Formulas Used

1. Bernoulli pmf (for $x \in \{0,1\}$):
(coin flip)

$$P(X=x \mid p) = p^x (1-p)^{1-x}$$

$$\Rightarrow P(X=1 \mid p) = p^1 (1-p)^0$$
$$= p$$

$$\Rightarrow P(X=0 \mid p) = p^0 (1-p)^1$$
$$= 1-p$$

2. Independence of i.i.d samples:

$$P(X_{1:n} \mid p) = \prod_{i=1}^{n} P(x_i \mid p)$$

e.g. => $P(2 \text{ heads in a row}) = (0.5)(0.5)$
$= 0.25$

3. Product Log Rule:

$$\log\left(\prod_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} \log a_i \, .$$

4. Exponent Log Rule:

$$\log(a^b) = b \log a$$

Exercise 2(a):

e.g. $x_1=0,\ x_2=1,\ x_3=1$

$$x_i \sim \text{Bernoulli}(p)$$

(i) Likelihood function $P(x_{1:n}\mid p)$

$$P(X=x_i\mid p) = p^{x_i}(1-p)^{1-x_i}$$

$$\Rightarrow P(X=0\mid p) = (1-p)$$

$$\Rightarrow P(X=1\mid p) = p$$

$$P(x_{1:n}\mid p) = \prod_{i=1}^{n} P(x_i\mid p)$$

We want $\boxed{P(x_{1:n}\mid p)}$

$$P(x_{1:n}\mid p) = \prod_{i=1}^{n} P(X=x_i\mid p)$$

$$\boxed{P(x_{1:n} \mid p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}}$$

(ii) Log-likelihood function

$$\log P(x_{1:n} \mid p)$$

$$= \log\left(\prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}\right)$$

$\log(ab) = \log a + \log b$

$$= \sum_{i=1}^{n} \log\left(p^{x_i}(1-p)^{1-x_i}\right)$$

$$= \sum_{i=1}^{n}\left(\log p^{x_i} + \log(1-p)^{1-x_i}\right)$$

$$\boxed{\log P(x_{1:n} \mid p) = \sum_{i=1}^{n} x_i \log p + (1-x_i)\log(1-p)}$$

(iii) Express $\log P(x_{1:n} | p)$ in terms of the total # of observed successes (1s) $S = \sum_{i=1}^{n} x_i$

First: why is # successes $\sum_{i=1}^{n} x_i$?

b/c $x_i = 1$ if trial i is a success

$x_i = 0$ if trial i is failure

$\Rightarrow \sum_{i=1}^{n} x_i$ literally counts the number of successes

Let

$$S = \sum_{i=1}^{n} x_i \text{ (total \# of 1s)}$$

$$\Rightarrow \sum_{i=1}^{n} (1 - x_i) = n - \sum_{i=1}^{n} x_i = n - S$$

$$\log P(x_{1:n} | p) = \sum_{i=1}^{n} x_i \log p + (1 - x_i) \log (1-p)$$

$$\boxed{\log P(x_{1:n} \mid p) = S \log p + (n - S) \log(1-p)}$$

where $S = \#$ successes

and $n = \#$ trials