CS 466, Spring 2007
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Assignment 4 (due July 11 Wed noon)

1. [*10 marks*] We are given a collection of m subsets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$, each of even size. We say that a subset $S \subseteq \{1, \ldots, n\}$ is a *splitter* if S intersects both A_i and its complement for all $i = 1, \ldots, m$, i.e., $S \cap A_i \neq \emptyset$ and $S \setminus A_i \neq \emptyset$. The problem of finding a splitter is NP-hard in general, but we will consider a special case:

   We say that a subset $P \subseteq \{1, \ldots, n\}$ is a *perfect splitter* if $|P \cap A_i| = |P \setminus A_i|$ for all $i = 1, \ldots, m$. For any input instance for which we know that a perfect splitter exists (but do not know the perfect splitter itself), design and analyze a Monte Carlo algorithm that finds a splitter in polynomial time. (The splitter found does not have to be perfect.)

   [Hint: imitate Papadimitriou's SAT algorithm. . .]

2. [*20 marks*] Given a complete graph $G = (V, E)$ with n vertices and a distance function $d : V \times V \to \mathbb{R}^+$ satisfying symmetry and the triangle inequality, consider the following problem called Metric-3-Clustering: find a partition of V into 3 subsets V_1, V_2, V_3, minimizing $c(V_1, V_2, V_3) = \max_{i \in \{1,2,3\}} \max_{u,v \in V_i} d(u, v)$.

   (a) [*10 marks*] Design and analyze a polynomial-time algorithm with approximation factor at most 2 for Metric-3-Clustering.

   [Hint: for each triple of vertices s_1, s_2, s_3, try $V_1 = \{v \in V \mid d(v, s_1) < d(v, s_2), d(v, s_3)\}$, $V_2 = \{v \in V \mid d(v, s_2) < d(v, s_1), d(v, s_3)\}$, $V_3 = V \setminus (V_1 \cup V_2)$,. . .]

   (b) [*10 marks*] Prove that if P $\neq$ NP, then there is no polynomial-time approximation algorithm with approximation factor strictly smaller than 2 for Metric-3-Clustering.

   You may use the fact that the well-known 3-Coloring problem (e.g., see [CLRS]) is NP-complete. In the 3-Coloring problem, we are given an unweighted graph $G = (V, E)$ and want to decide whether there exists a mapping $\varphi : V \to \{1, 2, 3\}$ such that for every $uv \in E$, $\varphi(u) \neq \varphi(v)$.

   [Hint: use distances that are all 1's and 2's.]

3. [*10 marks*] Given a set S of n circles in 2D (possibly of different radii), and we want to find a set P of points, minimizing $|P|$, such that each circle in S contains at least one point of P. Design and analyze a polynomial-time algorithm with approximation factor at most 25 (or better).

   [Hint: take the smallest circle, say, centered at (x, y) of radius r; take the 25 points $\{(x + ir, y + jr) \mid i = -2, \ldots, 2, j = -2, \ldots, 2\}$; remove all circles stabbed by these points; repeat. . .]

   [Bonus: 3 extra marks if you can bring the constant factor down to 7 or better with a rigorous proof.]