Problem 43 Solution
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Problem
Find a family of pairs of unary regular languages, $L_{1,n}, L_{2,n}$, both recognized by a DFA with $O(n)$ states, such that the shortest string that is in $L_{1,n}$ but not in $L_{2,n}$ is of length $\Omega(n^2)$.

Solution
Consider the languages

\begin{align*}
L_{1,n} &= \{ x : x \equiv 0 \mod n \} \\
L_{2,n} &= \{ x : x \not\equiv 0 \mod (n+1) \}
\end{align*}

$L_{1,n}$ has $n$ Myhill-Nerode equivalence classes since each $[0], [1], \ldots, [n-1]$ is in its own class. Similarly, $L_{2,n}$ has $n+1 \in O(n)$ equivalence classes. So both are recognized by a DFA with $O(n)$ states.

The shortest string that would be in $L_{1,n}$ but not in $L_{2,n}$ would be $n \times n + 1 = n^2 + n \in \Omega(n^2)$. Any other string with $x \equiv 0 \mod n$ would also be $x \not\equiv 0 \mod (n+1)$. 