CS 466/666, Spring 2011
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Assignment 4 (due July 20 Wednesday 5pm)

Please read `http://www.student.cs.uwaterloo.ca/~cs466/policies.html` first for general instructions.

1. [*10 marks*] Given a set P of n points in $\mathbb{R}^2$, we want to find a point $q \in \mathbb{R}^2$ that minimizes $\sum_{p \in P} d(p,q)$, where $d(p,q)$ denotes the Euclidean distance between p and q. (In contrast, the smallest enclosing circle problem from class is about minimizing $\max_{p \in P} d(p,q)$.)

   The exact problem turns out to be difficult (in fact, the exact answer may be irrational and not expressible by radicals). Thus, we turn to approximation algorithms.

   (a) [*4 marks*] Show that if we modify the problem and impose the constraint that q must be in P, then there is a polynomial-time algorithm. What is the running time?

   (b) [*6 marks*] Show that your algorithm in part (a) is an approximation algorithm for the original (unconstrained) problem, with factor at most 2.

   [Hint: let q^* be the optimal solution, let p_0 be the point in P closest to q^*, use triangle inequality...]

2. [*16 marks*] Given a set P of n points in 2D, let $G(P)$ denote the graph in which the vertices are the points in P, and there is an edge between points p and q iff $|p.x - q.x| \leq 1$ and $|p.y - q.y| \leq 1$. In this question, you will give a PTAS (polynomial-time approximation scheme) for finding the minimum vertex cover of this graph $G(P)$.

   Fix a constant $k > 2$. Fix $i, j \in \{0, \ldots, k-1\}$. Define R_{ij} to be the collection of all $k \times k$ squares of the form $s = \{(x,y) : ak + i \leq x < (a+1)k + i,\ bk + j \leq y < (b+1)k + j\}$ for some integers a, b. Define the *expansion* of s to be the $(k+2) \times (k+2)$ square $\widehat{s} = \{(x,y) : ak + i - 1 \leq x < (a+1)k + i + 1,\ bk + j - 1 \leq y < (b+1)k + j + 1\}$.

   (a) [*6 marks*] Given a $k \times k$ square s, let Q_s be the minimum vertex cover of the graph $G(P \cap \widehat{s})$. Show that Q_s can be computed in polynomial time. [Hint: the complement of a minimum vertex cover is a maximum independent set.]

   (b) [*8 marks*] Let Q_{ij} be the union of Q_s over all $s \in R_{ij}$. Let Q be the smallest Q_{ij} over all $i, j \in \{0, \ldots, k-1\}$. Prove that $k^2|Q| \leq (k+2)^2|Q^*|$, where Q^* denotes the minimum vertex cover of $G(P)$. [Hint: how many expanded squares $\widehat{s}$ can contain one given point?]

   (c) [*2 marks*] Conclude that there is a PTAS for the problem.

3. [*24 marks*] Consider the following problem: The input consists of n rectangles, where the i-th rectangle R_i has height $h_i \le 1$, width $w_i \le 1$, and area $a_i = h_i w_i$. The goal is to place the rectangles in a rectangular container of width 1 while minimizing the height of the container. Rotations of the rectangles are not allowed.

Consider the following greedy algorithm:

```
0.  sort the rectangles so that h_1 ≥ h_2 ≥ ··· ≥ h_n
1.  k_1 = 1
2.  for m = 1, 2, ... until k_m = n + 1 {
3.     let k_{m+1} be the largest index such that w_{k_m} + w_{k_m+1} + ··· + w_{k_{m+1}-1} ≤ 1
4.     place R_{k_m}, R_{k_m+1}, ..., R_{k_{m+1}-1} in a new row (a "shelf") of height h_{k_m}
    }
```

(a) [*12 marks*] Show that this greedy algorithm runs in polynomial time and has approximation factor at most 3 and asymptotic approximation factor at most 2.

[Hint: Prove the inequality $a_{k_m} + a_{k_m+1} + \cdots + a_{k_{m+1}} > h_{k_{m+1}}$. Then sum both sides.]

(b) [*12 marks*] Design and analyze an *online* algorithm that has asymptotic approximation factor upper-bounded by a constant. (The placement of R_i should depend only on $R_1, \ldots, R_i$, and the rectangles are not necessarily pre-sorted.)

[Hint: round the heights of the rectangles into numbers of the form $1/2^\ell$, and place rectangles of the same height in shelves by using the "first-fit" (online) bin packing algorithm...]