Lecture 01 - Introduction

May 7, 2025

Outline

1. Course Mechanics
2. Basic Theory of Linear Algebra

Course Mechanics

- See:
  - the unsecured course website: `https://student.cs.uwaterloo.ca/~cs475`, and
  - the course outline which will be linked there.

Basic Theory of Linear Algebra

Definition 1

Let A be a matrix. The **range** *of A defined as*

$$range(A) = \{y \mid Ax = y \text{ for some vector } x\}.$$

Basic Theory of Linear Algebra

- Let A be an $(m \times n)$ real matrix, with $m \geq n$.
- Then left multiplication by A affords a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.
- Recall that the **column vectors** of A are simply the columns of A, taken one at a time (so that they are $(m \times 1)$).

Theorem 2

$$
\begin{aligned}
range(A) &= \textit{the column space of } A \\
&= \textit{the space spanned by the column vectors of} \\
&\quad A = [a_1 \ \cdots \ a_n] \\
&= \{y = x_1 a_1 + \cdots + x_n a_n\} \\
&= \left\{ \sum_{j=1}^{n} x_j a_j \textit{ for scalars } x_j \right\}.
\end{aligned}
$$

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Q: What is the difference between *range*(A) and the column space of A?

A: There is no difference. The $\vec{x}$ in the definition of *range*(A) is precisely

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

from the definition of the column space.

Remark: We are working over the field $\mathbb{R}$. Otherwise our computational approach would not make sense.

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Definition 3

1. *column rank = dimension of the column space*
2. *row rank = dimension of the row space*

Theorem 4

column rank = row rank.

Thus we may simply refer to the rank of A.

Basic Theory of Linear Algebra

Definition 5

An $m \times n$ *matrix* A *is of* **full rank** *if*

$$rank(A) = \min\{m, n\}.$$

Thus, if $m \geq n$, then A is of full rank if it has n **linearly independent** column vectors.

Definition 6

A set $S = \{\vec{v_1}, \ldots, \vec{v_n}\}$ *of vectors is* **linearly independent** *of and only if, for any scalars* $c_1, \ldots, c_n,\ c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$ *implies* $c_1 = \cdots = c_n = 0$*, i.e. the only linear combination of* $\vec{v_1}, \ldots, \vec{v_n}$ *which equals* $\vec{0}$ *is the trivial one.*

Definition 7

A **nonsingular (invertible) matrix** *is a square matrix, of full rank.*

Basic Theory of Linear Algebra

Definition 8

The **null space** *of A is*

$$null(A) = \{x \mid Ax = \vec{0}\}.$$

Matrix Inverses

$$
\begin{aligned}
(AB)^{-1} &= B^{-1}A^{-1} \\
(A^{-1})^T &= (A^T)^{-1} \\
&\stackrel{def}{=} A^{-T}.
\end{aligned}
$$

Basic Theory of Linear Algebra

Theorem 9

$B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}$.

Proof.

$$
\begin{aligned}
& B\left[A^{-1} - B^{-1}(B - A)A^{-1}\right] \\
& BA^{-1} - BB^{-1}(B - A)A^{-1} \\
& \underbrace{BA^{-1} - BA^{-1}}_{=0} + \underbrace{AA^{-1}}_{=I} \\
& I.
\end{aligned}
$$

It is an exercise for you to check multiplication on the other side. □

Remarks:

1. This Theorem is useful to establish the Sherman-Morrison-Woodbury formula, below.

Basic Theory of Linear Algebra

Sherman-Morrison-Woodbury formula

$$(A + \underbrace{UV^T}_{\text{rank } k})^{-1} = A^{-1} - \underbrace{A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}}_{\text{rank } k},$$

where A is an invertible $n \times n$ matrix, and U, V are $n \times k$ matrices (usually $k \leq n$).
Thus a rank k correction to A (LHS) results in a rank k correction to the inverse (RHS):

- Assume UV^T (which is $(n \times n)$), has rank k.
- Further assume that $(I + V^T A^{-1} U)$ (which is $(k \times k)$) is invertible.
- Then $U(I + V^T A^{-1} U)^{-1} V^T$ also has rank k.
- A and A^{-1} have full rank, therefore $A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}$ also has rank k.

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Example: Let A be $n \times n$, say $A = [a_{ij}]$. Let $k = 1$.

$$
\begin{aligned}
u &= \begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
uv^T &= \begin{bmatrix} u_1 v_1 & 0 & \cdots & 0 \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
A + uv^T &= \begin{cases} a_{11} + u_1 v_1 & \text{if } i = 1 \text{ and } j = 1 \\ a_{ij} & \text{otherwise} \end{cases} \\
(A + uv^T)^{-1} &= A^{-1} - \underbrace{A^{-1}u}_{n \times 1}(1 + \underbrace{v^T A^{-1} u}_{1 \times 1})^{-1} \underbrace{v^T A^{-1}}_{1 \times n}
\end{aligned}
$$

Remarks:

1. A larger k would require more work.

Q & A

1. Will we need to know MATLAB for this course?
   A: Yes.
2. How should we learn MATLAB?
   A: I will run a MATLAB tutorial, well before the first Crowdmark assignment is released. I will also post a "Quick Reference" guide.
3. Will the instructor post the URL for the unsecured website on LEARN?
   A: Yes! This is already done.
4. Can you talk about the optional textbooks?
   A: Yes. See the details posted on the course outline.
5. Can you be specific about which textbook to read for each topic?
   A: Yes, absolutely.

Q & A

6. Will the mid-term be 100% written, or will there also be a MATLAB component?
   A: 100% written.
7. What types of questions will be on the marked quizzes on LEARN?
   A: The auto-marked types, e.g. MC, MS, MAT, TF, etc.
8. Will the Marked Quizzes and Crowdmark assignments be open book?
   A: Yes!