Is the Thue-Morse word \( t \) recurrent? That is, if \( x \) is a subword of \( t \), must \( x \) occur at infinitely many different positions of \( t \)?

We claim that yes, the Thue-Morse word \( t \) is recurrent.

Consider some character of the Thue-Morse word \( t_i \). Recall that \( t_i = 0 \) if the binary representation of \( i \) has an even number of 1s, and \( t_i = 1 \) if it has an odd number of 1s. Also recall that \( \bar{t}_i \) is the opposite (0 for odd, 1 for even).

Now consider \( k = \lfloor \log_2(i) \rfloor + 1 \). That is, \( 2^k \) is the smallest power of 2 that is greater than \( i \). Let \( u \) represent the binary representation of \( i \). Recall that we can represent a number in binary using \( \lfloor \log_2(i) \rfloor + 1 \) characters, so we know that \( |u| = \lfloor \log_2(i) \rfloor + 1 = k \).

Note that \( 2^k \) is represented in binary by \( 10^{|u|} \). That is, 1 followed \( |u| \) of zeros.

So the binary representation of \( 2^k + i \) can be represented as \( 1u \).

If we consider that \( 2^{k+1} \) is represented in binary by \( 100^{|u|} \), we can conclude that \( 2^{k+1} + 2^k + i = 11u \).

Since \( u \) is represented by \( t_i \) in the Thue-Morse word, then \( 1u \) is represented by \( \bar{t}_i \) since it adds one more 1, changing the parity of the number of ones. Then \( 11u \) is represented by \( \bar{t}_i = t_i \), since adding two ones does not change parity.

Thus we can conclude that \( t_i = t_{i+2^k+2^{k+1}} \), where \( k = \lfloor \log_2(i) \rfloor + 1 \).

So if we consider the subword \( x \), then for each character in \( x \), we can look at its index in the Thue-Morse word, and use the fact that \( t_i = t_{i+2^k+2^{k+1}} \), where \( k = \lfloor \log_2(i) \rfloor + 2 \).

So if we consider \( x_j \) for \( 1 \leq j \leq |x| \), and let \( n \) be the index of \( x_1 \) in the Thue-Morse word, \( t_{j+n} = t_{(j+n)+2^k+2^{k+1}} \), where \( k = \lfloor \log_2(j + n) \rfloor + 1 \).

To show that \( x \) appears infinitely many times, assume we have found the last occurrence of \( x \) starting at index \( m \). However, using the equation above, we know that \( t_{j+m} = t_{(j+m)+2^k+2^{k+1}} \), where \( k = \lfloor \log_2(j + m) \rfloor + 1 \). Thus we have found another occurrence of \( x \) further in the sequence. This is a contradiction, so we know there is no last occurrence of \( x \). Thus \( x \) occurs infinitely many times in \( t \).

Therefore, we can conclude that the Thue-Morse word is recurrent.