Here’s how these problem-solving sessions work.

Start by working on the problem assigned to your group. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it. Then write up your solution within one week, in a format I can post for the rest of the class, and send it to me. (A scanned handwritten version is fine, provided it is very neatly written.) If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the extra problems at the end.

Recall that a word $x$ is a palindrome if $x = x^R$, where $x^R$ denotes the reversal of $x$.

For a word $x \in \{0, 1\}^*$, let $\overline{x}$ denote the word obtained by changing each 0 to 1 and vice versa. Call a binary word $x$ an antipalindrome if $\overline{x} = x^R$. Thus, for example, 001011 is an antipalindrome.

Recall that $\text{III}$ denotes the “perfect shuffle”, so that, for example, $\text{clip III aloe} = \text{calliope}$.

New problems

31. Let $M$ be an NFA. Show that the set of all strings in $L(M)$ having exactly one accepting path in $M$ is a regular language. Hint: instead of using Boolean matrix multiplication, use ordinary matrix multiplication to compute the number of accepting paths.

33. Suppose $x \in \Sigma_k^* = \{0, 1, \ldots, k - 1\}^*$ and $y \in \Delta^*$ for some alphabet $\Delta$. If $|x| = |y|$, and $x = a_1 \cdots a_n$, $y = b_1 \cdots b_n$, we define $\text{rep}(x,y)$ to be $b_1^{a_{1}}b_2^{a_{2}}\cdots b_n^{a_{n}}$. Thus, for example,
rep(234, abc) = aabbbbcccc. Extend this definition to languages, as follows: if \( L_1 \subseteq \Sigma_k^* \) and \( L_2 \subseteq \Delta^* \) then \( \text{rep}(L_1, L_2) = \bigcup_{x \in L_1, y \in L_2} \{ \text{rep}(x, y) \} \). Show that if \( L_1 \) and \( L_2 \) are both regular, then so is \( \text{rep}(L_1, L_2) \).

35. Show that if \( L \) is a regular language, then so is

\[
\text{ROOT}(L) := \{ w : w^{|w|} \in L \}.
\]

**Additional Problems — if you already solved your group problem try these**

11. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

13. Let \( x, y \) be length-\( n \) words over a \( k \)-letter alphabet. How many solutions are there to the equation \( x \text{III} y = xy \)?

23. Show that if \( M \) is an \( n \)-state DFA, and accepts at least one string that is a palindrome, then it accepts a palindrome of length \( < 2n^2 \). Hint: using \( M \) design an NFA-\( \epsilon \) to accept the first halves of the palindromes in \( L(M) \).

29. Let \( L_1 \) and \( L_2 \) be languages, and define the “join” operation as follows:

\[
L_1 \bowtie L_2 = \{ xyz : \text{there exists } y \neq \epsilon \text{ such that } xy \in L_1 \text{ and } yz \in L_2 \}.
\]

This is similar to a “join” in database theory. Using morphisms and/or inverse morphisms show that if \( L_1 \) and \( L_2 \) are regular, so is \( L_1 \bowtie L_2 \).