CS 466/666 Spring 2013
Assignment 7
Due Noon, July 9, 2013

You are on your honour to present your own work and acknowledge your sources.

[13 marks] In class we looked at the set cover problem where we want a set cover of minimum number of subsets. A variant of the problem, which also has many applications, involves weights on the elements. The input consists of: a set U of n elements, where each u in U has a non-negative weight w(u); a collection $S_1, \ldots S_t$ where $S_i \subset U$; and a number k. The problem is to pick k of the subsets S_i to maximize the sum of the weights of the elements covered.
The goal of this question is to show that the obvious greedy algorithm has approximation factor 1-1/e (where e is the base of the natural logarithm). The greedy algorithm first picks a set that covers the maximum weight of elements, then deletes those elements and repeats until k sets have been chosen. Let w* be the weight of the elements covered by the optimum solution. Let w_i, for i = 1, …, k, be the weight of the elements covered by the first i sets of the greedy algorithm.
Thus w_k is the final weight of the greedy solution.
(a) [2 marks] Find an example with k = 2 and unit weights where $w_2 = \frac{3}{4} w^*$.
(b) [5 marks] Prove that $w_1 \geq w^*/k$. More generally, prove that $w_i - w_{i-1} \geq (w^* - w_{i-1})/k$.
(c) [6 marks] Prove by induction that $w_i \geq (1 - (1-1/k)^i)w^*$.
(d) [0 marks] From part (c), the weight of the greedy solution, w_k, is at least $(1 - (1-1/k)^i)w^*$.
Since $\lim_{k\to\infty} (1-(1-1/k)^k) = 1 - 1/e$ and $1- (1-1/k)^k$ is decreasing , thus $1- (1-1/k)^k \geq 1 - 1/e$, so the approximation ratio of the greedy algorithm is $1 - 1/e$.
In fact $1 - 1/e$ is the best approximation factor possible for this problem (unless P=NP).