CS 466/666, Fall 2007 due Friday November 30
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ASSIGNMENT 4

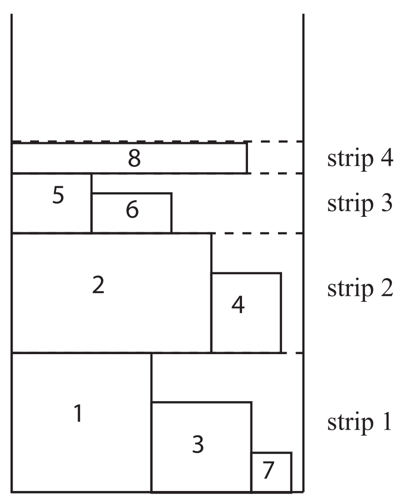
Do not copy from others, and acknowledge your sources.

1. [5 marks] Prove that if there is a constant k and a polynomial time approximation algorithm for the knapsack problem guaranteeing a solution of weight at least OPT $- k$ then P = NP. Do this by showing that such an approximation algorithm would solve the decision version of knapsack in polynomial time.

2. [15 marks] In the two-dimensional bin packing problem we must pack n rectangles of height h_i and width w_i, $i = 1, \ldots n$ into the minimum number of bins of height H and width W. We don't allow rectangles to be rotated. Suppose we normalize so that $W = 1$. Consider the following approximation algorithm: In Stage 1 we pack the rectangles into one big rectangle of width 1 and height T—we try to minimize T. We do the packing in strips $S_1, \ldots, S_k$, where the strips all have width 1, and strip S_j has height T_j. Each rectangle will be packed into one of the strips. In Stage 2 we pack the strips into the final $H \times 1$ bins. This is one-dimensional bin packing, since the strips and the bins all have the same width and only the heights matter, and so strategies such as First Fit will do well.

   For Stage 1, we use an idea similar to First Fit. Order the rectangles by height, so $h_1 \geq h_2 \geq \cdots \geq h_n$. We fill each strip from left to right with rectangles side by side, like packing books on a shelf. Pack rectangle i into the first strip where it fits. Note that "fitting" depends only on the width because every strip so far has height at least h_i. If rectangle i doesn't fit in any of the strips, then start a new strip of height h_i. See the example above.

   Let A be the total area of all the rectangles, i.e. $A = \sum_{i=1}^{n} h_i w_i$.



   (a) As a warm-up, and because you need it later, show that for 1-dimensional bin packing with items of sizes $b_1 \geq b_2 \geq \cdots \geq b_n$ First Fit uses at most $2(\sum_{i=1}^{n} b_i) + 1$ bins.

   (b) Stage 1 of the algorithm above packs the rectangles into a big rectangle of area T. Prove that $T \leq 2A + h_1$.

(c) Give further details for Stage 2 and argue that you can pack the strips into b bins where $b \leq 2(T - h_1) + 2$.

(d) Conclude that your algorithm achieves asyptotic approximation ratio 4 for the two-dimensional bin packing problem.

FURTHER INFORMATION: In fact, this algorithm achieves an asymptotic approximation factor better than 4. Other algorithms for the problem do even better.

3. [20 marks] Given a complete graph $G = (V, E)$ with n vertices and a distance function $d : V \times V \rightarrow R^+$ satisfying symmetry and the triangle inequality, consider the following METRIC-3-CLUSTERING problem: find a partition of V into 3 subsets V_1, V_2, V_3 minimizing $c(V_1, V_2, V_3) = \max_{i=1,2,3} \max_{u,v \in V_i} d(u, v)$, i.e. the maximum weight of an edge with both ends in the same V_i,

   (a) [10 marks] Design and analyze a polynomial time algorithm with approximation factor at most 2 for METRC-3-CLUSTERING. Hint: for each triple of vertices s_1, s_2, s_3 try $V_1 = \{v \in V | d(v, s_1) < d(v, s_2), d(v, s_3)\}$, $V_2 = \{v \in V | d(v, s_2) < d(v, s_1), d(v, s_3)\}$, and $V_3 = V - (V_1 \cup V_2)$.

   (b) [10 marks] Prove that if P $\neq$ NP then there is no polynomial time approximation algorithm with approximation factor strictly smaller than 2. You may use the fact that 3-COLOURING is NP-complete (see [CLRS]). Hint: use distances that are all 1's and 2's.