Imagine a polytope.
Skeleton graphs of polytopes

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Skeleton graph is a graph whose nodes are the nodes of the polytope and there is an edge between two nodes if there exists a 1-face that connects them.
Figure: The skeleton of an Icosidodecahedron
Figure: The skeleton of a four dimensional cube
Skeleton graphs of polytopes

Figure: The skeleton of a complicated four dimensional polytope
Skeleton graphs of polytopes

- What do these graphs look like?
- What are their properties?
- Given a graph, can we decide if it is the skeleton graph of some polytope?
Good news: It is decidable using Tarski’s algorithm for real closed fields.
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Bad news: No efficient algorithm is known.
Skeleton graphs of polytopes

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- Bad news: No efficient algorithm is known.
- Slightly good news: The skeleton graph of a d dimensional convex polytope is always d-connected (Balinski’s Theorem).
A graph is a skeleton graph of a two-dimensional polytope (or 2-polytope) if and only if it's a cycle.

A graph is a skeleton graph of a polyhedron if and only if no one knows.

It's not even known whether $K_{12}$ is the skeleton graph of some polyhedron.
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Polytopes in small dimensions

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- A graph is a skeleton graph of a polyhedron if and only if ...

Vinayak Pathak

Steinitz Theorems for Orthogonal Polyhedra
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It’s not even known whether $K_{12}$ is the skeleton graph of some polyhedron.
Convex polyhedra

A graph is the skeleton graph of a convex polyhedron if and only if it's 3-connected and planar.

Figure: Left: a convex polyhedron, Right: its 3-connected, planar skeleton

Branko Grunbaum says that Steinitz's theorem is "the most important and deepest known result on 3-polytopes."

Obvious next question: What about polyhedra that are not convex?

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  ![Convex Polyhedron and Skeleton Graph](image)

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Simple orthogonal polyhedra

▶ “Orthogonal” means each face is perpendicular to one of the coordinate axes.

▶ “Simple” means three things.

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Simple orthogonal polyhedra

- “Orthogonal” means each face is perpendicular to one of the coordinate axes.
- “Simple” means three things.
Simple orthogonal polyhedra

It should have the topology of a sphere.

Figure: Not allowed.
Simple orthogonal polyhedra

Exactly three mutually perpendicular edges should meet at each vertex.

Figure: Not allowed.
Simple orthogonal polyhedra

Each face should be simply connected.

Figure: Not allowed.
Simple orthogonal polyhedra

**Figure:** Some simple orthogonal polyhedra. (Simple means - 1. should have the topology of a sphere, 2. exactly three mutually perpendicular edges should meet at each vertex, 3. faces should be simply connected.)
Observations

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The skeleton must be

- planar
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The skeleton must be

- planar
- bipartite
Observations

**Figure:** Some simple orthogonal polyhedra. (Simple means - 1. should have the topology of a sphere, 2. exactly three mutually perpendicular edges should meet at each vertex, 3. faces should be simply connected.)

The skeleton must be

- planar
- bipartite
- 3-regular
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Figure: Some simple orthogonal polyhedra. (Simple means - 1. should have the topology of a sphere, 2. exactly three mutually perpendicular edges should meet at each vertex, 3. faces should be simply connected.)

The skeleton must be

- planar
- bipartite
- 3-regular
- 2-connected
Does the converse hold?

**Figure:** A planar, bipartite, 3-regular and 2-connected graph that is not the skeleton of a simple orthogonal polyhedron.
Theorem

A graph is the skeleton graph of an xyz polyhedron if and only if it is

- planar
- bipartite
- 3-regular
- 3-connected
An xyz polyhedron is a simple orthogonal polyhedron for which any axis-parallel line contains at most 2 vertices.

Figure: An xyz polyhedron.
Unfortunately, there are simple orthogonal polyhedra that are not xyz.

Figure: Not xyz.
A graph is the skeleton graph of a simple orthogonal polyhedron if and only if

- it is bipartite,
- it is planar,
- it is 3-regular,
- removal of any 2 of its vertices disconnects it into at most 2 components.

Note: This gives a polynomial time characterization.
Main result

A graph is the skeleton graph of a simple orthogonal polyhedron if and only if

- it is bipartite,
Main result

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Proof sketch for xyz polyhedra

**Theorem**

*A graph is the skeleton graph of an xyz polyhedron if and only if it is*

- planar
- bipartite
- 3-regular
- 3-connected

*Forward direction (polyhedron to graph) was proved in a previous paper.*

*For the other direction (graph to polyhedron), look at the dual graph.*

*Because of 3-regularity, the dual will be a triangulation (i.e. each face will be bounded by exactly three edges).*
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Forward direction (polyhedron to graph) was proved in a previous paper.
For the other direction (graph to polyhedron), look at the dual graph.
Because of 3-regularity, the dual will be a triangulation (i.e. each face will be bounded by exactly three edges).
Theorem

A graph is the skeleton graph of a corner polyhedron if
Corner polyhedra

Theorem
A graph is the skeleton graph of a corner polyhedron if
- it is planar
Theorem

A graph is the skeleton graph of a corner polyhedron if

▶ it is planar
▶ it is bipartite
Corner polyhedra

Theorem

A graph is the skeleton graph of a corner polyhedron if

- it is planar
- it is bipartite
- it is 3-regular
Theorem

A graph is the skeleton graph of a corner polyhedron if

- it is planar
- it is bipartite
- it is 3-regular
- its dual is 4-connected
A corner polyhedron is a simple orthogonal polyhedron for which all but three faces are oriented towards the vector \((1, 1, 1)\).

**Figure:** A corner polyhedron.

Note that a corner polyhedron is always xyz.
Proof sketch for xyz polyhedra

**Theorem**

*If a graph is planar, bipartite, 3-regular and 3-connected, then it can be represented as the skeleton of an xyz polyhedron.*

- **Check:** Is its dual 4-connected? If yes, then we are done.
Theorem

*If a graph is planar, bipartite, 3-regular and 3-connected, then it can be represented as the skeleton of an \(xyz\) polyhedron.*

- Check: Is its dual 4-connected? If yes, then we are done.
- If no, then split the dual along a separating triangle.

**Figure:** Splitting along separating triangles.
Proof sketch for xyz polyhedra

Theorem

*If a graph is planar, bipartite, 3-regular and 3-connected, then it can be represented as the skeleton of an xyz polyhedron.*

- Check: Is its dual 4-connected? If yes, then we are done.
- If no, then split the dual along a separating triangle.

![Diagram](image.png)

**Figure:** Splitting along separating triangles.

- Pick the one for which one of the components has no more separating triangles.
Proof sketch for xyz polyhedra

- The component with no separating triangle is 4-connected.
- So it gives a corner polyhedron.
Proof sketch for xyz polyhedra

- The component with no separating triangle is 4-connected.
- So it gives a corner polyhedron.
- The other component is an xyz polyhedron by induction.
Proof sketch for xyz polyhedra

- The component with no separating triangle is 4-connected.
- So it gives a corner polyhedron.
- The other component is an xyz polyhedron by induction.
- So glue them together.

Figure: Gluing two polyhedra together.
Prove that a planar, bipartite, 3-regular graph whose dual is 4-connected can be represented as a corner polyhedron, which is always an xyz polyhedron.
Summary of the proof

- Prove that a planar, bipartite, 3-regular graph whose dual is 4-connected can be represented as a corner polyhedron, which is always an xyz polyhedron.
- Use that and induction to give a characterization of xyz polyhedra.
Prove that a planar, bipartite, 3-regular graph whose dual is 4-connected can be represented as a corner polyhedron, which is always an xyz polyhedron.

Use that and induction to give a characterization of xyz polyhedra.

Use the characterization of xyz polyhedra to give a characterization of simple polyhedra. (Skipped. Uses techniques such as SPQR trees.)