CS 860 — Assignment 2

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Due October 30

1. Fix a matrix A, with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $x_1, \ldots, x_n$.

   (a) [Rating: 8]
   Suppose that B shares an eigenvector with A, say x_i, with corresponding eigenvalue μ. Show that x_i is also an eigenvector of $A + B$ and of AB. What is its corresponding eigenvalue in each case?

   (b) [Rating: 8]
   What are the eigenvalues and eigenvectors of the matrix $A + I$? (Note: if A is the adjacency matrix of a graph G, then $A + I$ is the adjacency matrix of the graph formed from G by adding a self-loop to each node.)

   (c) [Rating: 12]
   Under what conditions does adding a self-loop to each edge of a d-regular graph G improve the spectral gap?

2. [Rating: 12]
   Let A and B be $n \times n$ matrices, where A has eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $x_1, \ldots, x_n$, and B has eigenvalues $\mu_1, \ldots, \mu_n$ and corresponding eigenvectors $y_1, \ldots, y_n$.

   Prove that the matrix $A \otimes B$ has eigenvalues $\{\, \lambda_i \mu_j \mid 1 \leq i, j \leq n \,\}$. What are the corresponding eigenvectors?

3. Let $G = (V, E)$ be a graph. For $S \subseteq V$, its *frontier* is the set

   $$\delta(S) = \{\, u \notin S \mid \exists v \in S \ (u, v) \in E \,\} \ .$$

   For this problem, say that G *has* (α, β) *expansion* if $|\delta(S)| \geq \alpha |S|$ for all $S \subseteq V$ with $|S| \leq \beta n$.

   (a) [Rating: 10]
   Show that for all constants $\beta \geq 1/2$ and $\alpha > 0$, having (α, β) expansion guarantees that the graph is connected. Further show that it has diameter $O(\log n)$, where n is the number of nodes. Specify the constant factor.

   (The diameter of a graph is the maximum distance between two nodes.)

   (b) [Rating: 20]
   For every fixed $\alpha > 0$ and $0 < \beta < 1/2$, show that a connected graph that has (α, β) expansion has diameter $O(\log n)$. How does the constant factor depend on α and β?

4. Suppose that you need to construct an expander, on several million nodes, that meets the following criteria.

   - The graph must be regular and have spectral gap at least 0.25. (That is, $(d-\lambda) \geq d/4$, where d is the degree.)
   - Subject to achieving the required spectral gap, the lower the degree, the better. The number of edges must not exceed $2 \cdot 10^8$ (assuming each node name takes one 32-bit word, this uses 800MB of memory to store).
   - You may choose the exact size—between 2^{20} and 2^{25} nodes, perhaps—to best achieve the above.

   For this problem, use only techniques and bounds proved or discussed in Reingold-Vadhan-Wigderson.

   (a) [Rating: 12]
   The general construction starts with a (D^8, D, λ) graph, with $\lambda \leq 1/5$. Explain why one cannot start this way for the required graph.

   (b) [Rating: 12]
   What is the minimal degree for a large graph with spectral gap at least 0.25? (The Reingold-Vadhan-Wigderson paper has the answer to this, although not explicitly.)

   (c) [Rating: 30 (for a sufficient answer)]
   How would you create the required graph?

   (d) [Rating: 50 (a research problem)]
   How should one create such a graph?