CS 360 - MODULE 9 - ADDITIONAL NOTES

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1. Reductions Are Highly Directional

Here we exhibit a choice of decision problems P_1, P_2 (which are questions of membership in the languages L_1, L_2 respectively over $\Sigma = \{0,1\}$) such that there exists a reduction from P_1 to P_2, but there does **not** exist a reduction from P_2 to P_1.
Let

$$\begin{aligned} L_1 &= L(0^*) \\ L_2 &= \{M \mid L(M) \text{ is non-regular}\} = L_{nreg}. \end{aligned}$$

Note that:

- L_1 is regular. Therefore L_1 is a DCFL, a CFL, and a decidable language.
- $L_2 = L_{nreg}$ is undecidable. This is proved explicitly in Module 9, and is also easily proved directly using Rice's Theorem.
- Since some TMs have regular languages, while other TMs have non-regular languages, we have that $L_2 \neq \emptyset$ and $L_2 \neq \Sigma^*$.
- Then by Problem 2a on CM A06, there exists a reduction from P_1 to P_2.
- Now for a contradiction, assume that there exists a reduction from P_2 to P_1.
- Then, since P_2 is not decidable, Theorem 9.7 implies that P_1 is undecidable. This contradiction shows that no reduction from P_2 to P_1 exists.

2. If a Language and it Complement are Both CFLs, Does It Follow That the Language a DCFL?

Here we present a counterexample to show that this statement does **not** hold in general. Let

$$\begin{aligned} \Sigma &= \{0,1\} \\ L &= \{w \in \Sigma^* \mid w^R = w\}, \end{aligned}$$

i.e. L is the language of **palindromes** over Σ.
We have that L is a CFL, generated by the grammar $G : S \to \varepsilon|0|1|0S0|1S1$.
It is an exercise to prove that the complement L' is also a CFL. I suggest constructing a PDA to recognize even-length non-palindromes (an old assignment question for CS 360), and another PDA to recognize odd-length non-palindromes. This shows that both languages are CFLs, and then by the closure rules for CFLs, their union is also a CFL.
We argued informally in class that L is **not** the language of any DPDA, in other words, L is not a DCFL.