CS 466/666, Spring 2007
Timothy Chan

Midterm Examination

Last Name: ____________________ First Name: ____________________

Signature: ____

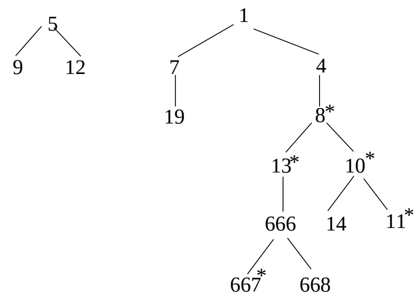
ID number: ____

- Date: June 12, 2007.
- Start Time: 7:00pm. End Time: 9:00pm.
- Number of pages (including cover and two blank pages): 12.
- No additional materials are allowed.
- Print your initials at the top of each page (in case a page gets detached).
- All answers should be placed in the spaces given. Backs of pages will not be marked and may be used as scratch papers. If you need more space to complete an answer, continue on the blank pages at the end.
- Cheating is an academic offense. Your signature on this exam indicates that you understand and agree to the University's policies regarding cheating on exams.

Q	Marks	Init.
1	/25	
2	/19	
3	/12	
4	/25	
5	/19	
Total	/100	

1. [*25 marks*] *Routine questions.*

   (a) [*7 marks*] Consider the following Fibonacci heap (* denotes marked nodes):

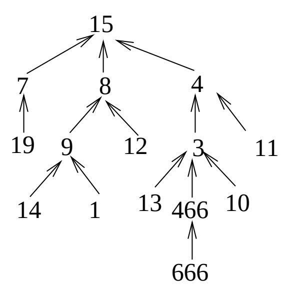


   Show the result of a delete-min operation.

   (b) [*7 marks*] Consider the same Fibonacci heap as drawn part (a) (but before the delete-min operation). Show the result of decreasing the key 666 to 466.

(c) [*4 marks*] Explain why the decrease-key algorithm does not work for the operation of increasing the key of an element.

(d) [*7 marks*] Consider the following union-find data structure by the tree method from class, implemented with weighted union heuristic and path compression:



Show the result of find(466).

2. [*19 marks*] *Short questions.*

   (a) [*3 marks*] True or False: The algorithm based on the high-low trick (by Alon, Yuster, and Zwick), which runs in $O(m^{1.41})$ time, is the fastest among the algorithms presented in class for the triangle finding problem for graphs of all sizes. Explain.

   (b) [*3 marks*] True or False: There is an $\Omega(n \log n)$ lower bound for the problem of computing the number of upper hull vertices for a given set of n points in 2D. Explain (you may quote results from class).

   (c) [*4 marks*] True or False: There is an $\Omega(n \log n)$ lower bound for the problem of the problem of determining whether the number of upper hull vertices is at most 666 for a given set of n points in 2D. Explain (you may quote results from class).

(d) [*3 marks*] Suppose algorithm A requires $O(n^2)$ decrease-key operations and $O(n)$ delete-min operations; all remaining steps takes $O(n)$ time. Suppose the decrease-key and delete-min operations are implemented using Fibonacci heaps. True or False: Algorithm A takes $O(n^2)$ amortized time. Explain.

(e) [*3 marks*] Consider the same scenario as part (d). True or False: Algorithm A takes $O(n^2)$ worst-case time. Explain.

(f) [*3 marks*] Suppose B is a Monte Carlo algorithm for the string matching problem with error probability exactly equal to 1/4. True or False: We can find a quarter of all input strings of length n for which algorithm B always fails. Explain.

3. [*12 marks*] *Simple amortized analysis.* Consider the following pseudocode, where all entries of the array $A[]$ are initially 0. [Note: It is not important to understand what the pseudocode actually does.]

```
insert(x):
    n = n + 1,  t = x
    for i = 0, 1, 2, ..., ⌈log n⌉ do
        if A[i] ≠ 0 then {
            t = t + A[i]
            A[i] = 0
        }
        else {
            A[i] = t
            stop and return
    }
```

Show that insert takes $O(1)$ amortized time by using a potential method.

4. [*25 marks*] *A randomized algorithm and a lower bound.* Given a set P of n points in 2D, a point $p \in P$ is said to be *central* if the x-coordinate of p has rank between $0.2n$ and $0.8n$ among the x-coordinates of P *and* the y-coordinate of p has rank between $0.2n$ and $0.8n$ among the y-coordinates of P. (Recall that the rank of an element z among a set refers to the number of elements smaller than z in the set. You may assume that coordinates are all distinct.)

   (a) [*5 marks*] Choose a random element r from P. Show that the probability that r is not central is strictly less than 1.

   (b) [*12 marks*] Design and analyze a Las Vegas linear-time algorithm for finding a central point in P. (Naturally, you may use part (a).)

(c) [*8 marks*] Prove the following lower bound result: any deterministic algorithm for finding a central point must require $\Omega(n)$ time.

5. [*19 marks*] *More amortized analysis.* Consider the tree method for the union-find problem, with weighted union heuristic and path compression from class. In class, we presented a (complicated) proof of the $\alpha(n)$ bound. In this question, you will explore a simpler proof of a weaker, but still strictly less than logarithmic, bound.

   Recall that $r[x]$ and $p[x]$ denote the rank and parent of x respectively. You may use the following properties without proof:

   (i) $r[p[x]] > r[x]$;

   (ii) $r[p[x]]$ can only increase but $r[x]$ stays the same if x has a parent;

   (iii) $r[x] \leq \log n$.

   Property (iii) was not officially stated in class but follows from the fact that tree heights are bounded by $\log n$ because of the weighted union heuristic.

   Fix a parameter d, to be determined later.

   (a) [*5 marks*] Say that a parent change of x from y to u is *far* iff $r[u] \geq r[y] + d$, and *near* otherwise. [Note: This is obviously different from the definition from class.]

   Fix one node x. Give an upper bound $f(n, d)$ on the number of far parent changes that x can undergo. Justify your answer. Your function $f(n, d)$ should be at most $\log n$ and should be decreasing in d. [Hint: use Property (iii).]

(b) [*5 marks*] Prove that each find operation can undergo at most $O(d)$ near parent changes.

(c) [*6 marks*] Use parts (a) and (b) to conclude that each makeset/union/find operation takes $O(f(n,d) + d)$ amortized time.

(d) [*3 marks*] With the particular function $f(n,d)$ you give for part (a) (regardless of whether your answer is correct for (a) or not), describe how to set the parameter d to make $O(f(n,d) + d)$ asymptotically as small as you can.

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