CS 466/666, Fall 2006 due Fri. Dec. 1 for CS 466, Fri. Nov. 24 for CS 666
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ASSIGNMENT 4

Do not copy from others, and acknowledge your sources.

1. Consider the weighted set cover problem with elements $E = \{1, \dots, n\}$, and sets $\mathcal{S} = \{S_1, \dots, S_m\}$. Let f be the maximum over all elements e of the number of sets containing e. Recall that the problem can be formulated as the integer version of the following linear program:

$$\min \sum_{S \in \mathcal{S}} w_S x_S$$

$$\begin{array}{ll} \forall e \in E & \displaystyle\sum_{\{S:\ e \in S\}} x_S \geq 1 \\ \forall S \in \mathcal{S} & x_S \geq 0 \end{array}$$

Consider the following approximation algorithm: Solve the LP, and then round up all variables (i.e. 0 stays 0, anything else goes to 1). Prove that this gives approximation factor f.

The easiest way I know to do this involves considering the dual LP:

$$\max \sum_{e \in E} y_e$$

$$\begin{array}{ll} \forall S \in \mathcal{S} & \displaystyle\sum_{e \in S} y_e \leq w_S \\ \forall e \in E & y_e \geq 0 \end{array}$$

You will need weak LP duality: that for any feasible x and y, $\sum w_S x_S \geq \sum y_e$. You will also need complementary slackness: that for any optimum x^* and y^*, if $x^*_S > 0$ for some S, then $\sum_{e \in S} y^*_e = w_S$. (You don't need to prove either of these things.)

2. Given a set of points in the plane, a *triangulation* is a maximal set of line segments joining pairs of points such that no two line segments intersect except at a common endpoint. Recall that maxim*al* means that no more line segments can be added without causing intersections. The interior regions formed by a triangulation are triangles. (This is a Lemma but you don't need to prove it.) The *Minimum Weight Triangulation Problem* is to find a triangulation that minimizes the sum of the lengths of the edges. This problem was listed as open in Garey and Johnston's NP-completeness book in 1976, and was only proved to be NP-hard this year. The *greedy triangulation* is formed by considering all the edges in order from shortest to longest, and choosing an edge if it causes no intersections with the edges chosen so far.

   (a) [4 marks] Show that the greedy triangulation is not always a minimum weight triangulation.

(b) [BONUS] Show that the approximation ratio of the greedy triangulation to the minimum weight triangulation is $\Omega(\sqrt{n})$.

(c) [3 marks] Prove or disprove: A minimum weight triangulation always contains a minumum weight spanning tree.

(d) [3 marks] Suppose that we change the problem by allowing the addition of extra points, as many as we like, anywhere in the plane. This is called a *Steiner triangulation*. Show that the minimum weight Steiner triangulation can be strictly less in weight than the minimum weight triangulation. Note: existence of a minimum weight Steiner triangulation is open—it is conceivable that in some situations you can add more and more extra points and make the weight go further and further down, never achieving the limit.

3. Devise a good approximation algorithm for the weighted max cut problem, where we are give weights on the edges of a graph and want to find a cut that maximizes the sum of weights of edges of the cut. Analyze the run time and the approximation factor. Recall that a cut is a partition of the vertices into two sets, and the edges of the cut are the edges that cross between the two vertex sets. [Hint: review the algorithm for the unweighted problem given in class.]

4. For the weighted minimum cluster problem we are given a graph $G = (V, E)$, and weights on the edges, and we want to find a partition $S, V - S$ that minimizes the sum of the weights of edges with both ends in S or both ends in $V - S$. Note that the complementary set of edges forms a maximum weight cut. Prove that there is no polynomial time approximation algorithm for the weighted minimum cluster problem that achieves a constant approximation ratio unless P = NP. [Hint: One method is to show that such an algorithm would yield a polynomial time algorithm for the NP-complete problem of whether a graph can be 3-coloured.]