CS 466/666, Spring 2009
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Assignment 4 (due July 15 Wednesday 5pm)

Please read `http://www.student.cs.uwaterloo.ca/~cs466/policies.html` first for general instructions.

1. [*10 marks*] We are given a collection of m subsets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$, each of even size. We say that a subset $S \subseteq \{1, \ldots, n\}$ is a *splitter* if S intersects both A_i and its complement for all $i = 1, \ldots, m$, i.e., $A_i \cap S \neq \emptyset$ and $A_i \setminus S \neq \emptyset$. The problem of finding a splitter is NP-hard in general, but we will consider a special case:

   We say that a subset $P \subseteq \{1, \ldots, n\}$ is a *perfect splitter* if $|A_i \cap P| = |A_i \setminus P|$ for all $i = 1, \ldots, m$. For any input instance for which we know the existence of a perfect splitter (but do not know the perfect splitter itself), design and analyze a Monte Carlo algorithm that finds a splitter in polynomial time. (The splitter found does not have to be perfect.)

   [Hint: imitate Papadimitriou's SAT algorithm. . .]

2. [*20 marks*] In this question, we analyze an approximation algorithm for the minimum vertex cover problem under the special case where the input graph has maximum degree 3. (This special case is still NP-hard.) The approximation factor we get will be better than 2 in this special case.

   (a) [*4 marks*] First, show that for graphs with maximum degree at most 2, the problem can be solved exactly in polynomial time.

   (b) [*3 marks*] Next, consider the following greedy algorithm to compute a vertex cover for a given graph G of maximum degree 3:

   ```
   A = ∅
   while there exist a vertex v of degree 3 do
       insert v to A, and remove v from the graph
   let B be the set of remaining vertices
   compute an exact optimal solution S_B for the subgraph G_B formed by B
   return S = A ∪ S_B
   ```

   Show that this algorithm returns a vertex cover S and runs in polynomial time.

   (c) [*4 marks*] Let S^* be the optimal solution. Let A^* be the subset of vertices of S^* that lie in A. For each $i \in \{0, 1, 2, 3\}$, let B_i^* be the subset of vertices of S^* that lie in B and have degree i in the subgraph G_B.
   Prove that $|S| \leq |A| + |B_1^*| + |B_2^*| + |B_3^*|$.

   (d) [*6 marks*] Prove that $3|A - A^*| \leq 3|B_0^*| + 2|B_1^*| + |B_2^*|$.
   [Hint: first argue that each vertex in A that is not in S^* is adjacent to exactly 3 vertices in B that are all in S^*.]

(e) [*2 marks*] Prove that $|S| \leq |A^*| + |B_0^*| + (5/3)|B_1^*| + (4/3)|B_2^*| + |B_3^*|$.

(f) [*1 marks*] Conclude that the above algorithm has approximation factor at most 5/3.

3. [*10 marks*] Given a set S of n (axis-aligned) unit squares in 2D, we want to find a subset $T \subseteq S$ such that every square in S intersects at least one square in T, while minimizing the number of squares chosen in T. Design and analyze a polynomial-time algorithm with approximation factor at most 4.

   [Hint: follow the same greedy algorithm from class but use a different analysis. How many unit squares in your solution can intersect a given unit square in the optimal solution?]