Module 3

Regular languages and regular expressions

What can a computer do with no memory?

CS 360: Introduction to the Theory of Computing
Spring 2024

Collin Roberts
University of Waterloo

Topics of Module 3

- Regular Expressions
- Regular Languages
- Kleene's Theorem

Regular Expressions

Recall our earlier rules for constructing new languages from existing languages:

1. Union $L \cup M$ of languages L and M
2. Concatenation LM of languages L and M
3. Closure L^* of a language L

Regular Expressions

Let Σ be a finite alphabet. We construct the regular expressions over Σ (and describe the language which each regular expression represents, i.e. the set of words that fit into the mold defined by the regular expression) recursively, as follows.

Base

1. $\emptyset$ is a regular expression, and $L(\emptyset) = \emptyset$.
2. ε is a regular expression, and $L(\varepsilon) = \{\varepsilon\}$.
3. If $a \in \Sigma$ is any symbol, then a is a regular expression and $L(a) = \{a\}$.

Induction

1. If E and F are regular expressions, then $E + F$ is a regular expression, and $L(E + F) = L(E) \cup L(F)$.
2. If E and F are regular expressions, then EF is a regular expression, and $L(EF) = L(E)L(F)$.
3. If E is a regular expression, then E^* is a regular expression, and $L(E^*) = (L(E))^*$.
4. If E is a regular expression, then (E) is a regular expression, and $L((E)) = L(E)$.

Remark: The regular expression for $L = \{w\}$ is just w itself.

Order of Precedence for Regular Expressions

As in algebra, there is an order of operations here:

1. Parentheses are used to override (or emphasize) the default order as needed.
2. $*$
3. Concatenation
4. $+$

Recommended Reading: Example 3.2 starting on p89 of the text (a regular expression over $\Sigma = \{0, 1\}$).

Regular Languages

A language L is regular if it obeys the following recursive definition:
Base

1. $L = \emptyset$ is regular.
2. $L = \{\varepsilon\}$ is regular.
3. $L = \{a\}$ for some alphabet letter $a \in \Sigma$ is regular.

Induction

1. $L = L_1 \cup L_2$, for regular languages L_1, L_2
2. $L = L_1 L_2$ for regular languages L_1, L_2
3. $L = L_1^*$, for some regular language L_1

No other languages are regular.
Remember that $\emptyset \neq \{\varepsilon\}$!

Beginnings of regular languages

First simple **Theorem**: All one-word languages L are regular.
Proof: Let $L = \{w\}$. The proof is by induction on $|w|$:

- Case 1: $|w| = 0$. Then $L = \{\varepsilon\}$, which is regular by a rule.
- Case 2: $|w| > 0$. The induction hypothesis is that the Theorem is true for all one-word languages $\{x\}$, where $|x| < |w|$.
  - Let $w = xa$, where a is a single character.
  - Then $L_1 = \{x\}$ is regular by the induction hypothesis.
  - And $L_2 = \{a\}$ is regular by a rule.
  - And $L = L_1 L_2$, which is regular by a rule.
  - We are done.

Finite languages are regular

Theorem: If L has a finite number of words, then L is regular.
Proof: By induction on $|L|$:

- Base case: $|L| = 0$. Then $L = \emptyset$, which is regular by a rule.
- Inductive case: Suppose all finite languages with k words are regular, and $|L| = k + 1$.
  - Then $L = \{w_1, w_2, \ldots, w_{k+1}\} = \{w_1, w_2, \ldots, w_k\} \cup \{w_{k+1}\}$.
  - By the induction hypothesis, the first language is regular.
  - By the previous Theorem, the second language is regular.
  - So L is the union of two regular languages, and thus L is regular.

Are all languages regular?

Every language is the union of a set of one-word languages.
Are all languages regular, using the proof model just shown?

- No!
- Finite unions are not infinite unions! We cannot do induction in that way!
- In fact, $\{\varepsilon, 01, 0011, 000111, 00001111, 0000011111, 000000111111, \ldots\}$ is not regular, which we will see in the next module.

Regular expressions

Regular expressions are another way of representing regular languages:

$L = \emptyset$	$\emptyset$
$L = \{\varepsilon\}$	ε
$L = \{a\}$	a
$L = L_1^*$	r_1^*
$L = L_1 L_2$	$r_1 r_2$
$L = L_1 \cup L_2$	$r_1 + r_2$

- (Yes, just like in `grep`. But you might have already known that.)
- If R is a regular expression, then $L(R)$ is the language of R, defined as we did earlier.

Examples of regular languages

- If $\Sigma = \{0, 1\}$, then $\Sigma^* = \{0, 1\}^*$.
  Regular expression: $(0 + 1)^*$
- Even-length sequences:
  Can be divided into 2-letter sub-words. $(00 + 11 + 01 + 10)^*$ or $((0 + 1)(0 + 1))^*$
- Sequences with length at most 3:
  $(0 + 1)(0 + 1)(0 + 1) + (0 + 1)(0 + 1) + (0 + 1) + \varepsilon$ or
  $(0 + 1 + \varepsilon)(0 + 1 + \varepsilon)(0 + 1 + \varepsilon)$ or
  $\varepsilon + 1 + 0 + 11 + 10 + 01 + 00 + 111 + \cdots$
- Sequences with at most two zeros:
  $1^*(0 + \varepsilon)1^*(0 + \varepsilon)1^*$
- And lots more.

Basic rules about regular languages

Basic rules that you can often use (not exciting, but true. . .):

- $\emptyset e = e\emptyset = \emptyset$ (If $w \in L(\emptyset e), w = w_1 w_2$ where $w_1 \in L(\emptyset)$. But nothing works for w_1.)
- $\emptyset^* = \{\varepsilon\}$ (might be surprising)
- $\{\varepsilon\}^* = \{\varepsilon\}$
- $x + x = x$ (remember, $+$ means union)
- $(x^*)^* = x^*$ (Taking closure twice equals taking closure once)
  - Side Note: an operation for which applying the operation twice equals applying the operation once is called idempotent.
  - Other natural examples of idempotents are projections in linear algebra.
- $x(y + z) = xy + xz$

Tons more of these, but we will not focus on them.
The first two might be surprising, and are thus important.

Kleene's Theorem

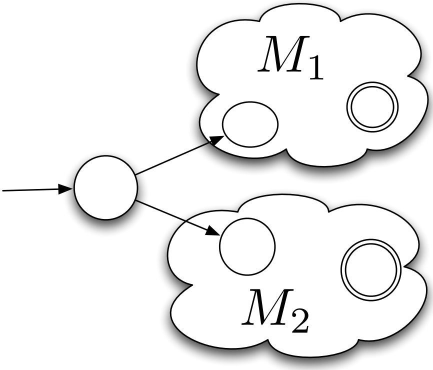
Theorem: Every regular language is the language of a DFA, and every DFA accepts a regular language.

- Recall: We saw that DFAs are equally powerful to ε-NFAs.
- We will show how to go from a regular language to an equivalent ε-NFA.
- We will find a regular language which coincides with the language of a DFA.
- Together these constructions will prove the Theorem.

How to accept $L_1 \cup L_2$?

What about for $L_1 \cup L_2$, if both are accepted by NFAs M_1 and M_2?

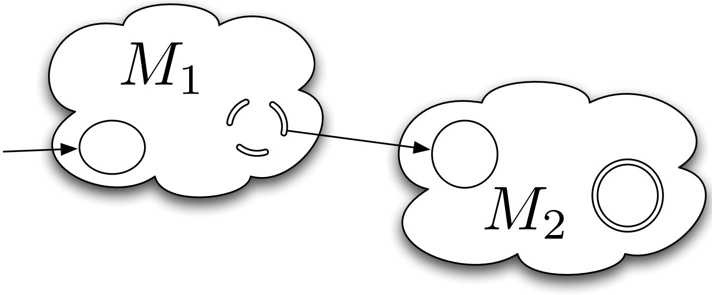
- Can we start out by going to both NFAs, M_1 and M_2, and just accept in either?
- Again this requires transitions that do not use letters from the input.
- The set of accepting states in the constructed machine is the union of the sets of accept states of M_1 and M_2.



How to accept $L_1 L_2$?

Suppose L_1 is accepted by NFA M_1, and L_2 is accepted by NFA M_2.
Find: A new ε-NFA M accepting $L_1 L_2$ (L_1 concatenated to L_2).

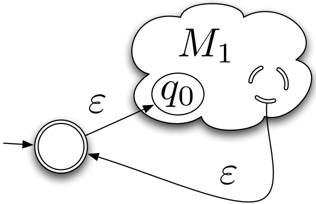
- We would kind of like:



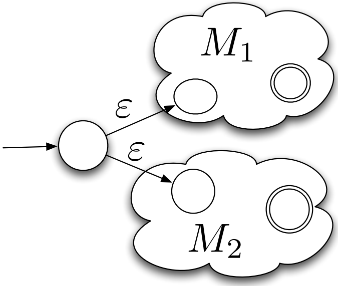
- ... where the combined machine M only accepts in M_2's accept states
- and where the new machine M jumps from accept states of M_1 to start state of M_2 without reading input characters.

Easy with an ε-NFA

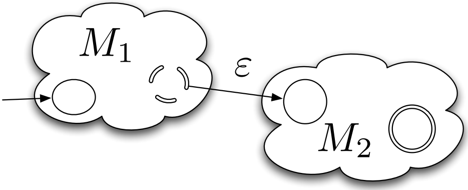
In both cases, and for the closure operator, we can do this with ε-transitions:



Kleene Star



Union



Concatenation

Proof of the Theorem

We want to prove:

- For every regular language L, there is an ε-NFA M, where $L(M) = L$.
- Recall: L is a regular language if L is any of:

  Base

  1. $\emptyset$
  2. $\{\varepsilon\}$
  3. $\{a\}$, for some alphabet character $a \in \Sigma$

  Induction

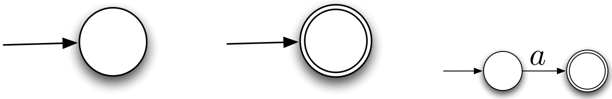
  1. $L_1 \cup L_2$ for regular languages L_1 and L_2.
  2. $L_1 L_2$ for regular languages L_1 and L_2
  3. L_1^* for regular language L_1

We will show an ε-NFA for the base cases, then prove the existence of the other three cases by structural induction.

Base cases

- The base cases are easy:

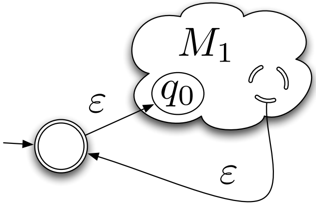
$\emptyset$ $\{\varepsilon\}$ $\{a\}$



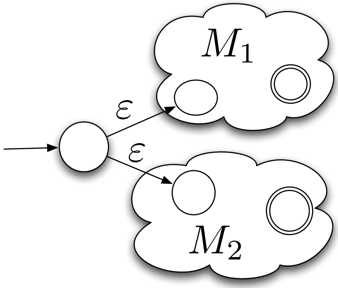
- Now, we must show the machines for the recursive definitions of regular languages.
- Assume L_1 and L_2 are regular languages, and that they are accepted by ε-NFAs M_1 and M_2, respectively.
- We must give ε-NFAs for $L_1 L_2$, L_1^* and $L_1 \cup L_2$.

Inductive cases

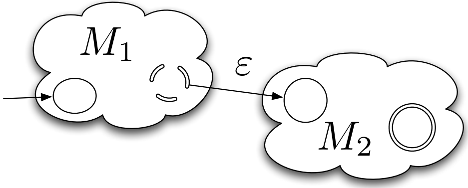
But, we already showed those, right?



Kleene Star



Union



Concatenation

We are not actually done yet

- It might seem like that is all we have to do.
- But must verify we have made the right construction, with a proof.
- One informal proof that this works, for the Kleene * operator:
  - Consider the Kleene * construction L_1^* for some regular language L_1.
  - Suppose x accepted by the ε-NFA, M, which we have constructed as above for L_1^* (i.e. suppose $x \in L(M)$).
  - Then there is a path in M for x ending in the start state.
  - This path can then be broken down into sub-paths, each of which begins and ends in the start state.
  - Each sub-path from the start state back to the start state corresponds exactly to a word from L_1 (M_1 accepts exactly L_1, and the only non-trivial path to the accept state of M is via an ε-transition from an accept state of M_1).
  - Therefore x is a concatenation of zero or more words in L_1, in other words $x \in L_1^*$. (Note that x can be ε.)
  - This shows that $L(M) \subseteq L_1^*$.

The other direction

Now, suppose that $x \in L_1^*$.

- Then $x = x_1 x_2 x_3 \cdots x_k$, with all $x_i \in L_1$.
- So there is a path from start state in the new machine to an accept state for each of the x_i.
- Join the paths together, following each with the ε-transition back to the start state to get a path for x from the start state back to itself.
- The start state is an accept state, so our new machine accepts x.
- So $x \in L(M)$.
- This shows that $L_1^* \subseteq L(M)$.

Now we are done.

- We have shown that $L(M) \subseteq L_1^*$ and $L_1^* \subseteq L(M)$.
- Therefore we have $L(M) = L_1^*$, as claimed.

The other arguments are left as exercises.

Where are we going, again?

To prove:

- The class of regular languages is the class of languages accepted by finite automata.

We have already seen:

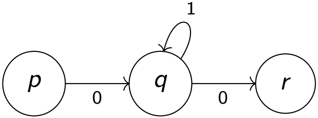
- From a regular language, we can produce an ε-NFA which recognizes the given language.
- Languages accepted by ε-NFAs = Languages accepted by DFAs (Module 2)
- Languages accepted by NFAs = Languages accepted by DFAs (also Module 2)

So given a regular language, we can produce a DFA which accepts the regular language.
Now we need to go the other way, i.e. given a DFA, produce a regular language which is the language of the DFA.

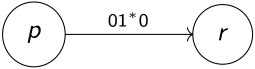
Find a regular expression for a DFA

Given a DFA D, we must find a regular expression for its language.
One idea:

- Paths through a state can be replaced by the regular expression that represents going from the previous state to the next one.



- Remove q, and represent this as:



State removal

More general:

- What about accept states and the start state?
- Make a new start state, s, with an ε-transition to the old start state
- Make a new accept state, f, with an ε-transition from all old accept states.

(Why? To avoid incoming/outgoing edges in start/final states.)

Removing states

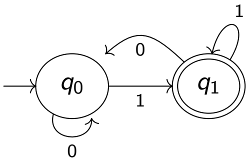
Edges in our "generalized FA" labelled with regular expressions.

- For each state q that is not s or f:
  - For all state pairs (p, r) that "flank" q: (p could be the same as r, but q cannot equal p or r)
    - Let e_1 = label of edge from p to q
    - Let e_2 = label of edge from q to r
    - Let e_3 = label of loop from q to itself (or $\emptyset$ if there is not one)
    - Let e_4 = label of edge from p to r (or $\emptyset$ if there is not one).
    - Make an edge from p to r with label $e_4 + e_1 e_3^* e_2$. Remember: $\emptyset^* = \varepsilon$ for our purposes here .
  - Then remove the node q.

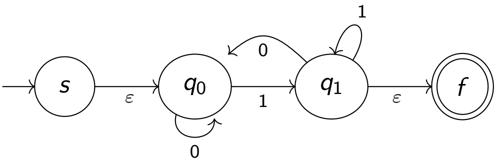
At the end of the process, we will have a single edge, from s to f.

And the label is the expression

- At the end of state elimination, the edge from s to f is labelled with a regular expression for the DFA's language.
- Let's do an example.



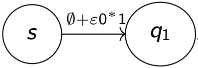
- First, make states s and f:



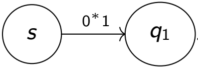
State elimination example

Now, eliminate state q_0.

- Consider all possible paths through q_0.
- The possible choices of flanking pairs for q_0 in this example are
  - (s, q_1): For this pair, we have $e_1 = \varepsilon, e_2 = 1, e_3 = 0, e_4 = \emptyset$, which means our new diagram must include

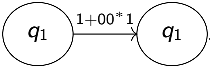


, or equivalently,



.

  - (q_1, q_1): For this pair, we have $e_1 = 0, e_2 = 1, e_3 = 0, e_4 = 1$, which means our new diagram must include

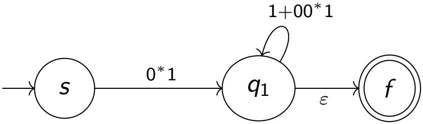


.

State elimination example

Now, finish eliminating state q_0.

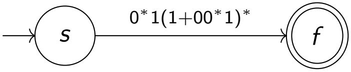
- We add an edge from s to q_1, and change the label of the self-loop on q_1:



End of example

Now, eliminate state q_1 in the same way.

- Paths from s to f via q_1 require a word in 0^*1, followed by any number of words in $(1 + 00^*1)$, followed by ε.
- This gives this new FA:



And the language of the original FA is $L(0^*1(1 + 00^*1)^*)$.
Note: this may not be the simplest possible form for this regular expression!
(What is? How long an expression can this generate?)

This is not a proof!

No, it is not. We can give a formal proof for state removal, but we will not.

We will give a formal proof, but with a different way of showing that $L(M)$ is regular, where M is an arbitrary DFA.

Recall:

- $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) \in F\} = \bigcup_{r \in F} \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) = r\}$
- Characterize all strings that take us from some state q to r: Define $L(q, r) = \{x \in \Sigma^* \mid \hat{\delta}(q, x) = r\}$.
- With this definition, $L(M) = \bigcup_{r \in F} L(q_0, r)$.
- This is the set of words that take us from q_0 to any accept state r.

Proving this language is regular

- $L(M) = \bigcup_{r \in F} L(q_0, r)$.
- This is a finite union, so if all $L(q_0, r)$ are regular, then so is $L(M)$. (Prove this, by induction on $|F|$!)

What we now want:

- **Theorem:** for any two states q and r in a DFA, $L(q, r)$ is regular.
- We will prove this using structural induction.
- How do we get from q to r?
- Do we use an intermediate state p?
- If so, we go from q to p, maybe from p to itself, then from p to r.
- One term of the regular expression for $L(q, r)$ might be $L(q, p)L(p, p)^* L(p, r)$.

But structural induction needs a base case!
[Remember: structural induction goes from “simple” structures to “complex”]

New definitions

How can we make the base case simple enough?

- Restrict the number of states that can come between q and r, and grow this set of states.
- Number the states of the DFA M $1, \ldots, n$.
- Let $L(q, r, k) = \{$ all words in $L(q, r)$ where all intermediate states between q and r are from $1, \ldots, k\}$.
  (Remember, M is a DFA, so each word has only one state path.)

More about the languages $L(q, r, k)$

- If the path from q to r for word x uses state k, then x is not in $L(q, r, k-1)$, or $L(q, r, k-2)$, or any other $L(q, r, i)$ for $i < k$.
- Formally: $L(q, r, k) = \{x \in \Sigma^* \mid \hat{\delta}(q, x) = r$ and $\hat{\delta}(q, w) \leq k$ for all proper prefixes of w of x where $w \neq \varepsilon$ and $w \neq x\}$.
- (Terminology: w is a proper prefix of x if w is a prefix of x and not equal to x.)
- Now it is enough to show $L(q, r, n)$ is regular, since $L(q, r) = L(q, r, n)$.
- Then it is enough show that $L(q, r, k)$ is regular for all k.
- The proof is by induction on k.

Goal, beginning of proof

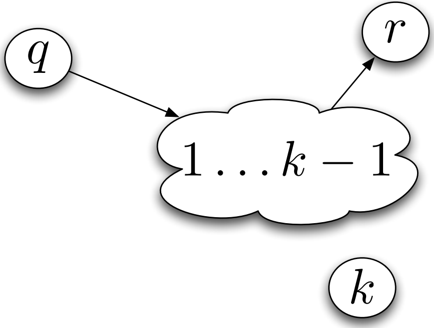
Goal: $L(q, r, k)$ is regular for all k: proof by induction on k.
Base case: $k = 0$:

- If $x \in L(q, r, 0)$, then $\hat{\delta}(q, w) \leq 0$ for all proper prefixes w of x.
- But that means there are no proper prefixes of the word x.
- Therefore, $|x| = 0$ or 1.
- So all words in $L(q, r, 0)$ are of length 0 or 1.
- As our alphabet is finite, this implies $L(q, r, 0)$ is finite (and thus regular by an earlier Theorem).

Inductive step

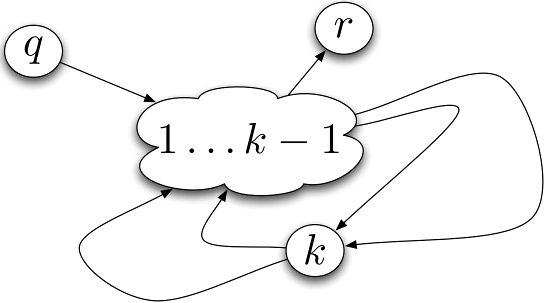
Induction step: $k = n \geq 1$:

- Inductive hypothesis: $L(q, r, k-1)$ is regular for all q and r.
- We must show that $L(q, r, k)$ is regular.
  - Let x be a word in $L(q, r, k)$.
  - If the path from q to r for x never touches the state k, then $x \in L(q, r, k-1)$.



Inductive step, continued

- Otherwise: k *is* on the path from q to r for x.
- Then there is a first and a last time we are in state k. Between the first and last time, we are only in states $1, \ldots, k$, regardless of whether $q > k$ or $r > k$.



Inductive step, finished

So we can divide x into:

- The part from q to k the first time,
- The first loop (if any) from k to k,
- The next loop from k to k,
- ...
- The last loop from k to k,
- and the part from k to r.

Words divided in this way are exactly the words in $L(q, r, k)$ that include state k on their state path. This set of words is $L(q, k, k-1)(L(k, k, k-1))^* L(k, r, k-1)$.
So the language $L(q, r, k)$ is the union of two languages:

- $L(q, r, k-1)$, which we know is regular by the induction hypothesis, and
- $L(q, k, k-1)(L(k, k, k-1))^* L(k, r, k-1)$, which is the concatenation of three languages (each of which is regular by the induction hypothesis), and thus is also regular.

Hence $L(q, r, k)$ is regular.

Wrapping it up

We are done, but that may not be obvious yet.

- We proved that $L(q, r, k)$ is regular for all k.
- We noted that $L(q, r) = L(q, r, n)$, so $L(q, r)$ is always regular for any $q, r \in Q$.
- Therefore $L(q_0, r)$ is regular for any $r \in F$.
- Thus $\bigcup_{r \in F} L(q_0, r)$ is regular (as it is the union of a finite number of regular languages).
- But $L(M) = \bigcup_{r \in F} L(q_0, r)$ is therefore regular.

So $L(M)$ is regular! (Again, how long is the expression for $L(q_0, r)$?)

Quite an achievement

This is the end of the proof of Kleene's Theorem:

- Given a DFA, we have shown that its language is regular.
- Given a regular language, we can produce an ε-NFA which recognizes it.
- NFAs and ε-NFAs have the same computing power as DFAs.

Next module: the boundaries of regular languages, and closure rules for them.