Here’s how these problem-solving sessions work.

Start by working on the problem assigned to your group. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it. Then write up your solution within one week, in a format I can post for the rest of the class, and send it to me. (A scanned handwritten version is fine, provided it is very neatly written.) If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the extra problems at the end.

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**New problems**

55. Write the language \( \{a^i b^j a^i b^j : i, j \geq 1 \} \) as the intersection of two CFL’s.

57. *Clickomania* is a game whose goal is to remove all the colored squares in an array. Squares are removed by clicking on a maximal contiguous block of at least two squares of the same color, and then these disappear. Consider a simplified version where the squares are arranged in a \( 1 \times n \) array and come in only two colors, \( a \) and \( b \). A string of \( a \)’s and \( b \)’s is *solvable* if there is some choice of moves that reduces it to the empty string. For example, \( abbaaba \) can be reduced to the empty string as follows, where the underline portion denotes the part that is removed at each step:

\[
abbaaba \rightarrow abbba \rightarrow aa \rightarrow \epsilon.
\]

Show that \( CL \), the language of all solvable strings, is a context-free language.
11. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

13. Let $x, y$ be length-$n$ words over a $k$-letter alphabet. How many solutions are there to the equation $x	ext{III}y = xy$?

23. Show that if $M$ is an $n$-state DFA, and accepts at least one string that is a palindrome, then it accepts a palindrome of length $< 2n^2$. Hint: using $M$ design an NFA-$\epsilon$ to accept the first halves of the palindromes in $L(M)$.

29. Let $L_1$ and $L_2$ be languages, and define the “join” operation as follows:

$$L_1 \bowtie L_2 = \{xyz : \text{there exists } y \neq \epsilon \text{ such that } xy \in L_1 \text{ and } yz \in L_2\}.$$ 

This is similar to a “join” in database theory. Using morphisms and/or inverse morphisms show that if $L_1$ and $L_2$ are regular, so is $L_1 \bowtie L_2$.

33. Suppose $x \in \Sigma_k^* = \{0, 1, \ldots, k-1\}^*$ and $y \in \Delta^*$ for some alphabet $\Delta$. If $|x| = |y|$, and $x = a_1 \cdots a_n$, $y = b_1 \cdots b_n$, we define $\text{rep}(x, y)$ to be $b_1^{a_1}b_2^{a_2}\cdots b_n^{a_n}$. Thus, for example, $\text{rep}(234, abc) = aabbbcccc$. Extend this definition to languages, as follows: if $L_1 \subseteq \Sigma_k^*$ and $L_2 \subseteq \Delta^*$ then $\text{rep}(L_1, L_2) = \bigcup_{x \in L_1, y \in L_2} \{\text{rep}(x, y)\}$. Show that if $L_1$ and $L_2$ are both regular, then so is $\text{rep}(L_1, L_2)$.

45. What are the Myhill-Nerode equivalence classes of the language $\{a^m b^n : 1 \leq m \leq n\}$?

53. We say a word $w \in \Sigma^*$ is good if $||u| - |v||_a \leq 1$ for all subwords $u, v$ of $w$, with $|u| = |v|$, and all $a \in \Sigma$; otherwise $w$ is bad. Show that the set of all bad words is context-free.