CS 860—Randomness and Complexity
Jonathan Buss
Fall 2008

Problem Set 1

The problems are rated on Knuth's scale:

10: "Simple"—solution time about one minute
20: "Medium"—solution time about fifteen minutes
30: "Moderately hard"—solution time more than two hours
40: A term project

Intermediate ratings are roughly logarithmic. All ratings are subject to the instructor's foibles and the students' individuality.

1. The Lubotsky-Philips-Sarnak expanders $LPS(p)$, for any for any prime $p > 2$, have p nodes, interpreted as the elements of the field $GF(p)$. The nodes adjacent to node x are $x+1$, $x-1$ and x^{-1} (where "0^{-1}"means 0).
   (a) [Rating: 12]
   What is the smallest p for which $LPS(p)$ is non-planar?
   (b) [Rating: 22]
   In any $LPS(p)$, the nodes 0, 1 and $p-1$ have self-loops. If we remove these nodes and connect 2 to $p-2$ (thus retaining degree 3), the result should be a slightly better expander.
   Show that, for all $p \geq 7$, the resulting graph contains a cycle of odd length and thus is non-bipartite. (You don't need any non-trivial number theory.)

2. [Proposed by Daniel Ivan. Instructor's rating: 25]
   For this problem, an (n, k, ℓ)-expander is a bipartite graph $G = (L \cup R, E)$, with $|R| = |L| = n$ and left degree d, such that for all $S \subseteq L$ with $|S| \leq k$, we have
   $$|N(S)| = \left|\{\, u \mid \exists v \in S\ (u, v) \in E) \,\}\right| \geq \ell \cdot |S| \tag{1}$$
   Show that for any fixed degree d, there exists a constant α_d $(0 < \alpha_d < 1)$ such that, for all n, $(n, \alpha_d n, d-2)$-expanders exist. Use a probabilistic construction.

3. [Rating: 30]
   Suppose that some one-tape Turing machine runs in time t. Show that another Turing machine can accept the same language in space $O(\sqrt{t})$.
   - The restriction to one tape for the original machine is important—proving the result for multi-tape machines would be a major achievement.
     You may use any number of tapes for the new machine. (Not necessary, but it makes the programming easier.)
   - For convenience, you may assume that $t = \Omega(n^2)$ (not strictly necessary).
   - Use the notion from Sipser's paper of splitting a tape into "blocks" of an appropriate size.
   - If you try to find this in the literature, you might succeed. Thus, don't try.