Problem 6:
Show that for every infinite string $w$ there must be some letter $a$ and some finite string $x$ such that $axa$ appears infinitely often as a subword of $w$. Furthermore, such an $x$ exists with $|x| \leq |\Sigma| - 1$, where $\Sigma$ is the alphabet.

Solution:
Since the string $w$ is infinite, it contains infinitely many subwords of length $n = |\Sigma| + 1$.

Now, since the alphabet $\Sigma$ is finite, there are only finitely many possible distinct subwords of length $n$ in $w$. In fact, the number of possible distinct subwords of length $n$ is $|\Sigma|^n$, clearly finite.

We claim that at least one of these subwords occurs infinitely many times within the string $w$. If we assume that all of these subwords occur finitely many times, then that would mean that $w$ must be finite as well, a contradiction. Thus, there exists a string $u$ of length $n$ that appears infinitely often as a subword of $w$.

Given that the length of the string $u$ is $n > |\Sigma|$, by the pigeonhole principle, every such subword must contain at least one letter $a$ that occurs more than once in the subword.

From this, we have that $u$ can be written as $vaxay$ for strings $v$, $x$, and $y$. Therefore, $u$ contains subword $axa$ where $|axa| \leq |u| = n$. It follows from this that $|x| \leq n - 2 = |\Sigma| - 1$.

Since the string $u$ contains $axa$ and appears infinitely many times in $w$ as a subword, the string $axa$ appears infinitely as a subword of $w$ as well.

Thus, for every infinite string $w$ there must be some letter $a$ and some finite string $x$ such that $axa$ appears infinitely often as a subword of $w$. Furthermore, such an $x$ exists with $|x| \leq |\Sigma| - 1$, where $\Sigma$ is the alphabet.