Module 4

Properties of regular languages

Not everything is regular.

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Collin Roberts
University of Waterloo

Topics for Module 4

- Proving languages non-regular: the Pumping Lemma
- Closure rules for regular languages
- Algorithms for decision problems about finite automata and regular languages.

Where are we?

- We have given definitions for regular languages, and shown their strong connection to FAs.
- If we apply certain operations to regular languages, we get back a regular language.
- Are all languages regular?
  - Obviously no: we are going to have 8 more weeks in the term, and we are getting to the end of regular languages.
  - In this section, we will think more about what makes a language regular.

Non-regular languages

- By Kleene's Theorem, a language L is not regular if for every DFA M, $L \neq L(M)$.
- So if we characterize languages of DFAs (that is, regular languages) very carefully, maybe we can also characterize those languages that are not regular.

How does a DFA M work?

- Suppose it has n states.
- Consider a word x in $L(M)$ with $|x| \geq n$.
- On its path from q_0 to an accept state, it must repeat a state somewhere along the path.
  - (Why? There are only n states in total, and the machine starts out in one of them, then reads $\geq n$ input characters.)
  - Arguments of this type use the pigeonhole principle.

Decompose the word into parts

Let's say that we repeat state r.

- Then the word x can be decomposed: $x = uvw$, where:
  - $u =$ the part from q_0 to the first time we reach r (i.e. after processing u, we are in state r).
  - $v =$ the loop from r to itself (i.e. after processing v, we are again in state r).
  - $w =$ The part from the second time we reach r that leads us to an accept state
  - Note: it is possible that either u or w is ε, but v cannot be ε.
- This decomposition is possible for any word x in $L(M)$ with $|x| \geq n$.
- Fact: $uvvw$ is also in $L(M)$. Why?
  - vv also takes M from r back to itself: $\hat{\delta}(r, vv) = r$.
- Another word in $L(M)$ is $uw = uv^0w$.
- We can show (by induction) that $uv^*w \subseteq L(M)$.

More about regular languages

We can decompose any word x in $L(M)$ of length at least n this way.

- If we choose the first time a state is repeated, then $|uv| \leq n$.
  - Why? The machine has n states, so we must have the first repeated state by the nth step.)
- And $|v| \geq 1$, since it is a DFA, and therefore has no ε-transitions.

Let's formalize this:

- Given a DFA M with n states, and a word x in $L(M)$, with $|x| \geq n$, x can be decomposed as $x = uvw$, where
  - $|uv| \leq n$,
  - $|v| \geq 1$ and
  - $uv^* w \subseteq L(M)$.

Pumping lemma

This fact is sometimes called the "Pumping Lemma":

- We can pump out many copies of v, and $uvvvvvvvvvvvw$ is still part of $L(M)$.

It can be seen as a statement about regular languages.

- Every regular language L is accepted by a DFA.
- For a given regular language L, there exists some smallest DFA (i.e. with the fewest states), M, that accepts L. Let's say M has n states.
- Therefore there is some n such that we can make the above statement about M.

Formal Pumping Lemma

For every regular language L, there exists some positive integer n such that all words $x \in L$ with $|x| \geq n$ can be decomposed into $x = uvw$, where:

- $|uv| \leq n$,
- $v \neq \varepsilon$, and
- $uv^iw \in L$ for all non-negative integers i.

You can think of n as being the number of states in a machine accepting L.

Again, this describes all long words in a regular language:

- For some definition of "long", all long words can be pumped.
- Note that, if L is finite (and therefore regular), then taking any $n > \max_{x \in L}\{|x|\}$ works (because with such an n, L contains no long words).

Non-regular languages

We know something about regular languages: long words can be pumped. Now let's describe some non-regular languages:

- Suppose that we have a language L.
- Suppose that no matter how we define "long", there are still long words in L that cannot be pumped.
- Then L is not regular, because all regular languages have a definition of "long" for which all long words can be pumped.

Formally

- Let L be a language.
- Suppose that for any positive integer n:
  - There exists a word $x \in L$ with $|x| \geq n$ such that
  - for any decomposition of x into $x = uvw$, with $|uv| \leq n$ and $v \neq \varepsilon$,
  - $uv^* w$ is not a subset of L.
- Then L is not a regular language.

That is a pile of negations and existences.

Again, the basis of the Pumping Lemma

- Language L is regular if it is accepted by some DFA.
- Suppose L is accepted by a DFA, M, with n states.
- Any word $x \in L$ with at least n letters includes a state cycle: some state r appears two times.
- This reuse of r corresponds to a substring v of x, so $x = uvw$. When we start v in state r, we also end in state r: $\hat{\delta}(r, v) = r$.
- If we got to the start of v (by reading in u), went through the cycle twice, and then finished with w we would wind up at the same accept state in M. So $uvvw$ and $uvvvw$, and all of uv^*w is in L.

Explaining Pumping Lemma proofs of non-regularity

Now, what about using the Pumping Lemma to prove a language L is not regular?

- "Suppose that for any value of $n > 0$, there exists a word $x \in L$ with $|x| \geq n$"... *(If there is always a long word in L)*
- "such that for any decomposition of x into $x = uvw$, with $|uv| \leq n$ and $v \neq \varepsilon$" ... *(that cannot be decomposed into three parts where the first 2 parts are not long and the middle part is non-trivial)*
- "$uv^*w \not\subseteq L$."... *(and the second part cannot be pumped,)*
- Then L is not a regular language.

An example

Let's show an example:

- Theorem: $L = \{0^i 1^i | i \geq 0\} = \{\varepsilon, 01, 0011, 000111, 00001111, \ldots\}$ is not a regular language.

Proof:

- For any $n > 0$, choose a word $x \in L$ whose length is at least n.
- We will choose $x = 0^n 1^n$. This is our long word.
- Now, consider all decompositions $x = uvw$, where $|uv| \leq n$, and $v \neq \varepsilon$.
- Fact: for any such decomposition, $uv = 0^k$ for some $0 < k \leq n$, because the first n characters of $x = uvw$ are all 0 (by the definition of x).
- Now, we must show that because of what we found, $uv^* w$ is not a subset of L. In particular, we must find an $i \geq 0$ such that $uv^i w \notin L$. (Typically, $i = 0$ or $i = 2$.)
- Let $i = 0$. Recall that v is all 0's. Then $uv^0 w$ will have fewer 0's than 1's. So $uv^0 w \notin L$.
- And hence the language L is not regular.

Again, how did that work?

Pumping lemma: to prove languages are not regular.

- For any definition of long, find a long word:
  Long: length $\geq n$. Our long word was $x = 0^n 1^n$.
- Consider all breakdowns of x into $x = uvw$, where uv is short and $v \neq \varepsilon$.
  For the long word x, if $x = uvw$, and uv is short, then uv is all 0's.
- If for all of these breakdowns $x = uvw$, we cannot pump v, then L is not regular.
  No matter what v is, it must be all 0's. So if we pump v, then $uvvw$ or uw both have the wrong number of 0's. So L is not regular.
- We can also prove L is not regular by thinking of possible DFAs for L and showing that they cannot exist.
- This is hard in general. The Pumping Lemma is better.

Another example

We saw that $\{0^i 1^i | i \geq 0\}$ is not regular.
Another case:
Theorem: The language $L = \{0^p | p \text{ is a prime }\}$ is not regular.

- (This language includes $00, 000, 00000, 0000000, 00000000000, \ldots$)
- Proof by Pumping Lemma. (Assume that there are infinitely many primes. There are many nice proofs of this fact.)
  - Choose a value of $n > 0$.
  - Choose $x = 0^p$, for a prime $p \geq n$.
  - Then x is a long word in L.
  - Now we must argue that no decomposition of x can be pumped.

Why can we not pump the primes?

So $x = 0^p$, for $p \geq n$, p a prime.
Consider all decompositions $x = uvw$, where $|uv| \leq n$ and $v \neq \varepsilon$.

- Then $v = 0^k$ for some $1 \leq k \leq n$.
- And $uv^*w = \{0^{p-k}, 0^p, 0^{p+k}, 0^{p+2k}, \ldots\}$.
- Is it possible that all of these are in L?
- No. One member of uv^*w is $0^{p+(pk)}$; it is the $(p+2)^{th}$ member in the above list.
- This word is not a member of L, since $p + pk = (1+k)p$ is composite (both factors are non-trivial, as $k \geq 1$).

For any n, we can find a long word, such that all decompositions of it cannot be pumped.
Therefore L is not regular.

Another example: palindromes

$L = \{s \mid s = s^R\}$ (This is the language of palindromes.)

- Examples: $0110, 01110, \varepsilon, 1111$, etc.

L is not regular.
Proof by Pumping Lemma.

- Given a value of $n > 0$, find a word in L of length at least n.
- How about $x = 0^n 1 0^n$?
- Now, consider all decompositions of this into $x = uvw$, where uv is short and v is not ε.
- Again, v must be 0^i for some $1 \leq i \leq n$.
- And the number of 0's before the only 1 in uv^2w is more than the number after it, so it cannot be a palindrome.
- So we cannot pump x, regardless of our choice of decomposition.
- So L is not regular.

One more example

Let $L = \{y!z \mid |y| > |z|, y, z \in \{0,1\}^*\}$.

- $\Sigma = \{0, 1, !\}$

This language includes words like $111!00, 1!, 10001!111$. Fact: L is not regular.

Proof by Pumping Lemma.

- Consider a value $n > 0$.
- The string $x = 0^n!0^{n-1}$ is long, and in L.
- We will show that uv^0w is not in the language.
  - Decompose $x = uvw$ with uv of length at most n and nonempty v.
  - For all such decompositions, $v = 0^k$ for some $k \geq 1$.
  - And $uv^0w = 0^{n-k}!0^{n-1}$.
  - This is not a word in L: the part before the ! character is too short.
  - So v is not pumpable, no matter how we do it.
- L is not regular.

What can go wrong?

It is easy to misuse the Pumping Lemma.

- The existence of one bad decomposition of x does not matter.
- We must show that all decompositions of $x = uvw$ with $|uv| \leq n$ and $v \neq \varepsilon$ cannot be pumped.

Example:

- Obviously, $L = (01)^*$ is regular.
- For any value of $n > 0$, $(01)^n$ is a long word in L.
- Decompose into $u = 0, v = 1, w = (01)^{n-1}$.
- Then $uv^2w = 011(01)^{n-1}$ is not in L.
- So we conclude that L is not regular (?!?!?)

Clearly we have done something wrong!

- Problem: We must show that no decomposition can be pumped.
- The decomposition $u = \varepsilon, v = 01, w = (01)^{n-1}$ is pumpable.

More Pumping Lemma: pitfalls

- The Pumping Lemma:
  - Long words in regular languages can be pumped.
- Its contrapositive:
  - If a language has long words that cannot be pumped, it is not regular.
- Note: the theorem does not give a definition of regular languages. The following is not true:
  - If all long words in a language can be pumped, it is regular.
- In fact, some non-regular languages can be pumped.

Closure rules

Regular languages are closed under $*$, union and concatenation. This is by definition:

- A class of languages is closed under a binary operation if applying that operation to 2 languages in the class always yields a language in the class
- A class of languages is closed under a unary operation if applying that operation to one language in the class always yields a language in the class.

Subsets of regular languages are not necessarily regular: $(0+1)^* = \Sigma^*$ is regular, so any language over $\Sigma = \{0, 1\}$ is the subset of a regular language! We just saw examples of languages over Σ which are not regular.

More closure rules

Regular languages are also closed under complement and intersection.
Theorem: If language L is regular, then so is its complement, L'.
Proof:

- Proof by Kleene's theorem.
- Since L is regular, it is the language of a DFA, M, with state set Q and accept states $F \subseteq Q$.
- Construct a new DFA, M' from M, as follows.
- Swap the accept and reject states in M.
- Then M', with accept set $Q \setminus F$ accepts all words which the old DFA, M, rejected and rejects all words which M accepted.
- M is a DFA, so there is only one path for a particular word.
- The same is true for M', so M' is a new DFA.
- M' accepts L'.
- By Kleene's theorem, L' is regular.

(This is much easier than exhibiting a regular expression for L'.)

Regular languages are closed under intersection

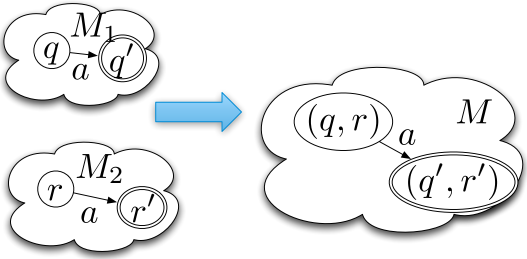
Theorem: If L_1 and L_2 are regular languages, then so is $L_1 \cap L_2$.
Proof:

- This can be proved in lots of ways.
- One easy one: $L_1 \cap L_2 = (L_1' \cup L_2')'$.
  (That can take a couple seconds to understand! Draw a suitable Venn diagram if it helps.)
- And L_1' is regular (by our last theorem); so is L_2'.
- By the definition of regular languages, so is $L_1' \cup L_2'$.
- And again, by our closure theorem, so is $(L_1' \cup L_2')' = L_1 \cap L_2$.

Or, by actually building a DFA

Theorem: Given regular languages L_1 and L_2, $L = L_1 \cap L_2$ is also regular.
Proof sketch:

- Let M_1 and M_2 be machines accepting L_1 and L_2 respectively.
- New machine M: one state for each pair of states in M_1 and M_2.
  - If M_1 is in state q, and M_2 is in state r, then M will be in (q, r).
  - If M_1 transitions from q to q' and M_2 from r to r' on letter a, then in M, $\delta_M((q,r),a) = (q', r')$.
  - Accept states are those that come from accept states in both machines.



Closure under reversal

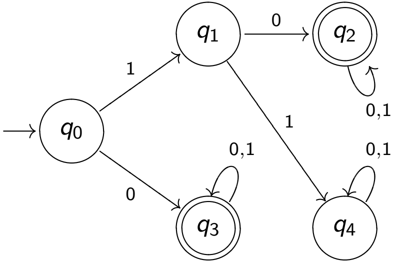
Recall: w^R is the reversal of the word w.
Given a language L, let L^R be the language that consists of all of the words of L, reversed.
Theorem: If L is regular, then so is L^R.
Proof sketch: Let M be a finite automaton whose language is L. Make a new finite automaton R with the same states as M, plus a new start state:

- All of the edges of R are the reversals of the edges of M.
- The sole accept state of R is the start state of M.
- The start state of R has an ε-transition to each accept state of M.

Then R reverses the automaton M: if we start at an accept state of M and work our way back to the start state of M (i.e. if M accepted x), then the new machine R accepts the word x^R, and vice versa.

Closure under reversal

Suppose we have the following DFA, M

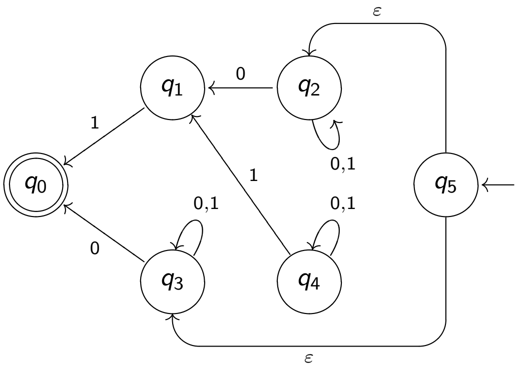


to accept the language

$$L = \{w \mid w \text{ begins with } 0 \text{ or with } 10\}.$$

Closure under reversal

The construction yields this ε-NFA, R



to accept the language

$$L = \{w \mid w \text{ ends with } 0 \text{ or with } 01\}.$$

Note that, although this ε-NFA R can be simplified, the construction is still correct.

Properly proving the reversal of a regular language is regular

We can prove this theorem by structural induction on the construction of the regular language L.

- Base cases: If $L = \emptyset, \{\varepsilon\}$ or $\{a\}$,then $L^R = L$ is regular.
- If $L = L_1 \cup L_2$ for regular languages L_1 and L_2, then $L^R = L_1^R \cup L_2^R$, and both of these languages are regular by induction.
- If $L = L_1 L_2$ for regular languages L_1 and L_2, then $L^R = L_2^R L_1^R$, and this is the concatenation of two regular languages and hence regular.
- If $L = L_1^*$, then $L^R = \left(L_1^R\right)^*$, since any word in L is the concatenation of a finite sequence of words of L_1: if $w \in L$, $w = w_1 \ldots w_n$, and $w^R = w_n^R \ldots w_1^R$. This is a sequence of words in L_1^R, and so L^R is the Kleene closure of the (by the induction hypothesis) regular language L_1^R.

Algorithmic questions about finite automata

We do not normally create algorithms in this class. Here is an exception: Is it possible to find algorithms for the following:

- Given a DFA M and a word x, does M accept x?
- Given a DFA M, is $L(M)$ empty?
- Given a DFA M, is $L(M)$ infinite?
- Given two DFAs M_1 and M_2, is $L(M_1) \cap L(M_2)$ empty?
- Given two DFAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?
- Given two DFAs M_1 and M_2, is $L(M_1) = L(M_2)$?
- Given two regular expressions e_1 and e_2, do they generate the same language?

Acceptance, empty language

- Given a DFA M and a word x, does M accept x?
  - Just simulate the DFA.
  - (This may seem obvious. But we will not be able to do this for Turing machines.)
- Given a DFA M, is $L(M)$ empty?
  More fun.
  - Suppose M has n states.
  - If M accepts any words, it must accept a word with fewer than n letters.
  - This is a consequence of the proof of the Pumping Lemma.
  - We prove it carefully on the next slide.

Acceptance, empty language

Lemma: If a DFA, M, having n states accepts any words, then it must accept a word with fewer than n letters.

Proof:

- Assume $L(M) \neq \emptyset$.
- By Kleene's Theorem, $L(M)$ is regular.
- Let $x_0 \in L(M)$ be arbitrary.
- If $|x_0| < n$, then we are finished.
- Otherwise, $|x_0| \geq n$ and by the proof of the Pumping Lemma, we can decompose $x_0 = u_0 v_0 w_0$, with $u_0 w_0 \in L(M)$ and $|v_0| \geq 1$.
- If $|u_0 w_0| < n$, then we are finished.
- Otherwise, $|u_0 w_0| \geq n$ and $u_0 w_0 \in L(M)$ and so by the proof of the Pumping Lemma, we can decompose $u_0 w_0 = u_1 v_1 w_1$, with $u_1 w_1 \in L(M)$ and $|v_1| \geq 1$.
- Continuing in this way we obtain a sequence of words in $L(M)$ having strictly decreasing lengths: $x_0, u_0 w_0, u_1 w_1, \ldots, u_j w_j, \ldots$.
- As x_0 has finite length, after at most $|x_0| - n + 1$ steps, we will obtain a word in $L(M)$ with length $< n$. $\square$

Acceptance, empty language

Now, back to the question

- Given a DFA M, is $L(M)$ empty?
  - Try every word of length less than n (finitely many since our alphabet is finite).
  - If no short word is accepted, then by the previous Lemma, $L(M) = \emptyset$.

Is the language of an FA finite?

- Given a DFA M, is $L(M)$ infinite?
  Theorem: If M is a DFA with n states, then $L(M)$ is infinite if and only if $L(M)$ includes a word x satisfying $n \leq |x| < 2n$.
  Proof:
  - Suppose $x \in L(M)$ and $n \leq |x| < 2n$.
  - From the Pumping Lemma, x must be pumpable.
  - The word $x = uvw$ can be used to generate the infinite language uv^*w, which is a subset of $L(M)$.
  - So $L(M)$ is infinite.

Other half of the proof

- Other direction: Assume that $L(M)$ is infinite.
  - For a contradiction, suppose that there does not exist any $x \in L(M)$ satisfying $n \leq |x| < 2n$.
  - In other words, every word $x \in L(M)$ with length at least n must have length at least $2n$.
  - Let $x \in L(M)$ be a shortest word with length at least n (so that $|x| \geq 2n$ by the above point).
  - (If there is no $x \in L(M)$ with length at least n, then $L(M)$ is finite, which cannot happen.)
  - Decompose $x = uvw$, where $v \neq \varepsilon$ and $|uv| \leq n$, so that $1 \leq |v| \leq n$.
  - By the Pumping Lemma, uw is also in $L(M)$.
  - We have only removed at most n letters by removing v, so $|uw| \geq n$, and by construction, $|uw| < |x|$.
  - Thus uw violates the choice of x as a shortest word in $L(M)$ with length at least n.
  - This contradiction completes the proof.
- If $L(M)$ is infinite, we must have a word in $L(M)$ with length between n and $2n$.
- To see if $L(M)$ is infinite, check all words between n and $2n$ in length.
- This runs in a finite amount of time.

Disjoint languages, subset

- Given two DFAs M_1 and M_2, is $L(M_1) \cap L(M_2)$ empty?
  - First, construct a DFA for $L(M_1) \cap L(M_2)$.
  - Then use the algorithm for testing for an empty language of an FA to see if it accepts the empty language.
- Given two DFAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?
  - If so, then $L(M_2)' \cap L(M_1)$ is empty.
    (There is nothing in $L(M_1)$ that is not in $L(M_2)$.)
  - Build the DFA for $L(M_2)'$.
  - Use it to build the DFA for $L(M_2)' \cap L(M_1)$.
  - Use the algorithm from before to test if its language is empty!

Two FAs with the same language

- Given two DFAs M_1 and M_2, is $L(M_1) = L(M_2)$?
  - Yes, if $L(M_1) \subseteq L(M_2)$ and $L(M_2) \subseteq L(M_1)$.
  - Use the algorithm for testing for subset twice.
- Given two regular expressions e_1 and e_2, do they represent the same language?
  - Construct the DFAs for each regular expression, using Kleene's Theorem.
  - Then use the algorithm for testing if two FAs have the same language.

(If you like this kind of stuff, take CS 462.)

End of regular language unit

Is a DFA a decent model of a computer?

- Yes, if resources are bounded.
- Regular languages: accepted by computers with very simple access to input and very little memory
- But $\{0^i 1^i \mid i \geq 0\}$ is a simple language. And yet no DFA can recognize it.

Except:

- Real computers have finite memory.
- Suppose a computer has q bits of memory.
- The computer can only be in 2^q possible states.

Often, we do not care, because that number is so huge.
Typically, we treat computers as having infinite memory, except when it really matters.
Next: Context-free languages (Module 5)
After that: Computers with infinite memory (Module 6)