Problem 27:
Call a language $L$ bounded if there exists a finite number of words $w_1, w_2, \ldots, w_n$ such that $L \subseteq w_1^* \cdots w_n^*$.

Give an example of regular language that is not bounded.

Solution:
Consider the alphabet $\Sigma := \{a, b, c\}$ on three symbols $a, b, c$.

Then we know that $\Sigma^\omega$, the set of infinite words over the alphabet has an infinite square-free string by Theorem 2.5.4 in the text by Shallit, “A Second Course in Formal Languages and Automata Theory”, 2ed.

Let $s \in \Sigma^\omega$ be this infinite squarefree string.

Now consider $L = \Sigma^*$ to be our candidate language that is both regular and not bounded. $L = \Sigma^*$ is then the set of all finite strings over the three-symbol alphabet $\Sigma = \{a, b, c\}$. We then must show that $L$ is both regular and not bounded.

Part 1: Proof of Regularity
We prove that $L$ is regular by giving a construction of a DFA $M$ that recognizes $L$, namely by using an automaton that recognizes all inputs:

$$
\begin{align*}
\begin{array}{ccc}
& & \\
 & \downarrow_{a,b,c} & \\
q_i & \rightarrow & \\
\end{array}
\end{align*}
$$

Figure 1: A diagram of the DFA $M$

Therefore, since the DFA $M = (\{q_i\}, \Sigma, \delta, q_i, \{q_i\})$ recognizes $L = \Sigma^*$, with $\delta(q_i, d) = q_i$ for all $d \in \Sigma = \{a, b, c\}$, $L$ is regular.

Part 2: Proof of Unboundedness
It now remains to prove that $L$ is not bounded. To do so, we proceed via proof by contradiction.
Now suppose on the contrary that $L$ is bounded. The for all words $w \in L$, there exists $w_1, w_2, \ldots, w_n \in L$ such that $w = w_1^{k_1}w_2^{k_2} \cdots w_n^{k_n}$.

Now since $L = \Sigma^*$, $w_1, w_2, \ldots, w_n \in \Sigma^*$ so that the lengths of $w_1, w_2, \ldots, w_n$ are finite. Call these lengths $|w_1|, |w_2|, \ldots, |w_n|$.

Then define $\ell_m := (\sum_{i=1}^n |w_i|) + 1$ to be the sum of the lengths of these words plus one so that $\ell_m$ is strictly greater than the sum of the lengths of the $w_i$.

We can then choose a prefix $p$ of fixed length of our infinite squarefree string $s$ such that $s = pq$, $q \in \Sigma^*$, with $|p| = \ell_m$ so that the finite word $p \in L = \Sigma^*$ is strictly longer than the sum of the lengths of the $w_i$ and is squarefree.

Since $p$ is longer than sum of the lengths of the $w_i$ it cannot be of the form $u = w_1^{k_1}w_2^{k_2} \cdots w_n^{k_n}$ where $k_i \in \{0, 1\}$ for $1 \leq i \leq n$ since $|u| < |p|$. Furthermore, none of the $k_i$ can be greater than one, since $p$ is squarefree, and if $k_i \geq 2$, then $u$ has a square so that $u \neq p$.

Hence $p$ cannot be written in the form $w_1^{k_1}w_2^{k_2} \cdots w_n^{k_n}$ with $k_i \in \{0, 1, 2, \ldots\}, 1 \leq i \leq n$.

Thus, $p \notin w_1^* \cdots w_n^*$ for any finite $n$ and choice of $w_1, \ldots, w_n \in L$. But $p \in L = \Sigma^*$, so $L \notin w_1^* \cdots w_n^*$. Thus, $L$ is not bounded.

**Part 3: Conclusion**

Thus, having proved both that $L$ is regular in Part 1 and that $L$ is not bounded in Part 2, it follows that $L = \Sigma^* = \{a, b, c\}^*$ is an example of a regular language that is not bounded.  

□