1. [10 marks] Let $A, B$ be languages. If $A$ reduces to $B$, and $B$ is regular, must $A$ be regular? Prove or disprove.

2. [10 marks] Show that the following decision problem is unsolvable: given two context-free grammars $G_1$ and $G_2$, is $L(G_1) \cap L(G_2) \neq \emptyset$? Hint: reduce from PCP.

3. [10 marks] I mentioned in class that the “holy grail” problem of software engineering is “does program $P$ have an infinite loop in it?”.

   In class we proved that the halting problem, defined by
   
   $$\text{HALT} = \{ e(T)e(w) : \text{DTM } T \text{ halts on input } w \},$$
   
   is not Turing-decidable; see https://student.cs.uwaterloo.ca/~cs360/unrec2.pdf. But the language $\text{HALT}$ doesn’t quite correspond to the “holy grail” problem, because it focuses on a particular input. What we really want to know is, is program $P$ guaranteed to never go into an infinite loop, for every single input? Let’s translate this into the language

   $$\text{ALWAYS-HALTS} = \{ e(T) : \text{DTM } T \text{ halts on all inputs} \}. $$

   Show that $\text{ALWAYS-HALTS}$ is not Turing-recognizable. This is slightly tricky and may require some ingenuity. Hint: use a reduction.