bigcup_i S_i\right),$$

(i.e. taking ε-closure commutes with taking set unions).

Proof:

$$\begin{aligned}
\bigcup_i \textsc{Eclose}\,(S_i) \quad & \underbrace{=}_{\text{Definition of } \textsc{Eclose}(S_i)} \quad \bigcup_i \left(\bigcup_{s \in S_i} \textsc{Eclose}(s) \right) \\
& \underbrace{=}_{S = \bigcup_i S_i} \quad \bigcup_{s \in S} \textsc{Eclose}(s) \\
& \underbrace{=}_{\text{Definition of } \textsc{Eclose}(S)} \quad \textsc{Eclose}(S) \\
& \underbrace{=}_{S = \bigcup_i S_i} \quad \textsc{Eclose}\left(\bigcup_i S_i \right). \ \square
\end{aligned}$$

The definition of the extended transition function

We can define $\hat{\delta}$ for ε-NFA's, also recursively.

- Base case: $\hat{\delta}(q, \varepsilon) = \textsc{Eclose}(q)$:
  states reachable from q with ε-transitions
- Inductive case: Suppose that $w = xa$, where $a \in \Sigma$.
  (Note: a cannot be ε, which is not a member of Σ.)
  - We know that $P = \hat{\delta}(q, x)$ is the set of all states in Q that we can get to by following either edges for the letters of x or ε-transitions (including ε-transitions at the end of x).
  - Then, we must follow the transitions for the alphabet symbol a: Let $R = \bigcup_{p \in P} \delta(p, a)$: then R has all of the states we can get to from P after following a transition for a.
  - Last, we might have some more ε-transitions.
- So, $\hat{\delta}(q, w) = \textsc{Eclose}(R) = \textsc{Eclose}\left(\bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)\right)$

Languages and power of ε-NFAs

- Language of an ε-NFA:
  $L = \left\{ w \in \Sigma^* \mid \hat{\hat{\delta}}(q_0, w) \cap F \neq \emptyset \right\}$
- That is, $\hat{\hat{\delta}}(q_0, x)$ includes an accept state.

Are ε-NFAs more powerful?

- No.
- Theorem: Given an ε-NFA E, there exists an ordinary DFA D such that $L(D) = L(E)$.
- This is not very surprising: we must show that we can include the ε-transitions of E in the transition function δ_D for D.
- (The other direction is just by definition: a DFA is an ε-NFA, once we make some trivial changes to the structure of δ so that it produces a 1-element set when it reads a symbol and the empty set when it reads in ε.)

To prove the theorem, we must construct a DFA D accepting language $L(E)$.

The equivalent DFA

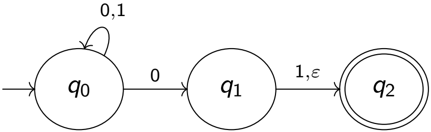
- Both machines use the same alphabet, of course.
- We use the subset construction, as when we built the DFA for an NFA.
- Starting state is $q_D = \text{ECLOSE}(q_0)$. Thus, we start having implicitly taken ε-transitions from the starting state q_0 of E.

The complexity comes in the transition function and the accept states.

- Transition function:
  - From one DFA state, S (corresponding to a set of states in E), if we process one letter a in the new DFA, we should mimic this behaviour from E:
    - follow any edges labelled a
    - take any ε-transitions
  - From any one state from E, say q, this then takes us to:
    - $\delta_E(q, a)$
    - $\text{ECLOSE}(\delta_E(q, a))$
- And we therefore want the union over all states $q \in S$: $\delta_D(S, a) = \bigcup_{q \in S} \text{ECLOSE}(\delta_E(q, a))$.

Example of Subset Construction

- Here we demonstrate one step in the subset construction for the earlier small example of an ε-NFA:



- The subset construction gives us

$$\begin{array}{rcl} Q_D & = & \{\emptyset, \{q_0\}, \{q_1\}, \{q_2\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\} \\ q_D & = & \textsc{Eclose}(q_0) = \{q_0\} \end{array}$$

- Now we determine $\delta_D(S, a)$, where $S = \{q_0, q_1, q_2\}$ (the state we reach from $\{q_0\}$ upon reading 0) and $a = 1$.
- Computing $\textsc{Eclose}(\delta_E(p, a))$ for each $p \in S$ gives

$$\begin{array}{rcl} \textsc{Eclose}(\delta_E(q_0, 1)) & = & \textsc{Eclose}(\{q_0\}) = \{q_0\} \\ \textsc{Eclose}(\delta_E(q_1, 1)) & = & \textsc{Eclose}(\{q_2\}) = \{q_2\} \\ \textsc{Eclose}(\delta_E(q_2, 1)) & = & \textsc{Eclose}(\emptyset) = \emptyset \end{array}$$

Example of Subset Construction

- Hence the target state coming from the subset construction is $\{q_0, q_2\}$.
- The construction says that we need to add to the transition function for D: $\delta_D(\{q_0, q_1, q_2\}, 1) = \{q_0, q_2\}$.
- Now to complete the transition function for D, we do this same construction for each of the 8 choices for S (on the previous slide) and each alphabet symbol form $\Sigma = \{0, 1\}$.

Picking the accept states, equality of languages

- We need to figure out which states are accepting states:
  - $F_D = \{S \mid S \in Q_D \text{ and } S \cap F_E \neq \emptyset\}$.
- Declare a word accepted by D if D is in an accept state (according to this recipe) when D finishes processing the word.
- Now, to show equality of the languages of the two automata, we must show that if x is accepted by E, then x is accepted by D, and vice versa.
  - (One concern: do we do the right thing for the word ε?)
  - To show $L(E) = L(D)$, can show that $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(q_D, x)$?
  - If so, then we will be in the same set of ε-NFA states after reading x, and our definitions of F_D and F_E will guarantee that both machines will accept exactly the same words.

Transition functions

We want to show: $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(q_D, w)$, for all strings w. The proof is by induction on $|w|$.

Base case ($|w| = 0$): In this case, $w = \varepsilon$. We have $\hat{\delta}_E(q_0, \varepsilon) = \text{ECLOSE}(\{q_0\})$, by definition of $\hat{\delta}_E$ in the ε-NFA.

- We therefore have

$$\begin{aligned}
\hat{\delta}_E(q_0, \varepsilon) &\underbrace{=}_{\varepsilon-NFA \text{ rule}} \text{ECLOSE}(\{q_0\}) \\
&\underbrace{=}_{\text{singleton set property}} \text{ECLOSE}(q_0) \\
&\underbrace{=}_{\text{Definition of } q_D} q_D \\
&\underbrace{=}_{D \text{ is a } DFA} \hat{\delta}_D(q_D, \varepsilon) \\
&\underbrace{=}_{\text{Definition of } q_D} \hat{\delta}_D(\text{ECLOSE}(q_0), \varepsilon).
\end{aligned}$$

- So the base case holds.

Transition functions

Inductive case ($|w| > 0$):

- We need to argue that $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(q_D, w)$.
- The induction hypothesis is that $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(q_D, x)$, for all strings x where $|x| < |w|$.
- Write $w = xa$, where a is a single character.
- The induction hypothesis applies to x.
- Thus we may let $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(q_D, x) = S$.
- Now we compute

Transition functions

$$
\begin{aligned}
\hat{\delta}_E(q_0, w) &\underbrace{=}_{w=xa} \hat{\delta}_E(q_0, xa) \\
&\underbrace{=}_{\text{Definition of } \hat{\delta}_E} \textsc{Eclose}\left(\bigcup_{p \in \hat{\delta}_E(q_0, x)} \delta_E(p, a) \right) \\
&\underbrace{=}_{\text{Definition of } S} \textsc{Eclose}\left(\bigcup_{p \in S} \delta_E(p, a) \right) \\
&\underbrace{=}_{\text{Lemma}} \bigcup_{p \in S} \textsc{Eclose}(\delta_E(p, a)) \\
&\underbrace{=}_{\text{Definition of } \delta_D} \delta_D(S, a) \\
&\underbrace{=}_{\text{Definition of } S} \delta_D(\hat{\delta}_D(\textsc{Eclose}(q_0), x), a) \\
&\underbrace{=}_{\text{Definition of } \hat{\delta}_D} \hat{\delta}_D(\textsc{Eclose}(q_0), xa) \underbrace{=}_{w=xa} \hat{\delta}_D(\textsc{Eclose}(q_0), w).
\end{aligned}
$$

Transition functions

- **Remark:** You should convince yourself that the base case handles the input word ε correctly.
- This argument did not use any special properties of q_0.
- We could re-run the argument with any state, q.
- Thus we could have proved, for any state q and any word $w \in \Sigma^*$:

$$\hat{\delta}_E(q, w) = \hat{\delta}_D(\textsc{Eclose}(q), x).$$

- We have proved by induction that the two extended transition functions agree on all input words.
- Therefore, as we argued earlier, this shows that the two automata accept precisely the same languages.
- So we are done.

End of module 2

Hence, the class of languages accepted by ε-NFAs is the same as the class accepted by ordinary NFAs, which is the same as the class of languages accepted by DFAs.

We have now seen a collection of types of automata:

- Deterministic finite automata
- Nondeterministic finite automata
- ε-NFAs

All three accept the same class of languages. But what is that class? They are the regular languages.