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Unit 1 Cardinality

Week 1 Cardinality I

1.1 Bijections

Definition. Injection: Let A, B be sets. A function $f : A \to B$ is *injective* (or an injection) if for all $a, b \in A$, we have $f(a) = f(b)$ if and only if $a = b$. Also called a one-to-one function.

Definition. Surjection: Let A, B be sets. A function $f : A \to B$ is *surjective* (or a surjection) if for all $b \in B$, there is a $a \in A$ such that $f(a) = b$. That is, $f(A) = B$. Also called an onto function.

Definition. Bijection: Let A, B be sets. A function $f : A \to B$ is *bijective* (or a bijection) if it is both an injective and surjective. Also called an isomorphism.

Proposition. Bijections and Invertibility: Let A, B be sets. A function $f : A \to B$ is a bijection if and only if there is a function $g : B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$ for all $a \in A$ and $b \in B$. We write $g = f^{-1}$ and denote g the inverse function of f.

Proof. ($\implies$) Note that f is surjective. Thus for all $b \in B$, there is an $a \in A$ such that $f(a) = b$. By way of contradiction suppose there are two distinct points $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$. Since f is injective, we must have $a_1 = a_2$. Therefore, for every point $b \in B$ there is a unique point $a \in A$ such that $f(a) = b$. Let $g : B \to A$ denote the mapping such that $g(b) = a$ where $a \in A$ is the unique point such that $f(a) = b$. Therefore, $f(g(b)) = b$ and $g(f(a)) = a$, as desired.

($\impliedby$) Suppose that $f : A \to B$ has an inverse function (as defined above) $g : B \to A$. Let $b \in B$ be arbitrary. Note $g(b) \in A$ is by definition such that $f(g(b)) = b$. Since such a $g(b) \in A$ must exist, f is surjective. Now let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, by definition

$$a_1 = g(f(a_1)) = g(f(a_2)) = a_2$$

and so f is injective, and thus a bijection. □

Proposition: Let A, B be non-empty sets. There exists an injection $f : A \to B$ if and only if there exists a surjection $g : B \to A$.

Proof. ($\implies$) Suppose there is an injection $f : A \to B$. Let $a^* \in A$ be given. Construct $g : B \to A$ as follows:

1. if $b \in f(A)$ then let $g(b)$ be the unique element in A such that $f(g(b)) = b$. Note uniqueness is guaranteed by the injectivity of f.

2. if $b \notin f(A)$ then let $g(b)$ be a^*.

Let $a \in A$ and consider $f(a) \in B$. By definition of g we have $g(f(a)) = a$, thus g is surjective.

($\Longleftarrow$) Suppose there is a surjection $g : B \to A$. For every $a \in A$ pick $b_a \in B$ such that $g(b_a) = a$. Note this must exist by the surjectivity of g. Define $f : A \to B$ by $f(a) = b_a$. Let $x, y \in A$ such that $f(x) = f(y)$, this is equivalent to saying $b_x = b_y$. Therefore, since g is well-defined $x = g(b_x) = g(b_y) = y$. Therefore f is injective. □

Remark: Note that in the above proof, we define $b_a \in B$ for all $a \in A$. By the surjectivity of g we know such a b_a exists, but how can we algorithmically find such a b_a or define such a b_a for every possible $a \in A$? Especially given that there may be several possible choice for b_a. This is because we used the Axiom of Choice.

Definition. Power Set: The power set of a set X is given by $\mathcal{P}(X) = \{A : A \subseteq X\}$. That is, it is the set of all subsets of X (including $\emptyset$).

Definition. Axiom of Choice: Let X be a non-empty set. Then there exists a (choice) function $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that $f(A) \in A$ for every $\emptyset \neq A \subseteq X$.

Remark: The Axiom of Choice is given its name because in a sense, it chooses an element from A. That is, the choice function f takes a non-empty subset of $A \subseteq X$ and returns a single element in A, thereby choosing one element from every possible subset.

Remark: In the context of the above proof, we use the Axiom of Choice to select b_a. Given a set of possible values in B which map to a (found by the surjectivity of g), a single point b_a is selected.

Note. Assumed Axioms: In this course and most courses, we assume the ZFC Axioms. In particular, this is set Zermelo-Fraenkel Axioms along with the Axiom of Choice.

1.2 Cardinality

Remark. Motivation of Cardinality: The goal of cardinality is to be able to classify sets based off of their "size" or the "number" of elements in the set.

For instance if $f : A \to B$ is injective, then $f : A \to f(A)$ is bijective (this is since functions are obviously surjective to their image). In this sense there is a "bijective copy" of A (namely $f(A)$) living in B. That is, our bijection is just relabelling elements in A to elements in B and so in a sense A exists inside of B. This means A is "smaller" than B.

Definition. Cardinality: Let A, B be sets.

1. We say A has *cardinality less than or equal* to B, written $|A| \leq |B|$, if there exists an injection $f : A \to B$.
2. We say A and B have *equal cardinality*, written $|A| = |B|$, if there is a bijection $f : A \to B$.

Remark: As mentioned previously, bijections are often viewed as simply relabelling elements from one set to another. This is because they are injective (no element map to the same as another element) and surjective (each element in the codomain is mapped to by some element in the domain) and so every single element uniquely maps to some other element. See also isomorphisms from MATH 146. This is why we say that the cardinality or "size" of sets is equal when we have a bijection between them. We can simply relabel the elements from domain to codomain.

Example. $|\mathbb{N}| = |\mathbb{Z}|$**:** Show $|\mathbb{N}| = |\mathbb{Z}|$ where $\mathbb{N} = \mathbb{Z}_{\geq 1}$.

Proof. Consider the bijection $f : \mathbb{Z} \to \mathbb{N}$ given by

$$f(n) = \begin{cases} 2n+2 & n \geq 0 \\ 2(-n)-1 & n < 0 \end{cases}$$

Note this means the non-negative integers cover the set of even natural numbers and the negative integers cover the set of odd natural numbers. In particular $f(-1) = 1$, $f(0) = 2$, $f(-2) = 3$, $f(1) = 4$ and so on so forth. □

Example. $|\mathbb{R}| = |(0,1)|$**:** Show $|\mathbb{R}| = |(0,1)|$.

Proof. Consider for instance the function $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ given by $f(x) = \arctan(x)$. Clearly f is bijective since arctan has inverse tan. Now consider the bijection $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0,1)$ given by $g(x) = \dfrac{x + \frac{\pi}{2}}{\pi}$. We know that composition of bijections are bijective, and thus $g \circ f : \mathbb{R} \to (0,1)$ is a bijection, showing $|\mathbb{R}| = |(0,1)|$, as desired.

Note that while the length of $(0,1)$ is 1, the length of $\mathbb{R}$ is infinite. Thus cardinality and length are separate. □

1.3 CSB Theorem

Lemma: Let X be a set. Let $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function such that $\varphi(A) \subseteq \varphi(B)$ whenever $A \subseteq B \subseteq X$. Then there exists a set $F \in \mathcal{P}(X)$ such that $\varphi(F) = F$.

Proof. Let

$$F = \bigcup_{\substack{A \subseteq X \\ A \subseteq \varphi(A)}} A \tag{*}$$

Note that $\emptyset$ is a subset of every set and so necessarily $\emptyset \subseteq \varphi(\emptyset)$, so this union must be well-defined. We will show $\varphi(F) = F$. Let $A \subseteq X$ be an arbitrary set such that $A \subseteq \varphi(A)$. We must have $A \subseteq F$ and so $\varphi(A) \subseteq \varphi(F)$. Therefore, we have $A \subseteq \varphi(A) \subseteq \varphi(F)$ and so since A was arbitrary and F is the union of all sets such that $A \subseteq \varphi(A)$, we must have $F \subseteq \varphi(F)$.

Now since $F \subseteq \varphi(F)$, we must have $\varphi(F) \subseteq \varphi(\varphi(F))$. This implies then that $\varphi(F)$ is one of the sets in the union (*). That is $\varphi(F) \subseteq F$. We have then that $F = \varphi(F)$ by construction of F. □

Theorem. Cantor-Schroeder-Berntein Theorem: Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Proof. Let $f : A \to B$ and $g : B \to A$ be injections. Note that if $X \subseteq Y \subseteq A$, then by the well-definedness of f and g and some set-theory, we have

$$\begin{aligned} X &\subseteq Y \\ f(X) &\subseteq f(Y) \\ B \setminus f(Y) &\subseteq B \setminus f(X) \\ g(B \setminus f(Y)) &\subseteq g(B \setminus f(X)) \\ A \setminus g(B \setminus f(X)) &\subseteq A \setminus g(B \setminus f(Y)) \end{aligned}$$

Now let $\varphi : \mathcal{P}(A) \to \mathcal{P}(A)$ be given by $\varphi(X) = A \setminus g(B \setminus f(X))$. We see that $\varphi(X) \subseteq \varphi(Y)$ whenever $X \subseteq Y$. By the above lemma, we know there is a subset $F \subseteq A$ such that

$$F = \varphi(F) = A \setminus g(B \setminus f(F))$$

Note this means that $A \setminus F = g(B \setminus f(F))$. Now restrict g such that $g : B \setminus f(F) \to A \setminus F$. We have, as mentioned, that $A \setminus F = g(B \setminus f(F))$ and so $g : B \setminus f(F) \to A \setminus F$ is in fact a bijection.

Notice now that $g^{-1} : A \setminus F \to B \setminus f(F)$ is necessarily a bijection since g is a bijection. Further the restriction $f : F \to f(F)$ is also clearly a bijection since f is injective and the co-domain has been restricted to the range of f. Defining $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & x \in F \\ g^{-1}(x) & x \in A \setminus F \end{cases}$$

is therefore also a bijection. We have then that $|A| = |B|$ as desired. □

Remark: The idea of the CSB theorem is that it is often easier to find two (potentially unrelated) injections $f : A \to B$ and $g : B \to A$ than it is to find an explicit bijection $h : A \to B$.

Example: Prove $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof. Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be given by $f(n) = (n, 1)$, it is trivial to see that f is an injection, so $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$. Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $g(n, m) = 2^n \cdot 3^m$. We know by the uniqueness of prime factorization that every product $2^n \cdot 3^m$ can be uniquely expressed this way and so g must be injective. Therefore we have $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$ and so $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$ by CSB as desired. □

Week 2 Cardinality II

2.1 Countable Sets

Definition. Finite Set: A set A is said to be finite if $|A| = |\{1, 2, 3, \ldots, n\}|$ for some $n \in \mathbb{N}$. In this case, we write $|A| = n$. Otherwise we say A is infinite.

Definition. Countably Infinite Set: A set A is said to be countably infinite if $|A| = |\mathbb{N}|$. In this case we write $|A| = \aleph_0$.

Definition. Countable Set: A set A is said to be countable if A is finite or A is countably infinite. Otherwise, we say A is uncountable.

Example: $\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}$ are all countable sets.

Proposition: If A is infinite then $|\mathbb{N}| \leq |A|$.

Proof. By the Axiom of Choice we may find a choice function $f : \mathcal{P}(A) \setminus \{\emptyset\} \to A$. I.e., for all $\emptyset \neq X \subseteq A$, we have $f(X) \in X$.

Let $a_1 = f(A) \in A$. Let $a_2 = f(A \setminus \{a_1\}) \in A \setminus \{a_1\}$. Let $a_3 = f(A \setminus \{a_1, a_2\}) \in A \setminus \{a_1, a_2\}$, and so on so forth. This process may go on infinitely since A is infinite.

Notice $\{a_1, a_2, a_3, \ldots\} \subseteq A$ is countably infinite. Therefore, there is an injection $g : \mathbb{N} \to A$ given by $g(n) = a_n$, so $|\mathbb{N}| \leq |A|$ □

Proposition: For any two countably infinite sets A, B, $|A \times B| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Proof. Notice A, B are countably infinite, therefore there exists bijections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Then $h : A \times B \to \mathbb{N} \times \mathbb{N}$ given by $h(a, b) = (f(a), g(b))$ is also a bijection (easy to show that h is invertible and therefore bijective). We have then $|A \times B| = |\mathbb{N} \times \mathbb{N}|$ and by an above example $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. □

Example. $\mathbb{Q}$ is countable: Prove $\mathbb{Q}$ is countable.

Proof. By an above proposition, we know $|\mathbb{N}| \leq |\mathbb{Q}|$. By CSB, it suffices to show $|\mathbb{Q}| \leq |\mathbb{N}|$. Note that every non-zero $q \in \mathbb{Q}$ can be uniquely written as $q = \frac{n}{m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $\gcd(n, m) = 1$.

This gives an injection $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Z} \times \mathbb{N}$ given by $f(\frac{n}{m}) = (n, m)$ as above. Therefore, $|\mathbb{Q} \setminus \{0\}| \leq |\mathbb{Z} \times \mathbb{N}|$. Now where $q = 0$, let $f(0) = (0, 1)$. Notice $n \neq 0$ for all $q \neq 0$, thus this mapping is unique. We have then extended $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ and so by an above proposition $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N}|$. □

Example. $\mathbb{R}$ is uncountable (Cantor's Diagonal Argument): Prove $\mathbb{R}$ is uncountable

Proof. Recall $|\mathbb{R}| = |(0,1)|$. By way of contradiction, suppose there is a bijection $f : \mathbb{N} \to (0,1)$. Note $(0,1) = f(\mathbb{N})$ by surjectivity. Suppose that in decimal form,

$$\begin{aligned} f(1) &= 0.d_{11}d_{12}d_{13}\cdots \\ f(2) &= 0.d_{21}d_{22}d_{23}\cdots \\ f(3) &= 0.d_{31}d_{32}d_{33}\cdots \\ &\vdots \end{aligned}$$

where d_{ij} is the jth decimal of $f(i)$. Note that for all $n \in \mathbb{N}$ $f(n) \neq 0.\overline{000} = 0$ and $f(n) \neq 0.\overline{999} = 1$ as these are excluded in our interval.

For every $i \in \mathbb{N}$, choose $b_i \in \{0, 1, 2, \ldots, 9\}$ such that $b_i \neq a_{ii}$ with not all $b_i = 0$ and not all $b_i = 9$. Let

$$x = 0.b_1b_2b_3\cdots \in (0,1)$$

(note $x \neq 0$ and $x \neq 1$). We have $x \in (0,1)$, but $x \notin f(\mathbb{N})$. This is because $f(i) \neq x$ for all $i \in \mathbb{N}$. In particular, decimal representation is unique and the ith decimal of $f(i)$ is different from the ith decimal of x ($a_{ii} \neq b_i$ by construction). Since $f(\mathbb{N}) \neq (0,1)$, we have a contradiction. □

Notation: We denote $|\mathbb{R}| = c$ where c is for continuum.

Remark: $n, \aleph_0, c$ are all examples of *cardinal numbers*. Think of them as symbols used to denote sizes particular sets may have.

Definition. Continuum Hypothesis: We take the following statement an axiom. If A is a set with $\aleph_0 \leq |A| \leq c$ then $|A| = \aleph_0$ or $|A| = c$.

Remark: This axiom is consistent with the normal ZFC axioms, but is also independent of them. That is, we cannot prove this statement from the ZFC axiom.

2.2 Power Sets

Proposition: If X is a set with cardinality $|X| = n \in \mathbb{N}$, then $|\mathcal{P}(X)| = 2^n$.

Proof. Notice there are $\binom{n}{k}$ ways to create a subset $x \subseteq X$ of k elements (e.g., $|x| = k$). Therefore, there are a total of

$$\sum_{i=0}^{n} \binom{n}{k} = \sum_{i=0}^{n} \binom{n}{k} 1^i 1^{n-i} = (1+1)^n = 2^n$$

possible subsets of X by the binomial theorem. This result could also be proved by induction without the use of combinatorics. □

Remark: Let A be a set. Let $A^{\mathbb{N}} = \prod_{i=1}^{\infty} A$ be the cartesian product of A with itself countably many times. Alternatively, this can be viewed as the set of sequences in A. We can also think of this set as

$$\{f \mid f : \mathbb{N} \to A\}$$

via the correspondance

$$(a_1, a_2, a_3, \ldots) \equiv f(i) = a_i$$

Notation: Let A, B be sets. We denote $A^B := \{f \mid f : B \to A\}$ to be the functions from B to A. Similarly, we denote $|A|^{|B|} := |\{f \mid f : B \to A\}$

Proposition: For all sets A, $|\mathcal{P}(A)| = 2^{|A|} = |\{f \mid f : A \to \{0, 1\}\}$.

Proof. Let $\varphi : \mathcal{P}(A) \to \{f \mid f : A \to \{0, 1\}\}$ given by $\varphi(X) = \varphi_X$ where

$$\varphi_X(a) = \begin{cases} 1 & a \in X \\ 0 & a \notin X \end{cases}$$

Notice φ is a bijection. We verify this by finding its inverse

$$\varphi^{-1} : \{f \mid f : A \to \{0, 1\}\} \to \mathcal{P}(A)$$

given by

$$\varphi^{-1}(f) = \{a : f(a) = 1\}$$

Notice for all $X \in \mathcal{P}(A)$ we have

$$\varphi^{-1}(\varphi(X)) = \varphi^{-1}(\varphi_X) = \{a \in A : a \in X\} = X$$

and for all $f : A \to \{0, 1\}$ we have

$$\varphi(\varphi^{-1}(f)) = \varphi(\{a \in A : f(a) = 1\}) = \begin{cases} 1 & a \in \{a \in A : f(a) = 1\} \\ 0 & a \notin \{a \in A : f(a) = 1\} \end{cases} = f$$

Hence φ is a bijection as desired. □

Proposition: If A is a set then $|A| < |\mathcal{P}(A)|$.

Proof. The injection $f : A \to \mathcal{P}(A)$ given by $f(a) = \{a\}$ proves $|A| \leq |\mathcal{P}(A)|$. Now by way of contradiction, suppose there exists a surjection $g : A \to \mathcal{P}(A)$. Now consider the set

$$B = \{x \in A : x \notin g(x)\} \subseteq A$$

Since g is surjective, there is an $a \in A$ such that $g(a) = B$. If $a \in g(a)$, then by definition $a \notin B$, but $B = g(a)$. If $a \notin g(a)$, then by definition $a \in B$, but again $B = g(a)$. So $a \in g(a) = B \iff a \notin B = g(a)$, a contradiction. So g cannot be surjective. Since g was general, no surjection and therefore no bijection from A to $\mathcal{P}(A)$ may exist. □

Remark: Notice $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$, so we can define infinitely many infinities.

Example: Prove $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, or equivalently $2^{\aleph_0} = c$.

Proof. Let $X = \{f \mid f : \mathbb{N} \to \{0, 1\}\}$. We know then that $|\mathcal{P}(\mathbb{N})| = |X|$. Now consider $\varphi : X \to \mathbb{R}$ given by $\varphi(f) = 0.f(1)f(2)f(3)\cdots$ where $f(i)$ is the ith decimal of $\varphi(f)$. We can see that φ is an injection, and so $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |X| \leq |\mathbb{R}| = c$. So by the continuum hypothesis, we know $|\mathcal{P}(\mathbb{N})| = |X| = |\mathbb{R}| = c$, as desired.

Alternatively, we can avoid invoking the continuum hypothesis by constructing an injection $h : \mathbb{R} \to \mathcal{P}(\mathbb{N})$. First define $g : \mathbb{N} \to \mathbb{Q}$ to be a bijection between the sets (we know this is possible since $|\mathbb{Q}| = |\mathbb{N}|$). Now define the injection $h : \mathbb{R} \to \mathcal{P}(\mathbb{N})$ given by $h(x) = \{n : h(n) < x\}$. By the density of the rationals in $\mathbb{R}$ we can see h is injective. □

2.3 Cardinal Arithmetic

Definition. Rules of Cardinal Arithmetic: Let A, B be sets. Then

1. If $A \cap B = \emptyset$ then we define $|A| + |B| := |A \cup B|$.
2. $|A| \cdot |B| = |A \times B|$.
3. $|A|^{|B|} = |\{f \mid f : B \to A\}$.

Note: The above definitions are consistent with usual arithmetic for finite cardinalities.

Example: We see from above $2^{\aleph_0} = c$.

Example: Show $\aleph_0 + \aleph_0 = \aleph_0$.

Proof. Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ be such that $A \cap B = \emptyset$. Consider the bijection $f : A \cup B \to \mathbb{N}$ given by $f(a_i) = 2i$ and $f(b_i) = 2i - 1$. Then $|A \cup B| = \aleph_0$. Note that this means even if $A \cap B \neq \emptyset$, so long as we can find other sets C, D such that $|A| = |C|$ and $|B| = |D|$ and $C \cap D = \emptyset$, then we can still define $|A| + |B| = |C| + |D| = |C \cup D|$. □

Example: Show $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Proof. Note $\aleph_0 \cdot \aleph_0 = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph_0$. □

Example: Show $\aleph_0 + c = c$.

Proof. Consider $|(0, 1)| = c$ and $|\mathbb{N}| = \aleph_0$. Notice $(0, 1) \subseteq \mathbb{N} \cup (0, 1) \subseteq \mathbb{R}$. So $c \leq \aleph_0 + c \leq c$ or $\aleph_0 + c = c$. □

Example: Show $c \cdot c = c$.

Proof. By the exponent rules of cardinal arithmetic (see next module) we have

$$c \cdot c = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = c$$

□

Example: Show $c^{\aleph_0} = c$.

Proof. By the exponent rules of cardinal arithmetic (see next module) we have

$$c^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$$

□

2.4 Cardinal Exponents

Proposition. Exponent Rules: Let A, B, C be sets. Then $\left(|A|^{|B|}\right)^{|C|} = |A|^{|B|\cdot|C|}$.

Proof. Let $X = \{f : B \to A\}$, let $Y = \{f : C \to X\}$, and let $Z = \{f : B \times C \to A\}$. We must show that there is a bijection $\varphi : Y \to Z$. Consider $\varphi : Y \to Z$ given by $\varphi(f)(b \times c) = f(c)(b)$. That is, φ is such that given $f : C \to X$, we have $\varphi(f) : B \times C \to A \in Z$ is given by $\varphi(f)(b \times c) = f(c)(b)$.

(Injectivity) For all $f, g \in Y$

$$\begin{array}{rrcll} & \varphi(f) & = & \varphi(g) & \\ \implies & f(c)(b) & = & g(c)(b) & \forall b \in B, \forall c \in C \\ \implies & f(c) & = & g(c) & \forall c \in C \\ \implies & f & = & g & \end{array}$$

(Surjectivity) Let $f \in Z$ so that $f : B \times C \to A$ is a function. Consider $g \in Y$ given by $g(c)(b) = f(b, c)$. Then $\varphi(g)(b, c) = g(c)(b) = f(b, c)$, so $\varphi(g) = f$. So since f was arbitrary, there must be $g \in Y$ such that $\varphi(g) = f$.

So φ is bijective, as desired. □

Proposition. Exponent Rules: Let A, B, C be sets such that $B \cap C = \emptyset$. Then $\left(|A|^{|B|}\right)\left(|A|^{|C|}\right) = |A|^{|B|+|C|}$.

Proof. Let $X = \{f : B \to A\}$, let $Y = \{f : C \to A\}$, and let $Z = \{f : B \cup C \to A\}$. We must show that there is a bijection $\varphi : X \times Y \to Z$. Consider $\varphi : X \times Y \to Z$ given by

$$\varphi(f, g)(x) = \begin{cases} f(b) & x \in B \\ g(c) & x \in C \end{cases}$$

(Injectivity) For all $f_1, f_2 \in X$ and $g_1, g_2 \in Y$

$$\begin{array}{rrcll} & \varphi(f_1, g_1) & = & \varphi(f_2, g_2) & \\ \implies & \varphi(f_1, g_1)(x) & = & \varphi(f_2, g_2)(x) & \forall x \in B \cup C \\ \implies & \varphi(f_1, g_1)(b) & = & \varphi(f_2, g_2)(b) & \forall b \in B \\ \implies & f_1(b) & = & f_2(b) & \forall b \in B \\ \implies & f_1 & = & f_2 & \end{array}$$

and

$$\begin{array}{rrcll} & \varphi(f_1,g_1) & = & \varphi(f_2,g_2) & \\ \implies & \varphi(f_1,g_1)(x) & = & \varphi(f_2,g_2)(x) & \forall x \in B\cup C \\ \implies & \varphi(f_1,g_1)(c) & = & \varphi(f_2,g_2)(c) & \forall c \in C \\ \implies & g_1(c) & = & g_2(c) & \forall c \in C \\ \implies & g_1 & = & g_2 & \end{array}$$

(Surjectivity) Let $f \in Z$ so that $f : B \cup C \to A$ is a function. Consider $g \in X$ given by $g(b) = f(b)$ for all $b \in B$ and $h \in Y$ given by $h(c) = f(c)$ for all $c \in C$. Then for any $x \in B \cup C$ we have

$$\varphi(g,h)(x) = \begin{cases} g(x) & x \in B \\ h(x) & x \in C \end{cases} = \begin{cases} f(x) & x \in B \\ f(x) & x \in C \end{cases} = f(x)$$

So for any $f \in Z$ there is $(g,h) \in X \times Y$ such that $\varphi(g,h) = f$. So φ is bijective as desired. □

Unit 2 Topology

Week 3 Metric Spaces

3.1 Metric Spaces

Remark: In MATH 137/147 we considered $\mathbb{R}$ equipped with $|\cdot|$. In MATH 247 we considered $\mathbb{R}^n$ equipped with $\|\cdot\|$ given by $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. In both cases, this gives us notions of

- Distance between points
- Convergent and Cauchy sequences
- Open and closed sets
- Compact sets
- Point wise and uniform convergence of functions
- Continuity and uniform continuity
- Spaces of continuous functions

and more. So equipping $\mathbb{R}$ with $|\cdot|$ allowed us to perform calculus, equipping $\mathbb{R}^n$ with $\|\cdot\|_2$ allowed us to perform multi-variable calculus. Calculus is just a subset of analysis though, so how can we perform analysis on a general space? Not just $\mathbb{R}$ or $\mathbb{R}^n$.

Definition. Metric Space: A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function (called the metric) such that

1. For all $x, y \in X$, we have $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. For all $x, y \in X$, we have $d(x, y) = d(y, x)$.
3. For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality).

Remark. Intuition of a Metric: The intuition behind the metric $d : X \times X \to \mathbb{R}$ is that $d(x, y)$ measures the distance between $x, y \in X$. This is just like $|x - y|$ for $x, y \in \mathbb{R}$ or $\|x - y\|_2$ for $x, y \in \mathbb{R}^n$. But rather, d is a distance calculating function on an abstract set X. This allows us to do usual mathematical analysis on X.

Definition. Normed Vector Space: Abbr. NVS. Let V be a vector space over a field $\mathbb{F}$ (in this class we will use $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A norm on V is a function $\| \cdot \| : V \to \mathbb{R}$ such that

1. For all $v \in V$, we have $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
2. For all $\alpha \in \mathbb{F}$ and $v \in V$, we have $\|\alpha v\| = |\alpha| \cdot \|v\|$.
3. For all $u, v \in V$, we have $\|u + v\| \leq \|u\| + \|v\|$ (Triangle Inequality).

Then we say $(V, \| \cdot \|)$ is a normed vector space.

Proposition: Let $(V, \| \cdot \|)$ be a normed vector space. Then the function $d : V \times V \to \mathbb{R}$ given by $d(u, v) = \|u - v\|$ is a metric on v.

3.2 Metric Space Examples

Notation: If $(a_1, a_2, a_3, \ldots)$ is a sequence with $a_n \in A$ for all $n \in \mathbb{N}$, then we write $(a_n)_{n=1}^{\infty} \subseteq A$ or simply $(a_n) \subseteq A$.

Example. Discrete Metric: Let X be a set. Then

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

is a metric on X. We call d the discrete metric on X.

Example: $(\mathbb{R}, | \cdot |)$ is a metric space (note $(\mathbb{R}, | \cdot |)$ is a NVS).

Definition. p-norm: For $p \in [1, \infty)$, the function $\| \cdot \|_p : \mathbb{R}^n \to \mathbb{R}$ given by

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

is called the p-norm.

Definition. Infinity-norm: For $p = \infty$, the function $\| \cdot \|_\infty : \mathbb{R}^n \to \mathbb{R}$ given by

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

is called the infinity-norm (or sup-norm) and is often referred to as a p-norm with $p = \infty$.

Definition. Holder Conjugate: Let $p \in (1, \infty)$. We define the Holder conjugate of p to be $q = \frac{p}{p-1}$. Notice that $\frac{1}{p} + \frac{1}{q} = 1$ and that the Holder conjugate of q is p. We define the Holder conjugate of 1 to be ∞ and vice versa.

Lemma. Young's Inequality: Let $p, q \in (1, \infty)$ be Holder conjugates. If $a, b > 0$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Consider $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ on $(0, \infty)$. We see that $f'(x) = x^{p-1} - 1 > 0$ for all $x > 1$. Moreover, $f'(x) < 0$ for all $0 < x < 1$. Since $f(1) = 0$, we have that $f(x) \geq 0$ for all $x > 0$ ($f(x)$ has minimum at $x = 1$). Considering $x = \frac{a}{b^{q-1}}$ we see that

$$\begin{aligned}
f(x) &\geq 0 \\
\frac{1}{p}\frac{a^p}{b^{(q-1)p}} + \frac{1}{q} - \frac{a}{b^{q-1}} &\geq 0 \\
\frac{1}{p}\frac{a^p}{b^{(q-1)p}} + \frac{1}{q} &\geq \frac{a}{b^{q-1}} \\
\frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q} &\geq \frac{a}{b^{q-1}} && \text{Since } q = pq - p \\
\frac{1}{p}a^p + \frac{1}{q}b^q &\geq ab
\end{aligned}$$

as desired. □

Theorem. Holder's Inequality: For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$$

Proof. Notice if $p = \infty$ then

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \|x\|_\infty |y_i| = \|x\|_\infty \sum_{i=1}^{n} |y_i| = \|x\|_\infty \|y\|_1 = \|x\|_p \|y\|_q$$

as desired. Assume then $p, q \in (1, \infty)$. Assume further that $x \neq 0$ and $y \neq 0$ as the result is trivial in this case. Replace x with $\frac{x}{\|x\|_p}$ and y with $\frac{y}{\|y\|_q}$ so that $\|x\|_p = \|y\|_q = 1$. By Young's Inequality we have

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \left(\frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \right) = \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_p$$

as desired. □

Theorem: For $1 \leq p \leq \infty$, the p-norm $\| \cdot \|_p$ is a norm on $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Proof. We only verify the triangle inequality (the other two requirements are obvious). For $1 \leq p < \infty$:

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\
&= \sum_{i=1} \sum_{i=1}^{n} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \cdot |x_i + y_i|^{p-1} \\
&\leq \|x\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{1/q} + \|y\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{1/q} \\
&= \|x\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^{p} \right)^{1/q} + \|y\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^{p} \right)^{1/q} \\
&= (\|x\|_p + \|y\|_p) \left(\sum_{i=1}^{n} |x_i + y_i|^{p} \right)^{1/q} \\
&= (\|x\|_p + \|y\|_p) \left(\sum_{i=1}^{n} |x_i + y_i|^{p} \right)^{1-\frac{1}{p}} \\
&= (\|x\|_p + \|y\|_p) \cdot \|x+y\|_p^{p(1-\frac{1}{p})} \\
&= (\|x\|_p + \|y\|_p) \cdot \|x+y\|_p^{p-1}
\end{aligned}$$

and so

$$\|x+y\|_p^p \leq (\|x\|_p + \|y\|_p) \cdot \|x+y\|_p^{p-1} \qquad \Longleftrightarrow \qquad \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

as desired. If $p = \infty$, then for any $x, y \in \mathbb{R}^n$ we see that

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_\infty + \|y\|_\infty$$

so by definition of the maximum

$$\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

□

Remark: Unless stated otherwise, we will always assume $\mathbb{R}^n$ is equipped with the 2-norm (i.e., the Euclidean norm).

Example. ℓ^p Spaces: Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all sequences of real numbers. For $1 \leq p \leq \infty$ and $(x_i) \in \mathbb{R}^{\mathbb{N}}$, let

$$\|(x_i)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

For $p = \infty$, let

$$\|(x_i)\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$$

Then for all $p \in [1, \infty]$ let

$$\ell^p = \{(x_i) \in \mathbb{R}^\mathbb{N} : \|(x_i)\|_p < \infty\}$$

Then $(\ell^p, \|\cdot\|_p)$ is a NVS.

Proof. Assume $p < \infty$. First let q be the Holder conjugate of p and let $x \in \ell^p$ and $y \in \ell^q$. Then

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q$$

Then the proof that $\|\cdot\|_p$ is a norm follows from the proof that it's a norm on $\mathbb{R}^n$. The proof where $p = \infty$ is also the same as in $\mathbb{R}^n$. □

Example: Let $C([a, b])$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$. For $1 \leq p < \infty$,

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

and

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$$

are norms on $C([a, b])$.

Proof. See A2 Q1 □

Example: Let $B([a, b])$ denote the set of bounded functions $f : [a, b] \to \mathbb{R}$. As above $\|\cdot\|_\infty$ is a norm on $B([a, b])$.

Definition. Subspace: Let (X, d) be a metric space. If $Y \subseteq X$ then (Y, d) is also a metric space and we call (Y, d) a subspace of (X, d).

Example: The function

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

is a metric on $\mathbb{R}^n$ which does not come from a norm.

Example. Baire Space: The function

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

is a metric on $\mathbb{N}^\mathbb{N}$. The metric space $(\mathbb{N}^\mathbb{N}, d)$ is called the Baire Space in set theory, however, there is a different meaning for a Baire space in topology which refers to a type of topological space (e.g., Banach space) rather than a specific set.

Example. Cantor Space: The function

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

is a metric on $2^{\mathbb{N}}$, the set of all $0-1$ sequences. The metric space $(2^{\mathbb{N}}, d)$ is called the Cantor Space.

Example. Hamming Distance: Let X be a finite set. Then

$$d(A,B) = |(A \Delta B)| := |(A \cup B) \setminus (A \cap B)|$$

is a metric on $\mathcal{P}(X)$.

Example. Hausdorff Metric: Let X be a closed subset of $\mathbb{R}^n$ and let $\mathcal{H}(x)$ denote the set of all non-empty, closed, bounded subsets of X. For $A \in \mathcal{H}(x)$ and $b \in X$, define $d(b, A) = \min_{a \in A} \|a - b\|$. Then

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b \in B} d(b,A)\right\}$$

is a metric on $\mathcal{H}(x)$. See page 5,6 of the course notes.

Example. p-Adic Metric: Let p be a prime number. For every $0 \neq q \in \mathbb{Q}$, we can write $q = p^a \frac{n}{m}$, where $a, n, m \in \mathbb{Z}$, $m \neq 0$, $\gcd(n,m) = \gcd(p,n) = \gcd(p,m) = 1$. We then define

$$|q|_p = p^{-a}, |0|_p = 0$$

Then $d(x,y) = |x-y|_p$ is a metric on $\mathbb{Q}$. See pages 6,7 of the course notes.

Example. Product Metric: Let (X,d) and (Y,d) be metric spaces. Then

$$d((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) = d(y_1,y_2)$$

is a metric on $X \times Y$.

Example. Infinite Product Metric: Let (X_i, d_i) be a metric space for every $i \in \mathbb{N}$. Then

$$d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{d_i(x_i,y_i)}{2^i(1 + d_i(x_i,y_i))}$$

is a metric on

$$\prod_{i=1}^{\infty} X_i$$

3.3 Convergence

Definition. Convergence: Let (X,d) be a metric space. A sequence $(x_n) \subseteq X$ converges to $x \in X$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. We denote this by $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition. Cauchy: Let (X, d) be a metric space. A sequence $(x_n) \subseteq X$ is Cauchy if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Proposition: Let (X, d) be a metric space. If $(x_n) \subseteq X$ converges to $x \in X$, then (x_n) is Cauchy.

Proof. Suppose $(x_n) \subseteq X$ is such that $x_n \to x$. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Let $n, m \geq N$. Then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So (x_n) is Cauchy. □

Example: Consider $(x_n) \subseteq \mathbb{R}$ given by $x_n = \frac{1}{n}$. Notice (x_n) converges in $(\mathbb{R}, |\cdot|)$, and so (x_n) is Cauchy in $(\mathbb{R}, |\cdot|)$, and so (x_n) is Cauchy in $((0, 1], |\cdot|)$, but (x_n) is divergent in $((0, 1], |\cdot|)$ since $0 \notin (0, 1]$.

Definition. Bounded Sequence: Let (X, d) be a metric space. We say $A \subseteq X$ is bounded if

$$\sup\{d(x, y) : x, y \in A\} < \infty$$

We say (x_n) is bouned if

$$\{x_1, x_2, \ldots\}$$

is bounded.

Definition. Ball: Let (X, d), let $x \in X$, and let $r > 0$. Then

- The open ball centred at x of radius r is

$$B_r(x) = \{a \in X : d(a, x) < r\}$$

- The closed ball centred at x of radius r is

$$B_r[x] = \{a \in X : d(a, x) \leq r\}$$

Proposition: Let (X, d) be a metric space. A set $A \subseteq X$ is bounded if and only if there is an $x \in X$ and $r > 0$ such that $A \subseteq B_r[x]$.

Proof. ($\Longrightarrow$) Let $r = \sup\{d(x, y) : x, y \in A\} < \infty$ and let $x \in A$ be arbitrary. Then for all $a \in A$, we necessarily have $d(a, x) \leq r$, so $a \in B_r[x]$. Therefore $A \subseteq B_r[x]$.

($\Longleftarrow$) Let $A \subseteq B_r[x]$ for some $r > 0$ and $x \in A$. Let $a, b \in A$ be arbitrary. Notice

$$d(a, b) \leq d(a, x) + d(b, x) \leq r + r = 2r$$

So necessarily $\sup\{d(x, y) : x, y \in A\} \leq 2r < \infty$. □

Proposition: Let (X, d) be a metric space. If $(x_n) \subseteq X$ is Cauchy, then (x_n) is bounded.

Proof. Since (x_n) is Cauchy, there is an $N \in \mathbb{N}$ such that $d(x_n, d_m) \leq 1$ for all $n, m \geq N$ and so $d(x_n, x_N) \leq 1$ for all $n \geq N$. Let

$$R = \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\}$$

then $(x_n) \subseteq B_R[x_N]$. By our above proposition (x_n) is bounded. □

3.4 Examples in Convergence

Example: Consider the metric space $(\mathbb{Q}, |\cdot|_2)$ where $|2^a \frac{n}{m}|_2 = \frac{1}{2^a}$. Let $x_n = \frac{1-(-2)^n}{3}$. Notice then

$$\left|x_n - \frac{1}{3}\right|_2 = \left|\frac{-(-2)^n}{3}\right|_2 = \left|2^n \frac{-(-1)^n}{3}\right|_2 = \frac{1}{2^n} \to 0$$

so $x_n \to \frac{1}{3}$. Notice that in $(\mathbb{Q}, |\cdot|)$ where $|\cdot|$ is the absolute value, x_n is clearly very divergent since it is unbounded and oscillating.

Example: Consider the Cantor Space $(2^{\mathbb{N}}, d)$ where $d((x_i), (y_i)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$. Let

$$x_n = (\underbrace{1, 1, \cdots, 1}_{n \text{ times}}, 0, 0, \ldots)$$

and let $x = (1, 1, 1, \ldots)$. Notice then

$$d(x_n, x) = \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}} \left(\frac{1}{1 - \frac{1}{2}}\right) = \frac{1}{2^n} \to 0$$

so $x_n \to x$.

Example: Consider the metric space $(\ell^p, \|\cdot\|_p)$ for $1 \leq p < \infty$. Let

$$x_n = (\underbrace{1, 1, \cdots, 1}_{n \text{ times}}, 0, 0, \ldots)$$

and let $x = (1, 1, 1, \ldots)$. Notice then

$$\|x_n - x\|_p = \sum_{i=1}^{\infty} (1-1)^n + \sum_{i=n+1}^{\infty} (1-0)^p = \sum_{i=n+1}^{\infty} 1 = \infty$$

and so clearly $|x_n - x| \not\to 0$, therefore $x_n \not\to x$. Suppose $p = \infty$, then

$$\|x_n - x\|_\infty = 1$$

so that $\|x_n - x\|_\infty \not\to 0$, therefore $x_n \not\to x$. Moreover, $\|x_n - x_m\|_\infty = 1$ for all $n \neq m$, so (x_n) is not Cauchy. Hence (x_n) diverges.

3.5 Completeness

Remark: It is often easier to prove a sequence is Cauchy than convergent. For instance, to prove convergence, we usually already need to have a pretty good idea of a candidate for the limit. So when are Cauchy sequences convergent?

Definition. Complete: Let (X, d) be a metric space. A subset $A \subseteq X$ is complete if every Cauchy sequence $(a_n) \subseteq A$ converges to a point in A. If X is complete in itself, we call (X, d) a complete metric space.

Definition. Banach Space: A complete normed vector space is called a Banach space.

Example: From MATH 247 we know $(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space.

Example: $(0, 1]$ is not a complete subset of $\mathbb{R}$. For instance $x_n = \frac{1}{n}$ is Cauchy but not convergent.

Example: If X is a set and d is the discrete metric on X, then (X, d) is complete.

Proof. Suppose $(x_n) \subseteq X$ is a Cauchy sequence in X. Then let $\varepsilon = 1$ and pick an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon = 1$ for all $n, m \geq N$. Note that the $d(a, b) < 1$ if and only if $a = b$, in which case $d(a, b) = 0$. So, we have that $x_n = x_m = x_N$ for all $n, m \geq N$. Then clearly for all $\varepsilon > 0$ we have $d(x_n, x) = 0 < \varepsilon$ for all $n \geq N$. So $x_n \to x_N$ as desired. □

Example: ℓ^p is a Banach space for all $1 \leq p \leq \infty$.

Proof. Suppose $p < \infty$. Let $(a_k) \subseteq \ell^p$ be a Cauchy sequence. Say $a_k = (a_k^{(1)}, a_k^{(2)}, \ldots)$ for each $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given, then there is an $N \in \mathbb{N}$ such that $\|a_k - a_m\| < \varepsilon$ for all $k, m \geq N$. Fix $i \in \mathbb{N}$. Since $|a_k^{(i)} - a_m^{(i)}| \leq \|a_k - a_m\|_p < \varepsilon$, we see that $(a_k^{(i)})_{k=1}^{\infty}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, we have that $a_k^{(i)} \to b_i$ for some $b_i \in \mathbb{R}$. We claim then that $a_k \to b = (b_1, b_2, \ldots)$.

For $k, m \geq N$, we see that

$$\sum_{i=1}^{M} |a_k^{(i)} - a_m^{(i)}|^p \leq \|a_k - a_m\|_p^p < \varepsilon^p$$

for every $M \in \mathbb{N}$. Taking $m \to \infty$ we have that

$$\sum_{i=1}^{M} |a_k^{(i)} - b_i|^p \leq \varepsilon^p$$

for every $M \in \mathbb{N}$. Moreover, taking $M \to \infty$ we see that

$$\sum_{i=1}^{\infty} |a_k^{(i)} - b_i|^p \leq \varepsilon^p$$

which means that we exactly have $\|a_k - b\|_p \leq \varepsilon$ for all $k \geq N$. Notice that $a_N, a_N - b \in \ell^p$ guarantees $b \in \ell^p$.

Suppose $p = \infty$. Let $(a_n) \subseteq \ell^\infty$ be Cauchy. Say $a_k = (a_k^{(1)}, a_k^{(2)}, \ldots)$ for each $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given, then there is an $N \in \mathbb{N}$ such that $\|a_n - a_m\|_\infty < \varepsilon$ for all $k, m \geq N$. Fix $i \in \mathbb{N}$. Then

$$|a_n^{(i)} - a_m^{(i)}| \leq \sup\{a_n^{(i)} - a_m^{(i)} : i \in \mathbb{N}\} = \|a_n - a_m\|_\infty < \varepsilon$$

We see that $(a_k^{(i)})_{k=1}^\infty$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, we have that $a_k^{(i)} \to b_i$ for some $b_i \in \mathbb{R}$. We claim then that $a_k \to b = (b_1, b_2, \ldots)$. Let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ such that $\|a_n - a_m\| < \frac{\varepsilon}{2}$ for all $n, m \geq N$. Then for all $i \in \mathbb{N}$ and $n, m \geq N$, we have

$$|a_n^{(i)} - a_m^{(i)}| \leq \|a_n - a_m\|_\infty < \frac{\varepsilon}{2}$$

So

$$\lim_{m\to\infty} |a_n^{(i)} - a_m^{(i)}| \leq \frac{\varepsilon}{2} \qquad \Longrightarrow \qquad |a_n^{(i)} - b_i| \leq \frac{\varepsilon}{2}$$

Since i was arbitrary, we have

$$\|a_n - b\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon$$

and

$$|a_k^{(i)} - b_i| < \varepsilon$$

for all $i \in \mathbb{N}$. Since this holds for all $i \in \mathbb{N}$, we have

$$\|a_k - b\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon$$

as desired. Notice that $a_N, a_N - b \in \ell^\infty$ guarantees $b \in \ell^\infty$. □

Example: Let

$$C_{00} := \{(x_n) \in \ell^\infty : \exists N, \forall n \geq N, x_n = 0\}$$

be the set of all zero terminated sequences in ℓ^∞. Then $(C_{00}, \|\cdot\|_\infty)$ is not a Banach space.

Proof. Consider $(x_n) \subseteq C_{00}$ given by

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right)$$

In $(\ell^\infty, \|\cdot\|_\infty)$, we can show that $x_n \to x := (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. In particular, let $\varepsilon > 0$ and let $N = \lceil \frac{1}{\varepsilon} \rceil \in \mathbb{N}$ so that $N \geq \frac{1}{\varepsilon}$. Then for all $n \geq N$ we have

$$x - x_n = \left(0, 0, 0, \ldots, 0, \frac{1}{n+1}, \frac{1}{n+2} \ldots\right)$$

and so $\|x - x_n\|_\infty = \|x_n - x\|_\infty = \sup\{\frac{1}{n+i} : i \in \mathbb{N}\} = \frac{1}{n+1}$. But $n + 1 \geq N \geq \frac{1}{\varepsilon}$, so $\|x - x_n\|_\infty = \frac{1}{n+1} \leq \varepsilon$, as desired.

Since $x_n \to x \notin C_{00}$, we have that (x_n) is Cauchy (since it is convergent in ℓ^∞) but not convergent in $(C_{00}, \|\cdot\|_\infty)$ by the uniqueness of limits. □

Example: If $p < \infty$, then

$$f_n(x) = \begin{cases} 0 & x \in \left[0, 1 - \frac{1}{n}\right] \\ 1 + n(x-1) & x \in \left[1 - \frac{1}{n}, 1\right] \\ 1 & x \in [1, 2] \end{cases}$$

is a Cauchy sequence in $(C([0,2]), \|\cdot\|_p)$ which does not converge. Each function looks like the constant zero function, then a linear portion (whose slope gets steeper as $n \to \infty$) and then the constant 1 function.

Week 4 Topology I

4.1 Topological Spaces

Remark. MATH 247 Topology: Let V, W be normed vector spaces. Then

- Open set: A subset $U \subseteq V$ is said to be open if for all $x \in U$, there is an $r > 0$ such that $B_r(x) \subseteq U$.
- Closed set: A subset $C \subseteq V$ is said to be open if $V \setminus C$ is open.
- Union of Open Sets: If $\{U_\alpha\}_{\alpha \in I}$ are open sets in V, then $\bigcup_{\alpha \in I} U_\alpha$ is open (note this union is of any size, including countable and uncountable unions).
- Intersection of Closed Sets: If $\{C_\alpha\}_{\alpha \in I}$ are closed sets in V, then $\bigcap_{\alpha \in I} C_\alpha$ is closed (note this union is of any size, including countable and uncountable unions).
- Finite Intersection of Open Sets: If $U_1, \ldots, U_n \subseteq V$ are open, then $U_1 \cap \cdots \cap U_n$ is open.
- Finite Union of Closed Sets: If $C_1, \ldots, C_n \subseteq V$ are closed, then $C_1 \cup \cdots \cup C_n$ is closed.
- The following are equivalent for $f : A \to W$ where $A \subseteq V$: (1) f is continuous, (2) f preserves convergence, (3) for all open $U \subseteq W$, we have $f^{-1}(U)$ is relatively open in A.
- The following are equivalent for $C \subseteq V$: (1) C is compact, (2) every sequence $(a_n) \subseteq C$ has a convergent subsequence $a_{n_k} \to a \in C$, (3) every open cover of C has a finite subcover. (Recall a cover of C is a collection open sets $\{U_\alpha\}_{\alpha \in I}$ such that $C \subseteq \bigcup_{\alpha \in I} U_\alpha$. A finite subcover is a subset of $\{U_\alpha\}_{\alpha \in I}$.)

Notice all of the above common tools in topology can be discussed in terms of open sets.

Definition. Topology: Let X be a set. A topology on X is a collection $T \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in T$,
2. If $\{U_\alpha\}_{\alpha \in I} \subseteq T$ then $\bigcup\limits_{\alpha \in I} U_\alpha \in T$,
3. If $U, V \in T$ then $U \cap V \in T$.

We call (X, T) a topological space and we call the elements of T the open (sub)sets of X.

Example: Let $X = \{a, b, c\}$. Then

$$\begin{aligned} T_1 &= \{\emptyset, X\} \\ T_2 &= \mathcal{P}(X) \\ T_3 &= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \\ T_4 &= \{\emptyset, X, \{a, b\}\} \\ &\vdots \end{aligned}$$

are all topologies on X.

Example. Discrete Topology: Let X be a set. Then $T = \mathcal{P}(X)$ is a topology, called the discrete topology.

Example. Trivial Topology: Let X be a set. Then $T = \{\emptyset, X\}$ is a topology, called the trivial or indiscrete topology.

Example. Finite-complement Topology: Let X be a set. Then

$$T = \{A \subseteq X : X \setminus A = X \text{ or } X \setminus A \text{ is finite}\}$$

is a topology, called the finite-complement topology.

Example. Countable-complement Topology: Let X be a set. Then

$$T = \{A \subseteq X : X \setminus A = X \text{ or } X \setminus A \text{ is countable}\}$$

is a topology, called the countable-complement topology.

Example. Subspace Topology: Let (X, T) be a topological space. Let $Y \subseteq X$. Then $T_Y = \{U \cap Y : U \in T\}$ is a topology, called the subspace topology on Y, relative to (X, T).

4.2 Metric Topology

Definition. Open Set: Let (X, d) be a metric space. A set $U \subseteq X$ is said to be open if for all $x \in U$ there is an $r > 0$ such that $B_r(x) \subseteq U$.

Proposition: Let (X, d) be a metric space. The collection $T_d = \{U \subseteq X : U \text{ open}\}$ is a topology on X.

Proof. Clearly $\emptyset, X \in T_d$. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in X. Let $x \in \bigcup\limits_{\alpha \in I} U_\alpha$ so that $x \in U_\alpha$ for some $\alpha \in I$. Since U_α is open, there is an $r > 0$ such that $B_r(x) \subseteq U_\alpha \subseteq \bigcup\limits_{\alpha \in I}$, so $\bigcup_{\alpha \in I} U_\alpha$ is open. Let $U, V \subseteq X$ be open and let $x \in U \cap V$. Then there are $r_1, r_2 > 0$ such that $B_{r_1}(x) \subseteq U$ and $B_{r_2}(x) \subseteq V$. Then $r = \min\{r_1, r_2\}$ is such that $B_r(x) \subseteq U \cap V$. So $U \cap V$ is open. □

Definition. Metric Topology: Let (X, d) be a metric space. We call T_d (as in the proposition above) the metric topology induced/generated by d.

Proposition: Let (X, d) be a metric space and let $Y \subseteq X$. Consider the subspace (Y, d') where $d' = d|_{Y \times Y}$ (restriction of d to $Y \times Y$). Then $T_{d'}$ is exactly the subspace topology of Y relative to (X, T_d).

Proof. Let $U \in T_{d'}$. Since U is open, this is equivalent to saying that for all $x \in U$ there is an $r(x) > 0$ such that $\{a \in Y : d'(x, a) < r(x)\} \subseteq U$ which is further equivalent to saying $Y \cap \{a \in X : d(x, a) < r(x)\} \subseteq U$. This is true if and only if $U = Y \cap (\bigcup_{x \in U} B_{r(x)}(x))$. However, $\bigcup_{x \in U} B_{r(x)}(x) \in T_d$, so this is true if and only if U is in the subspace topology on Y relative to (X, T). □

Definition. Hausdorff Topological Spaces: A topological space (X, T) is said to be Hausdorff if for all $x, y \in X$ with $x \neq y$ there is a $U, V \in T$ such that $x \in U$ and $y \in V$ but $U \cap V = \emptyset$.

Definition. Metrizable Topological Space: A topological space (X, T) is said to be metrizable if there is a metric d on X such that $T = T_d$.

Proposition: If (X, T) is metrizable then (X, T) is Hausdorff.

Proof. Suppose $T = T_d$ for some metric d. Let $x, y \in X$ with $x \neq y$. Let $r = d(x, y) > 0$ (since $x \neq y$). Then let $U = B_{r/2}(x)$ and $V = B_{r/2}(y)$. Clearly $x \in U$ and $y \in V$. Suppose $z \in U \cap V$. Then $d(x, z), d(y, z) < \frac{r}{2}$. But by the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r$$

a contradiction since $d(x, y) = r$. □

Example: Consider $X = \{a, b, c\}$ and $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then T is not metrizable since T is not Hausdorff. To see this pick c and (for instance) b.

4.3 Closed Sets

Definition. Closed Set: Let (X, T) be a topological space. We say $C \subseteq X$ is closed if $X \setminus C$ is open (i.e., $X \setminus C \in T$).

Remark: This also means $U \subseteq X$ is open if and only if $X \setminus U$ is closed.

Proposition: Let (X, T) be a topological space.

1. $\emptyset, X$ are closed.
2. If $\{C_\alpha\}_{\alpha\in I}$ are closed then $\bigcap_{\alpha\in I} C_\alpha$ is closed.
3. If $C, D \subseteq X$ are closed, then $C \cup D$ is closed.

Proof. The proof of all of these statements are immediate from De Morgan's laws and taking complements in the definition of a topology. □

Proposition: Let (X, T) be a topological space and let $Y \subseteq X$. Consider Y with the subspace topology T_Y. Then $C \subseteq Y$ is closed if and only if $C = Y \cap D$ for some closed $D \subseteq X$.

Proof. Note

$$\begin{array}{rrcl} & C & \subseteq & Y \text{ is closed} \\ \iff & Y\setminus C & \in & T_Y \\ \iff & Y \setminus C & = & Y \cap U,\ U \in T \\ \iff & C & = & Y \cap \underbrace{(X \setminus U)}_{D} \end{array}$$

where $X \setminus U = D$ is closed since $U \in T$. □

Definition. Limit Point: Let (X, T) be a topological space and let $A \subseteq X$. We say $x \in X$ is a limit point of A if for all $U \in T$ with $x \in U$, we have $U \cap A \neq \emptyset$.

Remark: If $x \in A$ and $x \in U \in T$, then $A \cap U \neq \emptyset$ since necessarily $x \in A \cap U$. Therefore x is a limit point in every set which contains it.

Example: Let $X = (\mathbb{R}, |\cdot|)$ and consider its standard metric topology. Let $A = \{1\} \cup (2, 3)$. Show 2 and 3 are limit points of A.

Proof. Let $U \subseteq X$ be an open set with $2 \in U$. Since U is open, there is an $r > 0$ such that $B_r(2) \subseteq X$. Then necessarily $2 + \frac{r}{2} \in U$, but for sufficiently small r we also have $2 + \frac{r}{2} \in (2, 3)$, thus $U \cap (2, 3) \neq \emptyset$. Since U was an arbitrary open set, 2 is a limit point of $(2, 3) \subseteq A$. A similar argument shows 3 is a limit point of A. We may in fact show that all the limit points of A are $\{1\} \cup [2, 3]$. □

Proposition: Let (X, d) be a metric space and let $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if there is a sequence $(a_n) \subseteq A$ such that $a_n \to x$.

Proof. ($\implies$) Suppose x is a limit point of A. Then $B_{1/n}(x) \cap A \neq \emptyset$ for all in $n \in \mathbb{N}$. In particular, for every $n \in \mathbb{N}$ let $a_n \in B_{1/n}(x) \cap A \neq \emptyset$. Then $d(a_n, x) < \frac{1}{n} \to 0$ so necessarily $a_n \to x$.

($\impliedby$) Suppose there is a sequence $(a_n) \subseteq A$ with $a_n \to x$. Let $U \subseteq X$ be an open set with $x \in U$. Then there is an $r > 0$ such that $B_r(x) \subseteq U$. Since $a_n \to x$, there is an a_N such that $d(a_N, x) < r$ meaning $a_N \in B_r(x)$ and necessarily $a_N \in A$, thus $\emptyset \neq B_r(x) \cap A \subseteq U \cap A$. □

Proposition: Let (X, T) be a topological space and let $A \subseteq X$. Then $A \subseteq X$ is closed if and only if A contains all of its limit points.

Proof. ($\implies$) Suppose $A \subseteq X$ is closed, then $X \setminus A$ is open. Then there can be no limit point x of A such that $x \in X \setminus A$ (i.e., $x \notin A$) since $(X \setminus A) \cap A = \emptyset$.

($\impliedby$) Suppose $A \subseteq X$ contains all its limit points. We claim $X \setminus A$ is open. To see this, let $x \in X \setminus A$ be arbitrary. Since $x \notin A$ then x is not a limit point of A, in particular, there is an open $U_x \subseteq X$ such that $x \in U_x$ and $U_x \cap A = \emptyset$. Note then that $U_x \subseteq X \setminus A$. Then by definition

$$X \setminus A \subseteq \bigcup_{x \in X \setminus A} U_x \subseteq X \setminus A \qquad \implies \qquad X \setminus A = \bigcup_{x \in X \setminus A} U_x$$

is open since arbitrary unions of open sets are open. □

Corollary: Let (X, d) be a metric space and let $A \subseteq X$. Then A is closed if and only if whenever $(a_n) \subseteq A$ is such that $a_n \to x \in X$, then $x \in A$.

4.4 Closure and Interior

Definition. Closure: Let (X, T) be a topological space and let $A \subseteq X$. Then the closure of A is

$$\overline{A} := \bigcap_{\substack{A \subseteq C \\ C \text{ is closed}}} C$$

Definition. Interior: Let (X, T) be a topological space and let $A \subseteq X$. Then the interior of A is

$$\text{Int}(A) := \bigcup_{\substack{U \subseteq A \\ U \text{ is open}}} U$$

Remark: Note that it is obvious that the closure of A is closed and the interior of A is open. This follows immediately from the fact that intersections of closed sets are closed and unions of open sets are open.

Remark:

1. $\overline{A}$ is the smallest closed set containing A.
2. $\text{Int}(A)$ is the largest open sets contained in A.
3. $\text{Int}(A) \subseteq A \subseteq \overline{A}$.
4. A is closed if and only if $A = \overline{A}$ and A is open if and only if $A = \text{Int}(A)$.

Proposition: Let (X, T) be a topological space and let $Y \subseteq X$. If $A \subseteq Y$ then $\overline{A}$ with respect to (Y, T_Y) is $Y \cap \overline{A}$ with respect to (X, T). (Note the definition of closure/interior depends on the topology in which we are working.)

Proof.

$$\begin{aligned}\overline{A} &= \bigcap\{C : A \subseteq C, C \text{ is closed w.r.t. } T_Y\} \\ &= \bigcap\{Y \cap C : A \subseteq C, C \text{ is closed w.r.t. } T\} \\ &= Y \cap \left(\bigcap\{C : A \subseteq C, C \text{ is closed w.r.t. } T\}\right) \\ &= Y \cap \overline{A}\end{aligned}$$

□

Proposition: Let (X, T) be a topological space and let $Y \subseteq X$. If $A \subseteq Y$ then $\text{Int}(A)$ with respect to (Y, T_Y) is $Y \cap \text{Int}(A)$ with respect to (X, T).

Proof.

$$\begin{aligned}\text{Int}(A) &= \bigcup\{U : U \subseteq A, U \text{ is open w.r.t. } T_Y\} \\ &= \bigcup\{Y \cap U : U \subseteq A, U \text{ is open w.r.t. } T\} \\ &= Y \cap \left(\bigcup\{U : U \subseteq A, U \text{ is open w.r.t. } T\}\right) \\ &= Y \cap \text{Int}(A)\end{aligned}$$

□

Proposition: Let (X, T) be a topological space and let $A \subseteq X$. Then

$$\overline{A} = \{x \in X : x \text{ is a limit point of } A\}$$

Proof. Let $L = \{x \in X : x \text{ is a limit point of } A\}$.

($\subseteq$) Let $x \in \overline{A}$. Let $U \in T$ (U is open) such that $x \in U$. By way of contradiction, suppose $A \cap U = \emptyset$, so that $A \subseteq X \setminus U$. Then necessarily since $X \setminus U$ is closed, $x \in \overline{A} \subseteq X \setminus U$. However, this is a contradiction since by construction $x \in U$. Therefore, $A \cap U \neq \emptyset$ and so x is a limit point of A (i.e., $x \in L$).

($\supseteq$) Suppose $x \in L$. Let C be closed, such that $A \subseteq C$. By way of contradiction, suppose $x \notin C$, then $x \in X \setminus C$ which is open. Then $(X \setminus C) \cap A \neq \emptyset$, however, $A \subseteq C$ and so $(X \setminus C) \cap A = \emptyset$, a contradiction. So $x \in C$ for any arbitrary closed C such that $A \subseteq C$. I.e.,

$$x \in \bigcap_{\substack{A \subseteq C \\ C \text{ is closed}}} C = \overline{C}$$

□

Definition. Interior Point: Let (X, T) be a topological space and let $A \subseteq X$. Then we say $x \in A$ is an interior point of A if there is an open $U \in T$ such that $x \in U \subseteq A$.

Remark: If (X, d) is a metric space, then $x \in A$ is an interior point if and only if there is an $r > 0$ such that $B_r(x) \subseteq A$.

Proposition: Let (X, T) be a topological space and let $A \subseteq X$. Then

$$\text{Int}(A) = \{x \in A : x \text{ is an interior point of } A\}$$

Proof. Let $I = \{x \in A : x \text{ is an interior point of } A\}$

($\subseteq$) Let $x \in \text{Int}(A)$ so that there is an open $U \in T$ such that $x \in U \subseteq A$. Then by definition $x \in I$.

($\supseteq$) Let $x \in I$ so that there is an open $U \in T$ such that $x \in U \subseteq A$. Then necessarily

$$x \in U \subseteq \bigcup_{\substack{U \subseteq A \\ U \text{ is open}}} U = \text{Int}(A)$$

□

Example: Let $(\mathbb{N}, |\cdot|)$ be the metric space of discussion. Notice $B_1(1) = \{1\}$ is closed. Further, note that

$$\overline{B_1(1)} = \overline{\{1\}} = \{1\} = B_1(1) \neq B_1[1] = \{1, 2\}$$

Similarly, note that $B_1[1] = \{1, 2\} = B_{1/2}(1) \cup B_{1/2})(2)$ is open and so

$$\text{Int}(B_1[1]) = \{1, 2\} = B_1[1] \neq B_1(1) = \{1\}$$

Remark: It is relatively easy to show, however, that in an NVS $(V, \|\cdot\|)$ we do in fact have

$$\overline{B_r(a)} = B_r[a] \qquad \text{and} \qquad \text{Int}(B_r[a]) = B_r(a)$$

Week 5 Continuity

5.1 Continuity

Definition. Continuous: Let (X, T_1) and (Y, T_2) be topological spaces. We say $f : X \to Y$ is continuous if

$$f^{-1}(U) := \{x \in X : f(x) \in U\} \in T_1$$

for all $U \in T_2$.

Proposition: Let (X, T_1) and (Y, T_2) be topological spaces and let $f : X \to Y$. The following are equivalent

1. f is continuous.
2. $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
3. If $C \subseteq Y$ is closed, then $f^{-1}(C) \subseteq X$ is closed.

Proof. (1 $\implies$ 2) Suppose f is continuous. Let $A \subseteq X$. We want to show $f(\overline{A}) \subseteq \overline{f(A)}$. Let $y \in f(\overline{A})$ so that $y = f(x)$ for some $x \in \overline{A}$. Let $U \subseteq Y$ be open with $y \in U$. Then $x \in f^{-1}(U)$ is open, and since x is a limit point of $\overline{A}$ there is an $a \in A$ with $a \in f^{-1}(U) \cap A \neq \emptyset$. Therefore $f(a) \in f(A) \cap U \neq \emptyset$. Since y was arbitrary, y is a limit point of $f(A)$ and so $y \in \overline{f(A)}$.

(2 $\implies$ 3) Suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $C \subseteq Y$ be closed. Let $A = f^{-1}(C)$. We want to show $A = \overline{A}$ (that A is closed). Let $x \in \overline{A}$. Then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C$$

Then clearly $x \in f^{-1}(C) = A$. Since $A \in \overline{A}$ was arbitrary, $A = \overline{A}$.

(3 $\implies$ 1) Assume $f^{-1}(C)$ for all closed $C \subseteq Y$. Let $U \subseteq Y$ be open. Then $C = Y \setminus U$ is closed and

$$f^{-1}(C) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$$

is closed. Therefore $f^{-1}(U)$ is open. □

Proposition: Let (X, d) and (Y, d') be metric spaces. Then $f : X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ whenever $(x_n) \subseteq X$ with $x_n \to x$.

Proof. ($\implies$) Suppose f is continuous. Let $(x_n) \subseteq X$ with $x_n \to x \in X$. Let $\varepsilon > 0$ be given and consider $U = B_\varepsilon(f(x))$. Since $x \in f^{-1}(U)$ is open, there is an $r > 0$ such that $B_r(x) \subseteq f^{-1}(U)$. Since $x_n \to x$, there is an $N \in \mathbb{N}$ such that $d(x_n, x) < r$ for all $n \geq N$. Then for all $n \geq N$ clearly

$$x_n \in B_r(x) \subseteq f^{-1}(U) \qquad \implies \qquad f(x_n) \in U = B_\varepsilon(f(x))$$

That is for all $n \geq N$ we have $d(f(x_n), f(x)) < \varepsilon$ so $f(x_n) \to f(x)$.

($\impliedby$) Suppose $f(x_n) \to f(x)$ whenever $x_n \to x$. Let $A \subseteq X$. If $x \in \overline{A}$ then there is a sequence $(a_n) \subseteq A$ with $a_n \to x$. Therefore $f(a_n) \to f(x)$ an so $f(x) \in \overline{f(A)}$. Since $x \in \overline{A}$ was arbitrary, we have $f(\overline{A}) \subseteq \overline{f(A)}$, and therefore f is continuous. □

5.2 Bounded Linear Maps

Definition. Bounded: Let V, W be normed vector spaces. Let $T : V \to W$ be a linear transformation. We say T is bounded if

$$\|T\|_{op} := \sup\{\|T(x)\| : \|x\| = 1\} < \infty$$

Definition. Operator Norm: $\|\cdot\|_{op}$ (as defined above) is a norm on the vector space of bounded linear maps $B(V, W)$. We call this norm the operator norm.

Proposition: Let V, W be normed vector spaces, and let $T : V \to W$ be linear. Then, T is continuous if and only if T is bounded.

Proof. ($\implies$) We prove by contrapositive. Suppose T is not bounded. Then for all $n \in \mathbb{N}$ there is an $x_n \in V$ with $\|x_n\| = 1$ such that $\|T(x_n)\| \geq n$. Then

$$\left\|\frac{x_n}{n}\right\| = \frac{1}{n}\|x_n\| = \frac{1}{n} \to 0$$

but

$$\left\|T\left(\frac{x_n}{n}\right)\right\| = \frac{1}{n}\|T(x_n)\| \geq \frac{1}{n} \cdot n = 1$$

So $T(\frac{x_n}{n}) \not\to T(0) = 0$, and therefore T is not continuous since it is not convergence preserving.

($\impliedby$) Suppose T is bounded. Let $(x_n) \subseteq V$ be such that $x_n \to x \in V$. If $x_n - x \neq 0$, then

$$\left\|\frac{x_n - x}{\|x_n - x\|}\right\| = 1$$

and therefore

$$\frac{1}{\|x_n - x\|}\|T(x_n) - T(x)\| = \left\|T\left(\frac{x_n - x}{\|x_n - x\|}\right)\right\| \leq \|T\|_{op}$$

this means

$$\|T(x_n) - T(x)\| \leq \|T\|_{op}\|x_n - x\| \to 0$$

by the squeeze theorem since $\|T\|_{op} < \infty$ is constant and $\|x_n - x\| \to 0$ since $x_n \to x$. Therefore T is continuous since T is convergence preserving. □

5.3 More Continuity

Definition. Uniform Continuity: Let (X, d) and (Y, d') be metric spaces. We say $f : X \to Y$ is uniformly continuous if for all $\varepsilon > 0$, there is a $\delta > 0$ such that $d'(f(a), f(b)) < \varepsilon$ whenever $a, b \in X$ are such that $d(a, b) < \delta$.

Remark: Notice uniform continuity means that for all choices of $\varepsilon > 0$ we can simultaneously show that f is continuous at every point $a \in X$ with the same δ for all points.

Definition. Lipschitz: Let (X, d) and (Y, d') be metric spaces. We say $f : X \to Y$ is Lipschitz if there is an $M > 0$ with $d(f(x), f(y)) \leq Md(x, y)$ for all $x, y \in X$.

Proposition: Let (X, d) and (Y, d') be metric spaces. If $f : X \to Y$ is Lipschitz then f is uniformly continuous.

Proof. Let f be such that $d(f(x), f(y)) \leq Md(x, y)$ for all $x, y \in X$. Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{M}$. Then whenever $x, y \in X$ are such that $d(x, y) < \delta$, we have

$$d(f(x), f(y)) \leq Md(x, y) < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

□

Example: Let $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. Show f is uniformly continuous but not Lipschitz.

Proof. Let $\varepsilon > 0$. Let $x, y \in [0,1]$ with $|x-y| < \delta := \varepsilon^2$. Then

$$|\sqrt{x}-\sqrt{y}|^2 = |\sqrt{x}-\sqrt{y}| \cdot |\sqrt{x}-\sqrt{y}| \leq |\sqrt{x}-\sqrt{y}| \cdot |\sqrt{x}+\sqrt{y}| = |x-y| < \varepsilon^2$$

so $|\sqrt{x}-\sqrt{y}| < \varepsilon^2$, proving f is uniformly continuous.

By way of contradiction, suppose f is Lipschitz so that there is an $M > 0$ with $|\sqrt{a}-\sqrt{b}| \leq M|a-b|$ for all $a, b \in [0,1]$. Without loss of generality, suppose $M > 1$ (if it holds for $M \leq 1$ it will also hold for $M' > 1 \geq M$). Then since $0, \frac{1}{M^4} \in [0,1]$ we have

$$\begin{aligned} \left|\frac{1}{\sqrt{M^4}} - 0\right| &\leq M\left|\frac{1}{M^4} - 0\right| \\ \frac{1}{M^2} &\leq \frac{1}{M^3} \\ M^2 &\geq M^3 \end{aligned}$$

Since $M > 1$, this is a contradiction. □

5.4 Isomorphisms

Remark: In a very broad sense, in mathematics we say two "objects" are isomorphic if they are the same "object" where one is just a relabelling of the other. These objects can include vector spaces, groups, rings, metric spaces, topological spaces, etc. Recall from MATH 146 that vector spaces are isomorphic if there exists an isomorphism (bijection) between them.

Remark. Motivation of Homeomorphisms: Let (X, T_1) and (Y, T_2) be topological spaces and let $f : X \to Y$. What does it mean for (X, T_1) to be isomorphic to or "the same" as (Y, T_2)? We clearly want f to bijective so that the set Y is a relabelling of the set X. We also want the open sets to be the same up to relabelling. In particular, whenever U is open we should have $f(U)$ is open. It is sufficient to impose then that f and f^{-1} both be continuous.

To see that having f and f^{-1} be continuous, notice that if $U \subseteq X$ is open then note that $f(U) = (f^{-1})^{-1}(U)$ is open by the continuity of f^{-1}. If $V \subseteq Y$ is open then note that $U = f^{-1}(V)$ is open by the continuity of f and $V = f(U)$ since f is bijective.

Definition. Homeomorphism: Let (X, T_1) and (Y, T_2) be topological spaces. We say $f : X \to Y$ is homeomorphism is f is bijective, f is continuous, and f^{-1} is continuous. If such an f exists, we say (X, T_1) and (Y, T_2) are homeomorphic. Homeomorphisms are the isomorphisms on the category of topological spaces.

Example: The topological space $(\{0,1\}, \{\emptyset, \{0,1\}, \{1\}\})$ is homeomorphic to the topological space $(\{a,b\}, \{\emptyset, \{a,b\}, \{b,\}\})$.

Example: Consider $f : [0, 2\pi) \to \{(x,y) : x^2+y^2 = 1\}$ where each is equipped with the usual norm to induce a topology given by $f(\theta) = (\cos\theta, \sin\theta)$. Then clearly f is a continuous bijection, however, these two spaces are not homeomorphic. To see this notice that $[0, 2\pi)$ is not compact, while $\{(x,y) : x^2+y^2 = 1\}$ is (more on compactness in week 7).

Remark. Motivation of Isometric Isomorphisms: Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. What does it mean for (X, d) to be isomorphic to or "the same" as (Y, d')? We again want f to be bijective so that Y is a relabelling of X. But we also want the metrics to be "the same". That is we want $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Definition. Isometry: Let (X, d) and (Y, d') be metric spaces. We say $f : X \to Y$ is an isometry if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Definition. Isometric Isomorphism: Let (X, d) and (Y, d') be metric spaces. We say $f : X \to Y$ is an isometric isomorphism if it is both an isometry and a bijection. In this case we say (X, d) and (Y, d') are isometrically isomorphic. Isometric isomorphisms are isomorphisms on the category of metric spaces.

Proposition: Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$ be an isometry. Then f is continuous and injective.

Proof. Since f is an isometry, we can clearly see f is Lipschitz with $M = 1$. We also see that f is injective since if $f(x) = f(y)$, then $d'(f(x), f(y)) = 0 = d(x, y)$ and so $x = y$ since d is a metric. □

Proposition: Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. If f is an isometric isomorphism then f^{-1} is an isometric isomorphism.

Proof. Let $x_1, x_2 \in X$ and let $y_1 = f(x_1), y_2 = f(x_2)$. Then notice that $d'(f(x_1), f(x_2)) = d(x_1, x_2)$ and so $d'(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$. Notice also inverses of bijections are bijective. Therefore f^{-1} is an isometric isomorphism. □

Proposition: Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. If f is an isometric isomorphism then f is a homeomorphism between (X, T_d) and $(Y, T_{d'})$.

Proof. By the above propositions, we know both f and f^{-1} are isometric isomorphisms and thus are both continuous. So since f is a bijection and f and f^{-1} are continuous, f is a homeomorphism. □

Example: Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$. Then f is a homeomorphism but not an isometric isomorphism. To see this, note for instance that $|0 - 2| = 2$ but $|f(0) - f(2)| = |0 - 8| = 8$.

Example: Let $B_1 = \{(a, b) : a, b \in \mathbb{R}\}$ and $B_2 = \{(a, b] : a, b \in \mathbb{R}\}$. Then $(\mathbb{R}, T_{B_1})$ and $(\mathbb{R}, T_{B_2})$ are not homeomorphic since $(\mathbb{R}, T_{B_1})$ is second-countable, whereas $(\mathbb{R}, T_{B_2})$ is not second-countable.

Example: Notice that ℓ^1 and ℓ^∞ are not isometrically isomorphic since ℓ^1 is separable but ℓ^∞ is not.

5.5 Urysohn's Lemma

Definition. Normal Topological Space: Let (X, T) be a topological space. We say (X, T) is normal if for all closed $C, D \subseteq X$ with $C \cap D = \emptyset$, there exists $U, V \in T$ such that $U \cap V = \emptyset$ and $C \subseteq U$ and $D \subseteq V$.

Proposition: Let (X, T) be a topological space. If (X, T) is metrizable, then (X, T) is normal.

Proof. See A3Q4. □

Theorem. Urysohn's Lemma: Let (X, d) be a metric space and let $A, B \subseteq X$ be closed with $A \cap B = \emptyset$. Then there exists a continuous $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Proof. Let (X, d) be a metric space. Let $A \subseteq X$ be closed. Define

$$d_A(x) := \inf\{d(x, a) : a \in A\}$$

Remark that if $d_A(x) = 0$, then for all $n \in \mathbb{N}$ there is a $a_n \in A$ with $d(x, a_n) < \frac{1}{n}$. Then $a_n \to x$ and so $x \in A$. That is, $d_A(x) = 0$ if and only if $x \in A$.

Remark that for all $x, y \in X$ and any $a \in A$ we have

$$\begin{array}{rrcl} & d_A(x) & \leq & d(x,a) \leq d(x,y) + d(y,a) \\ \Longrightarrow & d_A(x) - d(x,y) & \leq & d(y,a) \\ \Longrightarrow & d_A(x) - d(x,y) & \leq & d_A(y) \\ \Longrightarrow & d_A(x) - d_A(y) & \leq & d(x,y) \\ \Longrightarrow & |d_A(x) - d_A(y)| & \leq & d(x,y) \end{array}$$

Where the third inequality holds since $d_A(x) - d(x, y)$ formed a lower bound on $d(y, a)$ for an arbitrary $a \in A$ and so $d_A(y)$ being the greatest lower bound (infimum) must be greater. We have then that $d_A : X \to \mathbb{R}$ is Lipschitz.

Now let $A, B \subseteq X$ be closed with $A \cap B = \emptyset$ as above. We claim that

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}$$

is as above. It is obvious that if $a \in A$ then $d_A(x) = 0$ so $f(a) = 0$, and so since a was arbitrary $f|_A = 0$. Conversely if $b \in B$ then $d_B(b) = 0$ and so $f(b) = \frac{d_A(b)}{d_A(b)} = 1$, since b was arbitrary $f|_B = 1$. □

5.6 Completions

Definition. Dense: Let (X, T) be a topological space. We say $A \subseteq X$ is dense in X if $\overline{A} = X$.

Definition. Completion: Let (X, d) be a metric space. A completion of (X, d) is a complete metric space (Y, d') such that X is isometrically isomorphic to a dense subset of Y.

Example: Show $\mathbb{R}$ is a completion of $\mathbb{Q}$.

Proof. It is known that $\mathbb{Q}$ is dense in $\mathbb{R}$ and clearly $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ which is isometrically isomorphic to $\mathbb{Q}$. (Under the standard metric.) □

Lemma: Let (X, d) be complete. Then $A \subseteq X$ is complete if and only if A is closed.

Proof. ($\implies$) Let $A \subseteq X$ be complete. If $a_n \to x$ where $(a_n) \subseteq A$ and $x \in X$, then (a_n) is Cauchy. Since A is complete and limits are unique, $x \in A$. Since A contains its limit points, A is closed.

($\impliedby$) Let A be closed. If $(a_n) \subseteq A$ is Cauchy, then $(a_n) \subseteq X$ is Cauchy. Since X is complete, $a_n \to x \in X$. Since A is closed and limits are unique, $x \in A$. Since every Cauchy sequence in A is convergent, A is closed. □

Proposition: Let (X, d) be a metric space. Denote

$$C_b(x) := \{f : X \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$$

Define $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Then $(C_b(X), \|\cdot\|_\infty)$ is a Banach space.

Proof. We will prove this fact in a few weeks. □

Theorem. Completion Theorem: Every metric space has a completion.

Proof. Let (X, d) be a metric space. Fix $a_0 \in X$. Consider $\varphi : X \to C_b(x)$ where $\varphi(a) = f_a$ where $f_a(x) = d(x, a) - d(x, a_0)$. Notice $f_a \in C_b(X)$. To see this, let $a \in X$ then for all $x \in X$

$$f_a(x) = d(x, a) - d(x, a_0) \leq d(x, a_0) + d(a_0, a) - d(x, a_0) = d(a, a_0) < \infty$$

Since x was arbitrary $\sup\{f(x) : x \in X\} < \infty$, so f_a is bounded. To see that f_a is continuous, let $a \in X$ and let $(x_n) \subseteq X$ be such that $x_n \to x$ for some $x \in X$. Then

$$\begin{aligned} |f_a(x_n) - f_a(x)| &= |d(x_n, a) - d(x_n, a_0) - d(x, a) + d(x, a_0)| \\ &\leq |d(x_n, a) - d(x, a)| + |d(x_n, a_0) - d(x, a_0)| \\ &\leq |d(x_n, x)| + |d(x_n, x)| \qquad (*) \\ &= 2d(x_n, x) \to 0 \end{aligned}$$

where (*) holds by the reverse triangle inequality: $d(x, y) - d(y, z) \leq d(x, z)$ (this is a result immediate from the triangle inequality). Since f_a is convergence preserving, f is continuous.

Notice for all $a, b \in X$

$$\begin{aligned}\|\varphi(a) - \varphi(b)\|_\infty &= \sup\{|f_a(x) - f_b(x)| : x \in X\} \\ &= \sup\{|d(x,a) - d(x,a_0) - d(x,b) + d(x,a_0)| : x \in X\} \\ &= \sup\{|d(x,a) - d(x,b)| : x \in X\}\end{aligned}$$

However,

$$\begin{array}{rrcl} & d(x,a) & \leq & d(x,b) + d(a,b) \\ \implies & d(x,a) - d(x,b) & \leq & d(a,b) \\ \implies & |d(x,a) - d(x,b)| & \leq & d(a,b) \\ \implies & \|\varphi(a) - \varphi(b)\|_\infty & \leq & d(a,b) \end{array}$$

Where the third inequality holds since if $d(x,a) \leq d(x,b)$, swapping a and b would result in $d(x,a) \geq d(x,b)$ (and $d(a,b) = d(b,a)$). Note further that $|f_a(b) - f_b(b)| = d(a,b)$ and so

$$\|\varphi(a) - \varphi(b)\| = \sup\{|f_a(x) - f_b(x)| : x \in X\} = d(a,b)$$

So φ is an isometry. Therefore, X is isometrically isomorphic to $\varphi(X)$ since isometries are injective and $\varphi : X \to \varphi(X)$ is necessarily surjective. Since $\overline{\varphi(X)} \subseteq C_b(X)$ is closed and $C_b(X)$ is a Banach space, $\overline{\varphi(X)}$ is complete. Note also that necessarily $\varphi(X)$ is dense in $\overline{\varphi(X)}$, so $(\overline{\varphi(X)}, \|\cdot\|_\infty)$ is a completion for X. □

Week 6 Connectedness

6.1 Connectedness

Definition. Connected: Let (X, T) be a topological space. We say (X, T) (or X for short) is connected if there does not exist open, disjoint, non-empty sets $U, V \subseteq X$ such that $X = U \cup V$.

Remark: That is X is connected if you cannot break apart X into two open, disjoint sets.

Notation: Let (X, T) be a topological space and let $A \subseteq X$. If we say A is connected, we meant A is connected with respect to the subspace topology.

Example: Consider the standard topology on $\mathbb{Q}$. We will show $\mathbb{Q}$ is not connected. To see this, consider $((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cup ((\sqrt{2}, \infty) \cap \mathbb{Q})$. Each of these sets is open with respect to the subspace topology.

Example: Let $X = \mathbb{R}$ and let $T = T_B$ where $B = \{[a,b) : a < b\}$. Since we saw that all open sets in T_B are closed, we can separate $\mathbb{R} = [0,1) \cup (\mathbb{R} \setminus [0,1))$. So $\mathbb{R}$ is not connected with respect to T_B.

Notation: Many books will say a simultaneously open and closed set is clopen.

Proposition: Let (X, T) be a topological space. Then X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

Proof. ($\implies$) Let $A \subseteq X$ be open and closed and also $A \neq X$ and $A \neq \emptyset$. Then $X = A \cup (X \setminus A)$ which are disjoint, non-empty, and since A is both open and closed, both are open. Therefore X is not connected. Then by contrapositive the result holds.

($\impliedby$) Let X be not connected. Then there are open, disjoint, non-empty sets U, V such that $X = U \cup V$. Then $X \setminus U = V$, so since U open, V is open and closed. □

Proposition: Let $a < b$. Then $I = [a, b] \subseteq \mathbb{R}$ is connected. In fact, we may generalize that every interval in $\mathbb{R}$ is connected.

Proof. By way of contradiction, suppose I is not connected, so that there exists open $U, V \subseteq \mathbb{R}$ such that $I = (U \cap I) \cup (V \cap I)$, such that $(U \cap I) \cap (V \cap I) = \emptyset$, and such that $U \cap I, V \cap I \neq \emptyset$. Without loss of generality, suppose $a \in U \cap I$. Let $x = \sup\{t : [a, t] \subseteq U \cap I\}$. We consider two cases.

Case 1: Suppose $x \in U$. Since U is open, there is an $r > 0$ such that $(x - r, x + r) \cap I \subseteq U \cap I$. This contradicts the definition of x, unless $x = b$ in which case $V \cap I = \emptyset$, which is a contradiction since $V \cap I$ is non-empty.

Case 2: Suppose $x \in V$. Since V is open, there is an $r > 0$ such that $(x - r, x + r) \cap I \subseteq V \cap I$. Then since $x - \frac{r}{2} < x$, by definition of x we have $[a, x - \frac{r}{2}] \subseteq U \cap I$ and therefore $x - \frac{r}{2} \in U \cap I$. However, we also clearly have $x - \frac{r}{2} \in V \cap I$, therefore $U \cap V \neq \emptyset$, a contradiction.

We conclude I must be connected. Similar proofs extend this result to all intervals in $\mathbb{R}$. □

Proposition: If $A \subseteq \mathbb{R}$ is connected, then A is an interval.

Proof. We may assume $A \neq \emptyset$. Let $a = \inf A$ and $b = \sup A$ (note $a, b \in \overline{\mathbb{R}}$). By way of contradiction, suppose there is a $a < c < b$ with $c \notin A$. Then picking $U = (\infty, c) \cap A$ and $V = (c, \infty) \cap A$ we have U, V are open, disjoint, and non-empty, so that A is not connected. This is since for all $x \in A$ we have $x \in U = (\infty, c) \cap A$ or $x \in V = (c, \infty) \cap A$ but never $x = c$ since $c \notin A$. This is a contradiction since A is connected. □

Proposition: Let (X, T_1) and (Y, T_2) be topological space and let $f : X \to Y$. If f is continuous and X is connected, then $f(X)$ is connected. (Note we could also consider the subspace topology for any $A \subseteq X$).

Proof. Suppose there are $U, V \subseteq Y$ which are open such that $f(X) = (f(X) \cap U) \cup (f(x) \cap V)$ where each of $f(X) \cap U$ and $f(X) \cap V$ is disjoint. Note this means that $X = (X \cap f^{-1}(U)) \cup (X \cap f^{-1}(V))$ where $f^{-1}(U), f^{-1}(V) \subseteq X$. But then this means we have $X = f^{-1}(U) \cup f^{-1}(V)$. By the continuity of f and the openness of U and V, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open. Further, we know $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint since we can't have $x \in f^{-1}(U) \cap f^{-1}(V)$ which maps to $f(x) \in U \cap V = \emptyset$. Since X is connected, assume without loss of generality that $f^{-1}(U) = \emptyset$. Then $f(X) \cap U = \emptyset$ since if $u \in U$ then $f^{-1}(u) \in f^{-1}(U) = \emptyset$. So $f(X) \cap U = \emptyset$. □

Corollary. Intermediate Value Theorem: abbr. IVT. If $a, b \in \mathbb{R}$ with $a \leq b$ and $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is an interval.

Corollary. General Intermediate Value Theorem: If (X, T) is connected and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval.

Proposition: Let (X, T) be a topological space and let $A \subseteq X$. If A is connected then $\overline{A}$ is connected.

Proof. Let $\overline{A} = (\overline{A} \cap U) \cup (\overline{A} \cap V)$ where $(\overline{A} \cap U)$ and $(\overline{A} \cap V)$ are disjoint and where $U, V \subseteq X$ are open. Then $A = (A \cap U) \cup (A \cap V)$. Since $(\overline{A} \cap U)$ and $(\overline{A} \cap V)$ are disjoint, then $(A \cap U)$ and $(A \cap V)$ are disjoint. Since A is connected, suppose without loss of generality, suppose that $A \cap V = \emptyset$ so that $A = A \cap U$.

This means, however, that

$$A = A \cap U \subseteq \overline{A} \cap U = \overline{A} \cap (X \setminus V)$$

However, since V is open, $\overline{A} \cap (X \setminus V)$ is closed. By the smallness of closures, we have $\overline{A} \subseteq \overline{A} \cap (X \setminus V)$. Therefore, $\overline{A} \cap V = \emptyset$, and so $\overline{A}$ is connected. □

6.2 Path Connectedness

Definition. Path: Let (X, T) be a topological space and let $a, b \in X$. A path from a to b is a continuous function $f : [0, 1] \to X$ with $f(0) = a$ and $f(1) = b$.

Example: Intuitively in a space such as $\mathbb{R}^2$ this makes a graph (i.e., line) going from a to b.

Definition. Path Connectedness: Let (X, T) be a topological space. We say X is path connected if for all $a, b \in X$ there is a path in X from a to b.

Proposition: Let (X, T) be a topological space. If X is path connected then X is connected.

Proof. By way of contradiction, suppose X is path connected, but not connected. In particular, suppose $X = U \cup V$ where U and V are non-empty, disjoint, open sets. Since U, V are non-empty, let $a \in U$ and $b \in V$ so that there is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. Then $[0, 1] = f^{-1}(U) \cup f^{-1})(V)$. By the continuity of f, we have $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty, open, and disjoint. However, $[0, 1]$ is connected so we have a contradiction. □

Proposition: Every normed vector space is connected.

Proof. Let V be a normed vector space. Let $u, v \in V$ be arbitrary. Then there is a path in V from u to v given by $f(t) : [0, 1] \to V$ where $f(t) = tu + (1 - t)v$. □

6.3 Connected Components

Remark: The idea is that if a topological space X is not connected, we should be able to write it as a union of connected pieces, called X's components. That is, we can partition X is connected pieces.

Definition. Connected Components: Let (X, T) be a topological space and let $x \in X$. Define $C_x := \bigcup\{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}$. We call C_x the connected component of x.

Lemma: Let (X, T) be a topological space. Let $C_\alpha \subseteq X$ be connected where $\alpha \in I$ for some index set I. If $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$ then $C = \bigcup_{\alpha \in I} C_\alpha$ is connected.

Proof. Suppose $C = (C \cap U) \cup (C \cap V)$ where $C \cap U$ and $C \cap V$ are disjoint and $U, V \subseteq X$ are open. For $\alpha \in I$ we have then that $C_\alpha = (C_\alpha \cap U) \cup (C_\alpha \cap V)$. By the connectedness of C_α, suppose without loss off generality that $C_\alpha \cap V = \emptyset$ so that $C_\alpha = C_\alpha \cap U$. Let $\beta \in I$. We can't have then $C_\beta = C_\beta \cap V$. Because we have $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$ but this would imply $C_\alpha \cap C_\beta = \emptyset$. So we must be able to conclude that $C \cap V = \emptyset$, and therefore that C is connected. □

Corollary: Let (X, T) be a topological space and let $x \in X$. The connected component C_x is connected.

Proposition: Let (X, T) be a topological space and let $x \in X$. The connected component C_x is closed.

Proof. Since C_x is connected, $\overline{C_x}$ is connected and since $x \in \overline{C_x}$, we have that $\overline{C_x} \subseteq C_x$. Therefore C_x is closed. □

Proposition: Let (X, T) be a topological space. The connected components of X partition X.

Proof. Let $x, y \in X$ and let C_x, C_y be the corresponding connected components. Suppose $C_x \cap C_y \neq \emptyset$. Then $C_x \cup C_y$ is connected. Further, by the largeness of connected components $C_x \subseteq C_x \cup C_y \subseteq C_y$ and by the same argument $C_y \subseteq C_x$. □

Remark: Let (X, T) be a topological space and let $x \in X$. Define

$$P_x = \bigcup\{A : x \in A \subseteq X \ \ A \text{ is path connected}\}$$

to be the path connected component of x. Let $y, z \in P_x$, then we know there is a path from y to x and a path from x to z, so P_x is path connected. Notice the path connected components partition X. Finally note that for all $x \in X$, we have $P_x \subseteq C_x$ since path connected implies connected.

Example. Topologist's Sine Curve: Let $A = \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$ and further let $X = A \cup \{(0, 0)\} \subseteq \mathbb{R}^2$. We claim X is connected. To see this, note that A is path connected and therefore connected, but since $\lim_{x \to 0} \sin(\frac{1}{x}) = 0$, we have $\overline{A} = X$ and so X is connected.

We also claim X is not path connected. By way of contradiction, suppose there is a continuous $f : [0,1] \to X$ with $f(0) = (0,0)$ and $f(1) = (\frac{1}{\pi}, 0)$. Then we can write $f(t) = (a(t), b(t))$, where $a, b : [0,1] \to \mathbb{R}$ is continuous. By the continuity of a and the Intermediate Value Theorem there is a $0 < t_1 < 1$ such that $a(t_1) = \frac{2}{3\pi}$. By a second application of IVT there is a $t_2 < t_1$ such that $a(t_2) = \frac{2}{5\pi}$. By continuing this way we get a decreasing sequence $(t_n) \subseteq [0,1]$ such that $a(t_n) = \frac{2}{(2n+1)\pi}$. Note since $(t_n) \subseteq [0,1]$ is decreasing, by the monotone convergence theorem we know $t_n \to t \in [0,1]$. By continuity we know $b(t_n) \to b(t)$, however $b(t_n) = (-1)^n$ for all $n \in \mathbb{N}$ which does not converge. Hence we have a contradiction, the only assumption we made that X is path connected.

Note this also shows that path connected components are not necessarily closed. For instance we know that $(0,0)$ is a limit point of A which is not in A and we know A is path connected.

Week 7 Compactness I

7.1 Compactness in Topological Spaces

Remark: Recall some results of compactness

1. (Heine-Borel) $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.
2. $A \subseteq \mathbb{R}^n$ is compact if and only if every open cover of A has a finite sub-cover.
3. $A \subseteq \mathbb{R}^n$ is compact if and only if every $(a_n) \subseteq A$ has a subsequence $(a_{n_k}) \subseteq (a_n)$ with $a_{n_k} \to a \in A$.
4. (Extreme Value Theorem) Let $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be continuous, then f attains its maximum and minimum.
5. Continuous functions on compact domains are uniformly continuous.
6. Images of compact sets on continuous functions are compact.
7. If $K \subseteq \mathbb{R}^n$ is compact, then $C(K)$ is a Banach space with $\|\cdot\|_\infty$.

Definition. Open Cover: Let (X, T) be a topological space. An open cover of X is a collection $\{U_\alpha : \alpha \in I\} \subseteq T$ where I is an index set such that $X = \bigcup_{\alpha \in I} U_\alpha$.

Definition. Subcover: Let (X, T) be a topological space. A subcover of a cover $\{U_\alpha : \alpha \in I\} \subseteq T$ is a collection $\{U_\alpha : \alpha \in J\}$ where $J \subseteq I$ and where $X = \bigcup_{\alpha \in J} U_\alpha$. If J is finite, we call this a finite subcover.

Definition. Compactness: Let (X, T) be a topological space. We say X is compact if every open cover of X has a finite subcover.

Example: Consider $(0, 1] \subseteq$ with the standard subspace topology. Then $(0,1] = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1]$ is an open cover $(0, 1]$ with no finite subcover.

Example: Consider $\mathbb{R}$ with the standard topology. Then notice $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$ is an open cover of $\mathbb{R}$ with no finite subcover.

Example: Let $A = \{x \in \ell^\infty : \exists n \in \mathbb{N}, x = e_n\}$ where $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ is the sequence with a 1 in the ith component and 0's elsewhere. Then $A = \bigcup_{n=1}^{\infty} B_1(e_n)$ is an open cover of A. Notice for all $i, j \in \mathbb{N}$ we have $\|e_i - e_j\|_\infty = 1$, so no finite subcover contains all e_i.

Lemma. Shortcut Lemma: Note shortcut lemma is not a commonly used name. Let (X, T) be a topological space and let $Y \subseteq X$. Then Y is compact (with respect to T_Y) if and only if whenever $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ where $U_\alpha \subseteq X$ is open in X then there are $\alpha_1, \ldots, \alpha_n \in I$ such that $Y \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. That is, if every open cover of Y in X has a finite sub-cover.

Proof. Let $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ this is true if and only if $Y = \bigcup_{\alpha \in I}(Y \cap U_\alpha)$. Then if we have that one can be finitely covered clearly the other can be finitely covered. □

Proposition: Let (X, T) be a compact topological space. If $Y \subseteq X$ is closed then Y is compact.

Proof. Let $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$ where $U_\alpha \in X$ is open. Therefore $X = (\bigcup_{\alpha \in I} U_\alpha) \cup (X \setminus Y)$ since $X \setminus Y$ is open. Since X is compact there is a finite sub-cover for X, i.e., there is $\alpha_1, \ldots, \alpha_n$ such that $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X \setminus Y)$ (note $X \setminus Y$ might not be necessary for this cover). Therefore, we also have that $Y \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. So by the shortcut lemma Y is compact. □

Proposition: Let (X, T) be a Hausdorff topological space. If $Y \subseteq X$ is compact, then Y is closed.

Proof. We will show $X \setminus Y$ is open. Fix $x \in X \setminus Y$. Since X is Hausdorff, for all $y \in Y$ there are open sets $x \in U_y$ and $y \in V_y$ such that $U_y \cap V_y = \emptyset$. Then clearly $Y \subseteq \bigcup_{y \in Y} V_y$. And so, there are $y_1, \ldots, y_n$ such that $Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$. Then we know $x \in U_x := U_{y_1} \cap \cdots \cap U_{y_n}$ and that U_x is open. We know that $U_x \subseteq X \setminus Y$ because for all $y \in U_x \cap Y$ we must have that $y \in V_{y_i}$ for some i and also that $y \in U_{y_i}$, a contradiction by our Hausdorff assumption. Then $X \setminus Y = \bigcup_{x \in X \setminus Y} U_x$, and so $X \setminus Y$ is open. □

Example: Consider $X = \mathbb{R}$ and let $T = \{A \subseteq \mathbb{R} : \mathbb{R} \setminus A \text{is finite}\} \cup \{\emptyset\}$. Note the closed sets are $\mathbb{R}$ and all finite sets. Let $x, y \in X$. Let $A \subseteq \mathbb{R}$. Suppose $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ (without loss of generality suppose $U_\alpha \neq \emptyset$ for all $\alpha \in I$). Take $\alpha_0 \in I$. Then $\mathbb{R} \setminus \alpha_0$ is finite. Notice then we have $A \setminus U_{\alpha_0}$ is finite. In particular suppose $A \setminus U_{\alpha_0} = \{a_1, \ldots, a_n\}$. Then find α_i so that for all $1 \leq i \leq n$ we have $a_i \in U_{\alpha_i}$. So $A \subseteq U_{\alpha_0} \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

Proposition: Let (X, T_1) and (Y, T_2) be topological spaces. Let $f : X \to Y$ be continuous. If X is compact then $f(X)$ is compact.

Proof. Let $f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$. Since f is compact, $X \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$ where each $f^{-1}(V_\alpha)$ is open. Since X is continuous $X \subseteq f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$. Therefore $f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. □

Proposition: Let (X, T_1) be a compact topological space. Let (Y, T_2) be a Hausdorff topological space. Let $f : X \to Y$ be bijective and continuous. Then f is a homeomorphism. (I.e. f^{-1} is continuous.)

Proof. We want to show f^{-1} is continuous. Let $C \subseteq X$ be closed. Since C is closed, it is compact. Then $(f^{-1})^{-1}(C) = f(C)$ is compact. Since $f(C) \subseteq Y$ is compact and Y is Hausdorff, we know $f(C)$ is closed. □

Proposition. Generalized Nested Intervals: Let (X, T) be a compact topological space. Let $C_1 \supseteq C_2 \supseteq \cdots$ be closed and non-empty. Then $C = \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Proof. By way of contradiction, suppose $C = \emptyset$ so that $X \setminus C = X$. However, $X = X \setminus C = \bigcup_{n \in \mathbb{N}} (X \setminus C_n)$ where each $X \setminus C_n$ is open. Then there are $n_1 < n_2 < \cdots < n_m$ so that

$$X = (X \setminus C_{n_1}) \cup \cdots \cup (X \setminus C_{n_m}) = X \setminus C_{n_m}$$

So $C_{n_m} = \emptyset$, a contradiction since we assumed each C_n was non-empty. □

7.2 Compactness in Metric Spaces

Definition. Totally Bounded: Let (X, d) be a metric space. We say X is totally bounded if for all $\varepsilon > 0$ there are finitely $a_1, \ldots, a_n \in X$ such that $X = B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_n)$.

Example: Let $X = \{x \in \ell^\infty : \|x\|_\infty = 1\}$. Then X is clearly bounded, but is not totally bounded. In particular, let $\varepsilon = 1$. Then any $B_\varepsilon(a_i)$ can only contain a single e_i and therefore there cannot be a finite union of balls which covers X.

Definition. Subsequence: A subsequence of a sequence $(a_n) \subseteq A$ is a sequence of the form $(a_{n_k})_{k=1}^\infty$ where $n_1 < n_2 < \cdots$.

Lemma: Let (X, d) be a metric space. Then X is totally bounded if and only if every sequence in X has a Cauchy subsequence.

Proof. ($\implies$) Suppose X is totally bounded. Let $(x_n) \subseteq X$. Since X is totally bounded, there is a $b_1 \in X$ such that $T_1 := \{n : x_n \in B_1(b_1)\}$ is infinite. Since (x_n) is infinite and by the total boundedness of X there are finitely many sets which cover X, so such a set must exist. Similarly, there is a $b_2 \in X$ such that $T_2 := \{n \in T_1 : x_n \in B_{1/2}(b_2)\}$, this is again since T_1 is infinite. Continuing, we may pick $n_1 < n_2 < \cdots$ such that $x_{n_k} \in B_{1/k}(b_k)$. For $k > \ell$ we have that $x_{n_k}, x_{n_\ell} \in B_{1/\ell}(B_\ell)$ since $T_k \subseteq T_\ell$. Then

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, b_\ell) + d(b_\ell, x_{n_\ell}) \leq \frac{1}{\ell} + \frac{1}{\ell} = \frac{2}{\ell} \to 0$$

($\Longleftarrow$) Suppose by way of contrapositive X is not totally bounded. Then $\exists \varepsilon > 0$ such that $X = \bigcup_{x \in X} B_\varepsilon(x)$ has no finite subcover (since otherwise X would be totally bounded by definition). Fix $x_1 \in X$. Then there is an $x_2 \notin B_\varepsilon(x_1)$, as otherwise $B_\varepsilon(x_1)$ is a finite subcover. There is again an $x_3 \notin B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ as otherwise $B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ is a finite subcover. Continuing infinitely we have $(x_n) \subseteq X$ such that $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$. Then (x_n) cannot have any Cauchy subsequence. □

Remark: Recall from MATH 247 we proved that a NVS V is compact if and only if every sequence has a convergent subsequence.

Example: Show that if X is totally bounded then X separable. For all $n \in \mathbb{N}$ we may write $X = B_{1/n}(a_1^{(n)}) \cup \cdots \cup B_{1/n}(a_{k_n}^{(n)})$. Then $D := \{a_i^{(j)}\}$ is a countable dense subset of X.

Definition. Sequentially Compact: Let (X, d) be a metric space. We say X is sequentially compact if and only if every sequence in X has a convergent subsequence.

Lemma. Lebesgue Number Lemma: Let (X, d) be a sequentially compact metric space. Let $X = \bigcup_{\alpha \in I} U_\alpha$ be an open cover. Then there is an $R > 0$ (called the Lebesgue number) such that for all $x \in X$, $B_R(x) \subseteq U_\alpha$ for some $\alpha \in I$.

Proof. By way of contradiction, suppose no such R exists. In particular, for all $n \in \mathbb{N}$, there is an $a_n \in X$ such that for all $\alpha \in I$ then $B_{1/n}(a_n) \not\subseteq U_\alpha$. By the sequential compactness of X there is a subsequence with $a_{n_k} \to a \in X$. Say $a \in U_{\alpha_0}$. Then for a sufficiently large N, $B_{1/N}(a) \subseteq U_{\alpha_0}$ (since U_{α_0} is open). Then for sufficiently large k, we have $B_{1/n_k}(a_{n_k}) \subseteq B_{1/N}(a) \subseteq U_{\alpha_0}$ by the convergence of a_{n_k}. This is a contradiction by our assumption. □

Theorem. Characterization Theorem of Compactness: Let (X, d) be a metric space. The following are equivalent:

1. X is sequentially compact,
2. X is complete and totally bounded,
3. X is compact.

Proof. ($1 \iff 2$) Note X is totally bounded if and only if every sequence has a Cauchy subsequence and X is sequentially compact if and only if every subsequence has a convergent subsequence. These are equivalent where X is complete.

($1 \implies 3$) Let $X = \bigcup_{\alpha \in I} U_\alpha$ be an open cover. Let R be the Lebesgue number for this open cover. Since X is totally bounded (by 2), we may say $X = B_R(x_1) \cup \cdots \cup B_R(x_n)$. But for all x_i, there is an $\alpha_i \in I$ such that $B_R(x_i) \subseteq U_{\alpha_i}$, so we may write $X \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

($3 \implies 1$) Let $(x_n) \subseteq X$. For all $n \in \mathbb{N}$, define $C_n := \overline{\{x_k : k \geq n\}}$. Since we know C_n is closed, we know it is compact. By the generalized nested intervals, we know $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Take $x \in \bigcap_{n \in \mathbb{N}} C_n$. We may then find n_1 such that $d(x_{n_1}, x) < 1$. We may also find $n_2 > n_1$

such that $d(x_{n_2}, x) < \frac{1}{2}$. We can repeatedly do this to get a subsequence $(x_{n_k})_{k=1}^{\infty}$ where $x_{n_k} \to x$. Note that we know this is the case because x is as close as we may want to all tail ends our sequence. □

Proposition: Let (X, d) and (Y, d') be metric spaces. Let $f : X \to Y$. If X is compact and f is continuous, then f is uniformly continuous.

Proof. By way of contradiction, suppose f is not uniformly continuous. In particular, there is an $\varepsilon > 0$ and sequences $(a_n), (b_n) \subseteq X$ such that $d(a_n, b_n) < \frac{1}{n}$ but $d'(f(a_n), f(b_n)) \geq \varepsilon$ since no $\frac{1}{n}$ for $n \in \mathbb{N}$ is a valid choice of δ. Since X is compact, say $a_{n_k} \to a \in X$. Then by the triangle inequality

$$d(b_{n_k}, a) \leq d(b_{n_k}, a_{n_k}) + d(a_{n_k}, a) \to 0$$

so $b_{n_k} \to a$. But by the continuity of f we have $f(a_{n_k}) \to f(a)$ and $f(b_{n_k}) \to f(a)$. But by our assumption we said that $d'(f(a_n), f(b_n)) \geq \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. □

Week 8 Compactness II

8.1 Compactness in $\mathbb{R}^n$

Lemma: Let $a < b \in \mathbb{R}$. Then $[a, b]$ is totally bounded.

Proof. Let $\varepsilon > 0$. Let n be such that $\frac{b-a}{n} < \varepsilon$. Then let $x_i = a + \frac{b-a}{n} \cdot i$ for $0 \leq i \leq n$. Then $[a, b] \subseteq \bigcup_{i=0}^{n} B_\varepsilon(x_i)$.

Note technically we need $[a, b] = \bigcup_{i=0}^{n} B_\varepsilon(x_i)$, but we can just intersect each $B_\varepsilon(x_i)$ with $[a, b]$ to get the desired result. In general $Y \subseteq \bigcup_{i=0}^{n} B_\varepsilon(x_i)$ if and only if $Y = \bigcup_{i=0}^{n} (Y \cap B_\varepsilon(x_i))$. □

Remark: We may similarly show $[a, b]^n \subseteq \mathbb{R}^n$ is totally bounded.

Remark: Note that $A \subseteq \mathbb{R}^n$ is bounded if and only if $A \subseteq [-R, R]^n$ if and only if A is totally bounded.

Theorem. Heine-Borel Theorem: $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Proof. Since $\mathbb{R}^n$ is complete, A is closed if and only if A is complete (since closed subsets of complete spaces are complete). By our remark we also have that A is bounded if and only if it is totally bounded. But we know A is complete and totally bounded if and only if it is compact. □

Theorem. Extreme Value Theorem: Abbr. EVT. If (X, d) is a compact metric space and $f : X \to \mathbb{R}$ is continuous, Then f attains its max and minimum on X.

Proof. Since f is continuous, $f(X)$ is compact, and so $f(X)$ is closed and bounded. Let $m = \inf f(x)$ and $M = \sup f(x)$. Then for all $\varepsilon > 0$ there is an $f(x_1) < m + \varepsilon$ (since $m + \varepsilon$ is not a greatest lower bound) and an $f(x_2) > M - \varepsilon$ (since $M - \varepsilon$ is not a least upper bound). Then for any $\varepsilon > 0$ with $|f(x_1) - m| < \varepsilon$ and $|f(x_2) - M| < \varepsilon$. We could form sequences $(a_n), (b_n)$ such that $a_n \to x_1$ and $b_n \to x_2$. So we know that $m, M \in \overline{f(X)} = f(X)$, as desired. □

8.2 Finite Dimensional Normed Vector Spaces

Remark: Let V be a normed vector space. Recall from A%, two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on V are strongly-equivalent if and only if there are $C, D > 0$ such that

$$C\|x\|_a \leq \|x\|_b \leq D\|x\|_a$$

for all $x \in V$.

Theorem: Let V be a finite dimensional normed vector space. Any two norms on V are strongly-equivalent.

Proof. Assume V is a vector space over $\mathbb{R}$. Let $\{b_1, b_2, \ldots, b_n\}$ be a basis for V. Note since V is n-dimensional, it is isomorphic to $\mathbb{R}^n$ (i.e., there is an invertible linear map $T : V \to \mathbb{R}^n$). Recall any $v \in V$ may be uniquely written in the form $v = \sum_{i=1}^{n} c_i b_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Define

$$\|v\|_2 := \left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2} = \|T(v)\|_2$$

is a norm on V for some $T : V \to \mathbb{R}^n$ such that $T(b_i)e_i$ for all $1 \leq i \leq n$. Moreover, we have that $(V, \|\cdot\|_2)$ is isometrically isomorphic to $(\mathbb{R}^n, \|\cdot\|_2)$. Let $\|\cdot\|$ be an arbitrary norm on V. We have then

$$\|v\| \overset{TI}{\leq} \sum_{i=1}^{n} |c_i| \cdot \|b_i\| \overset{CS}{\leq} \left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2} \cdot \underbrace{\left(\sum_{i=1}^{n} \|b_i\|^2\right)^{1/2}}_{D} = D\|v\|_2$$

by the triangle inequality and the Cauchy-Schwartz inequality (note we can view $(|c_1|, \ldots, |c_n|)$ and $(\|b_1\|, \ldots, \|b_n\|)$ as vectors in $\mathbb{R}^n$, hence why we can view the sum as a dot product). Consider $f : (V, \|\cdot\|_2) \to \mathbb{R}$ where $f(x) = \|x\|$. Notice f is defined on V equipped with $\|\cdot\|_2$ but is giving $\|x\|$ (different norms). Then

$$|f(x) - f(y)| = \big|\|x\| - \|y\|\big| \leq \|x - y\| \leq D\|x - y\|_2$$

by the reverse triangle inequality. So f is Lipschitz and therefore continuous. Let $S = \{v \in V : \|v\|_2 = 1\}$ be the unit circle. Then we know S is closed and bounded in $(V, \|\cdot\|_2)$. But since V is isometrically isomorphic to $\mathbb{R}^n$, we have by Heine-Borel that S is compact. Then by the extreme value theorem, $f|_S : S \to \mathbb{R}$ achieves its minimum. In particular, there is a $v_0 \in S$ (e.g., $\|v_0\|_2 = 1$) such that

$$\|v_0\| = \min f(S) = \min\{\|x\| : \|x\|_2 = 1\} =: C$$

(Note we know $v_0 \neq 0$ since $\|v_0\|_2 = 1$.) For any $0 \neq v \in V$,

$$\left\| \frac{v}{\|v\|_2} \right\| \geq C$$

since $\frac{v}{\|v\|_2}$ is a vector with norm 1 and we found above that $\|u\| \geq C$ for all vectors with $\|u\|_2 = 1$. Then $\|v\| \geq C\|v\|_2$, as desired. □

Remark: Suppose V is a finite dimensional normed vector space over $\mathbb{R}$ with norm $\|\cdot\|$. We know that

$$C\|v\|_2 \leq \|v\| \leq D\|v\|_2$$

Now consider

$$f : (V, \|\cdot\|_2) \to (V, \|\cdot\|) \qquad v \mapsto v$$

Clearly f is invertible and therefore a bijection. We also have

$$\|f(v) - f(w)\| = \|v - w\| \leq D\|v - w\|_2$$

so f is Lipschitz. We can similarly show f^{-1} is Lipschitz. So $(V, \|\cdot\|)$ is homeomorphic to $(V, \|\cdot\|_2)$. We also saw in the above proof that $(V, \|\cdot\|)$ is homeomorphic to $\mathbb{R}^n$. That is, every finite dimensional normed vector space over $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$.

8.3 The Cantor Set

Definition. Cantor Set: Let $C_0 = [0, 1]$. Let $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. In general, for each C_i we remove the open middle third of each set in the union.

Let $C = \bigcap_{n=1}^{\infty} C_n$ be the Cantor set.

Remark: Note that each C_i is closed because its the finite union of closed sets. Since C is the infinite intersection of these sets, C is closed. We also know that C is non-empty, as for instance $0, 1 \in C$. Since $C \subseteq \mathbb{R}$ is closed and bounded, C is compact.

Remark: The Cantor set contains no non-empty open intervals. In particular, no matter what open interval U we pick, we can find a C_n such that $U \not\subseteq C_n$. This means $\text{Int}(C) = \emptyset$, or otherwise we say C is nowhere dense.

Remark: For all $x \in C$, the connected component $C_x = \{x\}$. We say that C is totally disconnected.

Notation: Think of the number 0 as left, the number 2 as right. We can also write C_1 as

$$C_1 = \underbrace{[0, \tfrac{1}{3}]}_{I_0} \cup \underbrace{[\tfrac{2}{3}, 1]}_{I_2} = I_0 \cup I_2$$

we can do this also for C_2:

$$C_2 = \underbrace{[0, \tfrac{1}{9}]}_{I_{00}} \cup \underbrace{[\tfrac{2}{9}, \tfrac{1}{3}]}_{I_{02}} \cup \underbrace{[\tfrac{2}{3}, \tfrac{7}{9}]}_{I_{20}} \cup \underbrace{[\tfrac{8}{9}, 1]}_{I_{22}}$$

In general, when constructing C_{n+1} from C_n, for each set

$$I_{a_1 a_2 \cdots a_n} \subseteq C_n$$

we use the set

$$I_{a_1 a_2 \cdots a_n 0} \cup I_{a_1 a_2 \cdots a_n 2} \subseteq C_{n+1}$$

Remark: We can show that $I_{a_1 \cdots a_n} = [[0.a_1 a_2 \cdots a_n]_3, [0.a_1 a_2 \cdots a_n]_3 + \frac{1}{3^n}]$ where $[0.a_1 a_2 \cdots a_n]$ is base three number with decimals points $a_1 a_2 \cdots a_n$. Then

$$C = \bigcap_{n \in \mathbb{N}} \left[[0.a_1 a_2 \cdots a_n]_3, [0.a_1 a_2 \cdots a_n]_3 + \frac{1}{3^n} \right] = \{[0.a_1 a_2 \cdots]_3 : a_i \in \{0, 2\}\}$$

So there is a bijection between C and the set of sequences of 0's and 2's. In particular, there is a bijection between C and the sequence of 0's and 1's. Therefore $|C| = |2^{\mathbb{N}}| = c$. So as a topological space the Cantor set is small but as a set it is large.

Theorem: If (X, d) is compact, then there is a continuous surjection $\varphi : C \to X$, where C is the Cantor set.

Week 9 Arzela-Ascoli Theorem

9.1 Uniform Convergence

Definition. Function Convergence: Let (X, d) and (Y, d') be metric spaces. Let $f_n : X \to Y$ be a sequences of functions.

1. We say (f_n) converges to $f : X \to Y$ pointwise if $f_n(x) \to f(x)$ for all $x \in X$.
2. We say (f_n) converges to $f : X \to Y$ uniformly if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $\|f_n - f\|_\infty := \sup\{d'(f_n(x), f(x)) : x \in X\} < \varepsilon$.

Remark. Uniform Norm: The function $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ is called the uniform norm, but note this is not necessarily an actual norm. E.g., there is no guarantee that $\|f\|_\infty < \infty$.

Remark:

1. Suppose $f_n \to f$ uniformly. Then for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $x \in X$ we have that $d'(f_n(x), f(x)) \leq \|f_n - f\|_\infty < \varepsilon$. Therefore, $f_n \to f$ pointwise and given $\varepsilon > 0$ we can find an N that proves $f_n(x) \to f(x)$ for all x (uniformly) at the same time.

2. Recall
$$C_b(X) = \{f : X \to \mathbb{R} \mid f \text{ is bounded and continuous}\}$$
where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Then $(C_b(X), \|\cdot\|_\infty)$ is a normed vector space where $\|\cdot\|_\infty$ is the uniform norm. Note this is a well defined norm due to the boundedness of functions in $C_b(X)$. Therefore, we see that $f_n \to f$ uniformly if and only if $f_n \to f$ in $(C_b(X), \|\cdot\|_\infty)$.

Proposition: Let (X, d) and (Y, d') be metric spaces. Let $f_n : X \to Y$ be a sequence of continuous functions. Then

1. If $f_n \to f$ uniformly, then f is also continuous.
2. If $(f_n) \subseteq C_b(X)$ and $f_n \to f$ uniformly, then $f \in C_b(X)$.

Proof. (1) Let $(x_n) \subseteq X$ with $x_n \to x \in X$. We claim that $f(x_n) \to f(x)$ and therefore that f is continuous. Let $\varepsilon > 0$. Since $f_n \to f$, there is an $N \in \mathbb{N}$ such that $\|f_N - f\|_\infty < \frac{\varepsilon}{3}$. Therefore, since f_N is continuous, there is an $M \in \mathbb{N}$ such that $d'(f_N(x_n), f_N(x)) < \frac{\varepsilon}{3}$ for all $n \geq M$. So, by the triangle inequality

$$\begin{aligned} d'(f(x_n), f(x)) &\leq d'(f(x_n), f_N(x_n)) + d'(f_N(x_n), f_N(x)) + d'(f_N(x), f(x)) \\ &\leq \|f - f_N\|_\infty + d'(f_N(x_n), f_N(x)) + \|f_N - f\|_\infty \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

(2) If $(f_n) \subseteq C_b(x)$ and $f_n \to f$ uniformly, then by (1) f is continuous. Recall since $f_n \to f$ uniformly there is an $N \in \mathbb{N}$ such that $\|f_N - f\|_\infty < 1$. In particular we have that

$$\|f\|_\infty \leq \underbrace{\|f - f_N\|_\infty}_{<1} + \underbrace{\|f_N\|_\infty}_{<\infty} < \infty$$

□

Example: Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = x^n$. Let $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{otherwise} \end{cases}$$

Then $f_n \to f$ pointwise, but each f_n is continuous whereas f is not.

Proposition: Let (X, d) be a metric space. Then $(C_b(x), \|\cdot\|_\infty)$ is a Banach space.

Proof. Let $(f_n) \subseteq C_b(X)$ be Cauchy. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$ for all $n, m \geq N$. In particular, for all $x \in X$ and $n, m \geq N$ we have that $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon$. Therefore, $(f_n(x))_{n=1}^\infty$ is Cauchy, but since $\mathbb{R}$ is complete this implies there is an $f(x) \in \mathbb{R}$ such that $f_n(x) \to f(x)$. Therefore, we may construct a function $f : X \to \mathbb{R}$ such that $f_n \to f$ pointwise (picking the function values to the be limit point $f(x)$).

Since limits are unique, we know that if $f_n \to f$ uniformly, it must be the f given above. For $n \geq N$ we have that

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty = \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \varepsilon$$

(since $\|f_n - f_m\| < \varepsilon$ for $n, m \geq N$). Since $x \in X$ was arbitrary, we have that $\|f_n - f\|_\infty \leq \varepsilon$ and so $f_n \to f$ uniformly. Therefore, by our previous result $f_n \to f \in C_b(X)$. □

Remark: Recall this proposition is something we called a fact in the proof for the completion theorem (module 5.6).

Remark: If (X, d) is a compact metric space, then $C_b(X) = C(X)$ where $C(X)$ is the set of continuous functions from X to $\mathbb{R}$.

9.2 Compactness in $C(X)$

Example: Let $A = \{f_n(x) = x^n : n \in \mathbb{N}\} \subseteq C([0, 1])$. For all $n \in \mathbb{N}$, we have that $\|f_n\|_\infty = 1$. Therefore we see that A is bounded. We claim A is closed. To see this, let $(g_n) \subseteq A$ be convergent. We will show $g_n \to g \in C([0, 1])$. Note we have for all n that $g_n = f_{n_k}$ for some $k \in \mathbb{N}$ (e.g., $g_n \in A$).

Consider two cases. Suppose there is an $m \in \mathbb{N}$ such that $g_n = f_m$ for infinitely many g_n, then (g_n) has a subsequence $(f_m, f_m, f_m, \ldots)$. Since subsequences and sequences must share limits, and clearly $f_m \to f_m$ then $g_n \to f_m \in A$.

Otherwise, there is a subsequence of (g_n) which is a subsequence of (f_n) as to have infinitely many different f_m terms in (g_n), we need eventually that $m \to \infty$. Therefore, since

$$f_n \to f = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{otherwise} \end{cases}$$

Then again since subsequences share limits with their sequences, every subsequence of (f_n) has $f_n \to f$ pointwise. Therefore, $g_n \to f$ pointwise. But recall we mentioned that the convergence $f_n \to f$ is pointwise but not uniform, therefore $g_n \to f$ pointwise but not uniformly. In particular, this means $g_n \not\to f$ in $C([0, 1])$.

Note also that A is not compact. To see this, note that (f_n) has no convergent subsequence, since again such a subsequence would need to converge to f which is not possible. So, despite $[0, 1]$ being compact by Heine-Borel, $C([0, 1])$ is not.

Remark: Since $C(X)$ is complete, $K \subseteq C(X)$ is compact if and only if K is complete and totally bounded. Since $C(X)$ is complete, K is compact if and only if it closed and totally bounded.

Remark. Investigation of Compact Subsets of $C(X)$: We want to try and remove the condition that $K \subseteq C(X)$ needs to be totally bounded to be compact. In particular, we want to find some property such that K is compact if and only if it closed, bounded, and satisfies our property.

Suppose K is compact. Let $\varepsilon > 0$. Since K is totally bounded then there are $f_1, \dots, f_n \in K$ such that $K \subseteq B_\varepsilon(f_1) \cup \dots \cup B_\varepsilon(f_n)$. Since each $f_i \in C(X)$, it is uniformly continuous. So for all $1 \leq i \leq n$ there is a $\delta_i > 0$ such that if $a, b \in X$ are such that $d(a,b) < \delta_i$ then $|f_i(a) - f_i(b)| < \varepsilon$. Let $\delta = \min\{\delta_1, \dots, \delta_n\}$. Let $f \in K$ be arbitrary. Let $a, b \in X$ with $d(a,b) < \delta$. Then since $f \in K$ and $K \subseteq B_\varepsilon(f_1) \cup \dots \cup B_\varepsilon(f_n)$ there is an f_i such that $\|f - f_i\|_\infty < \varepsilon$ and so

$$|f(a) - f(b)| \leq |f(a) - f_i(a)| + \underbrace{|f_i(a) - f_i(b)|}_{<\varepsilon} + |f_i(b) - f(b)| \leq 2\|f - f_i\|_\infty + \varepsilon < 3\varepsilon$$

So for all $f \in K$ we have shown f is uniformly continuous. Furthermore, we proved so uniformly, that is for all $\varepsilon > 0$ there is a $\delta > 0$ which proves that f is uniformly continuous for all $f \in K$.

Definition. Equicontinuous: Let (X, d) be a compact metric space. We say $A \subseteq C(X)$ is equicontinuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $f \in A$ we have that

$$d(a,b) < \delta \qquad \Longrightarrow \qquad |f(a) - f(b)| < \varepsilon$$

Proposition: Let (X, d) be a compact metric space. If $K \subseteq C(X)$ is compact, then K is equicontinuous.

Proof. See investigation above. □

Theorem. Arzela-Ascoli Theorem: If (X, d) is a compact metric space, then $K \subseteq C(X)$ is compact if and only if K is closed, bounded, and equicontinuous.

Proof. ($\Longrightarrow$) This follows from our above proposition and an earlier remark.

($\Longleftarrow$) Suppose $K \subseteq C(X)$ is closed, bounded, and equicontinuous. Note that since K is closed and $C(X)$ is complete, K is complete. It remains to show that K is totally bounded.

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be so that

$$d(a,b) < \delta \qquad \Longrightarrow \qquad |f(a) - f(b)| < \frac{\varepsilon}{4}$$

as per the equicontinuity of K. Since X is compact and therefore totally bounded, there are $x_1, \dots, x_n \in X$ such that $X = B_\delta(x_1) \cup \dots \cup B_\delta(x_n)$. Consider

$$T : C(X) \to (\mathbb{R}^n, \|\cdot\|_\infty) \qquad f \mapsto (f(x_1), f(x_2), \dots, f(x_n))$$

Note then that

$$\|T(f)\|_\infty = \max\{|f(x_1)|, \dots, |f(x_n)|\} \leq \sup\{|f(x)| : x \in X\} = \|f\|_\infty$$

Since K is bounded, by definition $\|f\|_\infty$ is bounded for all $f \in K$ and therefore $T(K)$ is bounded. This means that $\overline{T(K)} \subseteq \mathbb{R}^n$ is also bounded, and therefore since it is closed, it compact by Heine-Borel. Therefore, $\overline{T(K)}$ is also totally bounded and so there are $f_1, f_2, \ldots, f_m \in K$ such that

$$\overline{T(K)} \subseteq B_{\varepsilon/4}(T(f_1)) \cup \cdots \cup B_{\varepsilon/4}(T(f_m))$$

Note this follows since for all $a \in \overline{T(K)}$, there is a $T(f) \in T(K)$ which is arbitrarily close to a. Let $f \in K$ be arbitrary. Then there is a j such that $\|T(f) - T(f_j)\|_\infty < \frac{\varepsilon}{4}$ since $T(f) \in \overline{T(K)}$ and we saw $\overline{T(K)}$ is totally bounded. Let $y \in X$ be arbitrary. Then since X is totally bounded there is an i with $d(x_i, y) < \delta$.

Notice that since $d(x_i, y) < \delta$, we have $|f(y) - f(x_i)| < \frac{\varepsilon}{4}$ and $|f_j(y) - f_j(x_i)| < \frac{\varepsilon}{4}$. Notice also that

$$\begin{aligned}|f(x_i) - f_j(x_i)| &\leq \max\{|f(x_1) - f_j(x_1)|, \ldots, |f(x_n) - f_j(x_n)|\} \\ &= \|T(f) - T(f_j)\|_\infty < \frac{\varepsilon}{4}\end{aligned}$$

Therefore,

$$\begin{aligned}|f(y) - f_j(y)| &\leq |f(y) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}\end{aligned}$$

Therefore since $y \in X$ was arbitrary

$$\|f - f_j\|_\infty = \sup\{|f(y) - f_j(y)| : y \in X\} \leq \frac{3\varepsilon}{4} < \varepsilon$$

Therefore since $f \in K$ was arbitrary, we have that $K \subseteq B_\varepsilon(f_1) \cup \cdots \cup B_\varepsilon(f_m)$. So K is totally bounded, and as remarked previously K is complete, thereby showing K is compact, as desired. □

Example: Fix $M > 0$. Let $K = \{f \in C([0,1]) : \forall x, y |f(x) - f(y)| \leq M|x - y|\}$. That is, K is the set of all $C([0,1])$ with the same Lipschitz constant M. Notice, however, K is not bounded as it is unbounded.

Fix $L > 0$. Let $K' = \{f \in C([0,1]) : \forall x, y |f(x) - f(y)| \leq M|x - y|, |f(0)| \leq L\}$. We show K is compact.

(Equicontinuous) Let $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{M}$. If $f \in K$ and $x, y \in [0,1]$ with $|x - y| < \delta$ then

$$|f(x) - f(y)| \leq M|x - y| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

(Bounded) If $x \in [0,1]$ then by the reverse triangle inequality,

$$|f(x)| - |f(0)| \leq |f(x) - f(0)| \leq M|x - 0| \leq M$$

so $|f(x)| \leq L + M$ since $|f(0)| \leq L$. Therefore, for $f \in K$ we have $\|f\|_\infty \leq L + M$.

(Closed) Let $(f_n) \subseteq K$ such that $f_n \to f \in C([0,1])$ uniformly. So $f_n(0) \to f(0)$. Then $|f_n(0)| \leq L$ for all $n \in \mathbb{N}$ we necessarily also have $|f(0)| \leq L$. For $x, y \in [0,1]$ and $n \in \mathbb{N}$ we have that

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq M|x-y| \\ |f(x) - f(y)| &\leq M|x-y| && \text{Taking } n \to \infty \end{aligned}$$

Note this follows since limits preserve order. We can see then that $f \in K$, so K is closed.

Week 10 Baire Category Theorem

10.1 First and Second Category

Definition. F_σ Set: Let (X,d) be a metric space. We say $A \subseteq X$ is a F_σ set if $A = \bigcup_{n=1}^{\infty} C_n$, where $C_n \subseteq X$ is closed.

Definition. G_δ Set: Let (X,d) be a metric space. We say $A \subseteq X$ is a G_δ set if $A = \bigcap_{n=1}^{\infty} U_n$, where $U_n \subseteq X$ is open.

Note: Notice that we can see a set $A \subseteq X$ is F_σ if and only if its complement $X \setminus A$ is G_δ

Definition. Nowhere Dense Set: Let (X,d) be a metric space. We say $A \subseteq X$ is nowhere dense if $\text{int}(\overline{A}) = \emptyset$.

Note: Notice this means that a closed $A \subseteq X$ is nowhere dense if and only if its interior is empty.

Definition. First Category: Let (X,d) be a metric space. We say $A \subseteq X$ is of first category if $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \subseteq X$ is nowhere dense.

Definition. Second Category: Let (X,d) be a metric space. If $A \subseteq X$ is not first category, we say it is of second category in X.

Definition. Residual Set: Let (X,d) be a metric space. We say $A \subseteq X$ is residual if $X \setminus A$ is of first category.

Example: Consider $A = [0,1)$. We may write this as $A = \bigcup_{n=1}^{\infty} [0, 1 - \frac{1}{n}]$, therefore A is F_σ. However, we may also write this as $A = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, 1)$, therefore A is G_δ.

Example: Let $C \subseteq X$ be closed. Let $U_n = \bigcup_{x \in C} B_{1/n}(x)$ so that U_n is open. Prove that $C = \bigcap_{n=1}^{\infty} U_n$.

Proof. Clearly, $C \subseteq \bigcap_{n=1}^{\infty} U_n$ since each $x \in C$ is in U_n for all $n \in \mathbb{N}$. Let $x \in \bigcap_{n=1}^{\infty} U_n$. So there is a $c_n \in C$ such that $d(x, c_n) < \frac{1}{n}$. We see then that $c_n \to x$, so $x \in C$. □

Example: Let C be the cantor set. Recall C is closed and notice $\text{int}(C) = \emptyset$.

Example: Clearly $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$. Since $\{q\}$ is nowhere dense for all $q \in \mathbb{Q}$, we have that $\mathbb{Q}$ is of first category. Notice then the irrationals $\mathbb{R} \setminus \mathbb{Q}$ are thus residual.

Remark: Let (X, d) be a metric space and let $A \subseteq X$. Recall from 247 that $\overline{X \setminus A} = X \setminus \text{Int}(A)$ and $\text{Int}(X \setminus A) = X \setminus \overline{A}$.

Suppose A is nowhere dense in X. Consider $\overline{X \setminus \overline{A}} = X \setminus \text{Int}(\overline{A}) = X \setminus \emptyset = X$. So $X \setminus \overline{A}$ is dense in X.

Remark: (Idea around first category sets) First category sets are in a sense "topologically thin" in that they are the union of nowhere dense sets. Residual sets are therefore "topologically fat" and very big.

Theorem. Baire Category Theorem: Let (X, d) be a complete metric space. Let $U_n \subseteq X$ be open and dense in X for $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X.

Proof. Let $x \in X$. Let $\varepsilon > 0$ be given. We may find $x_1 \in X$ and $0 < r_1 < 1$ such that $B_{r_1}[x_1] \subseteq B_\varepsilon(x) \cap U_1$ (since $B_\varepsilon(x) \cap U_1$ is open and since U_1 is dense in X we know r_1 exists). Similarly, we may find $x_2 \in X$ and $0 < r_2 < \frac{1}{2}$ such that $B_{r_2}[x_2] \subseteq B_{r_1}(x_1) \cap U_2$. Continuing, we construct a sequence $B_{r_{k+1}}[x_{k+1}] \subseteq B_{r_k}(x_k) \cap U_{k+1}$ with $0 < r_{k+1} < \frac{1}{k+1}$.

By an assignment, we know there is an $x_0 \in \bigcap_{k=1}^{\infty} B_{r_k}[x_k] \neq \emptyset$. Then we see that $x_0 \in \bigcap_{n=1}^{\infty} U_n$ since each $B_{r_k}[x_k] \subseteq U_k$. Recall also that $x_0 \in B_{r_1}[x_1] \subseteq B_\varepsilon(x)$. That is, for all $x \in X$ and $\varepsilon > 0$, we may find an $x_0 \in \bigcap_{n=1}^{\infty} U_n$ with $d(x_0, x) < \varepsilon$, thereby showing $x_0 \in \bigcap_{n=1}^{\infty} U_n$ is dense. □

Corollary: A complete metric space (X, d) is of second category in itself.

Proof. By way of contradiction, suppose $X = \bigcup_{n=1}^{\infty} A_n$ where $\text{Int}(\overline{A_n}) = \emptyset$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} X &= \bigcup_{n=1}^{\infty} A_n \\ &= \bigcup_{n=1}^{\infty} \overline{A_n} \\ \emptyset &= \bigcap_{n=1}^{\infty} X \setminus \overline{A_n} \end{aligned}$$

Recall from our remark that $X \setminus \overline{A}$ is dense in X if A is nowhere dense in X and clearly $X \setminus \overline{A}$ is open. Therefore by the Baire Category Theorem $\emptyset$ is dense in X, which is a contradiction. □

10.2 Applications

Proposition: $\mathbb{Q} \subseteq \mathbb{R}$ is not a G_δ set.

Proof. By way of contradiction, suppose $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ where $U_n \subseteq \mathbb{R}$ is open. Since $\mathbb{Q} \subseteq U_n$ is dense in $\mathbb{R}$, then $\mathbb{R} \subseteq \overline{\mathbb{Q}} \subseteq \overline{U_n} \subseteq \mathbb{R}$ and so U_n is dense in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, suppose $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Set $V_n = U_n \setminus \{q_n\}$ so that V_n is still open and dense. Therefore we know that $\bigcap_{n=1}^{\infty} V_n = \emptyset$, but by the Baire Category theorem, $\bigcap_{n=1}^{\infty} V_n$ is dense in $\mathbb{R}$, a contradiction. □

Definition. Oscillation: Let (X, d) be a metric space and let $f : X \to \mathbb{R}$. We define the oscillation of f at x to be

$$\omega_f(x) = \inf_{\delta > 0} \sup \left\{ |f(a) - f(b)| : a, b \in B_\delta \right\}$$

Remark: The oscillation measures how much f can bounce around as you get closer and closer to x.

Note: f is continuous at x if and only if $\omega_f(x) = 0$.

Lemma: Let (X, d) be a metric space and let $f : X \to \mathbb{R}$. Let $\varepsilon > 0$ be given. Then $U = \{x \in X : \omega_f(x) < \varepsilon\}$ is open.

Proof. Let $x \in U$. For some $\delta > 0$, we have that $\sup\{|f(a) - f(b)| : a, b \in B_\delta(x)\} < \varepsilon$. Take $y \in X$ such that $r := d(y, x) < \delta$. Then the $B_{\delta - r}(y) \subseteq B_\delta(x)$. To see this, let $z \in B_{\delta-r}(y)$, then

$$d(z, x) \leq d(z, y) + d(y, x) < \delta - r + r = \delta$$

Therefore,

$$\sup\{|f(a) - f(b)| : a, b \in B_{\delta-r}(y)\} \leq \sup\{|f(a) - f(b)| : a, b \in B_\delta(x)\} < \varepsilon$$

Therefore, $\omega_f(y) < \varepsilon$. And so, for any $y \in B_\delta(x)$ we have $y \in U$. Therefore, $B_\delta(x) \subseteq U$, showing that is U is open. □

Proposition: There is no $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every rational but discontinuous at each irrational.

Proof. By way of contradiction, suppose such a function f exists. Let $C_n = \{x \in \mathbb{R} : \omega_f(x) \geq \frac{1}{n}\}$. We know by our lemma that C_n is closed. Then since f is continuous at the rationals but discontinuous at the rationals, we know $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n \in \mathbb{N}} C_n$. Therefore, by De Morgan's law $\mathbb{Q} = \bigcap_{n \in \mathbb{N}} \mathbb{R} \setminus C_n$ where $\mathbb{R} \setminus C_n$ is open. This implies $\mathbb{Q}$ is G_δ, a contradiction. □

Example: Consider Thomae's function

$$f:\mathbb{R}\to\mathbb{R} \qquad x\mapsto\begin{cases}0 & x\notin\mathbb{Q}\\ \frac{1}{n} & x=\frac{m}{n}\in\mathbb{Q}\\ 1 & x=0\end{cases}$$

which is continuous at each irrational but discontinuous at each rational.

Proposition: Let $(f_n)\subseteq C([a,b])$. If $f_n\to f$ pointwise then f is continuous on a residual subset of $[a,b]$

Proof. Let $C_n=\{x\in[a,b]:\omega_f(x)\geq\frac{1}{n}\}$. Recall C_n is closed since its complement is open. Therefore f is discontinuous on $\bigcup_{n=1}^{\infty}C_n$, therefore if this set is of first category, then f is continuous on a residual set. We are then required to show $\text{Int}(C_n)=\text{Int}(\overline{C_n})=\emptyset$.

By way of contradiction, suppose there is a $K\in\mathbb{N}$ and an open interval I such that $I\subseteq C_K$. (Technically we mean open with respect to the $[a,b]$ but open intervals in $[a,b]$ are simply open intervals.) If there is an open interval contained in C_K then C_K is not nowhere dense. Let $0<\varepsilon<\frac{1}{3K}$.

For all $i,j\in\mathbb{N}$, let $X_{ij}=\{x\in\overline{I}:|f_i(x)-f_j(x)|\leq\varepsilon\}$. Also let $E_n=\bigcap_{i,j\geq n}X_{ij}$. Fix $x\in\overline{I}$. We know that $f_n(x)\to f(x)$. In particular, we know there is an $N\in\mathbb{N}$ such that $|f_n(x)-f(x)|<\frac{\varepsilon}{2}$ for $n\geq N$. By the triangle inequality, for $i,j\geq N$ then $|f_i(x)-f_j(x)|\leq|f_i(x)-f(x)|+|f(x)-f_j(x)|<\varepsilon$. So we have that for any $x\in\overline{I}$ there is a $N\in\mathbb{N}$ so that $x\in E_N$ (since for all $i,j\geq N$ we have $x\in X_{i,j}$ as shown above). So $\overline{I}=\bigcup_{n=1}^{\infty}E_n$. Note also that $\overline{I}$ is complete by Heine-Borel ($\overline{I}$ is a closed, bounded interval). Note that by the continuity of f_i,f_j we have that $x_{ij}=\underbrace{|f_i-f_j|}_{\text{continuous}}^{-1}([0,\varepsilon])$ and so x_{ij} is closed. Then since E_n is an infinite union of the X_{ij}, then E_n is closed for all $n\in\mathbb{N}$.

By the Baire Category Theorem there is an n_0 such that $\text{Int}(E_{n_0})\neq\emptyset$ we may then find an open interval $J\subseteq I$ (note again this is with respect to $\overline{I}$ subspace topology, but if there is an open interval with respect to I) such that $J\subseteq E_{n_0}$. Therefore, for $n\geq n_0$ and $x\in J$ we have that $|f_n-f_{n_0}|\leq\varepsilon$ (since $x\in E_{n_0}$). Take $n\to\infty$, then since limits preserve order $|f(x)-f_{n_0}(x)|\leq\varepsilon$.

Now f_{n_0} is uniformly continuous since it is continuous on $[a,b]$ which is compact by Heine-Borel. Then there is a $\delta_0>0$ such that if $|y-z|<\delta_0$ then $|f_{n_0}(y)-f_{n_0}(z)|<\varepsilon$. Fix $x\in J$. Let $C=[a,b]\setminus J$. Let $\delta_x=\min\{\frac{\delta_0}{2},d_C(X)\}$. Note that $\delta_x\neq 0$ since $x\in J$. Note if $|x-y|<\delta_x$ then $y\in J$ since $d_C(x)$ is the minimum distance between points not in J and x. In particular, $B_{\delta_x}(x)\subseteq J$ and if $y,z\in B_{\delta_x}(x)$ then $|y-z|<2\delta_x\leq\delta_9$ and so $|f_{n_0}(y)-f_{n_0}(z)|<\varepsilon$.

For $y,z\in B_{\delta_x}(x)$ we have

$$|f(y)-f(z)|\leq|f(y)-f_{n_0}(y)|+|f_{n_0}(y)-f_{n_0}(z)|+|f_{n_0}(z)-f(z)|<3\varepsilon<\frac{1}{K}$$

(Recall that we saw eons ago that $|f(x) - f_{n_0}(x)| \leq \varepsilon$.) Therefore, for $x \in J$ we see that $\omega_f(x) < \frac{1}{K}$. So $J \subseteq [a,b] \setminus C_K$. But $J \subseteq I \subseteq C_K$. □

Corollary: Let $(f_n) \subseteq C([a,b])$. If $f_n \to f$ pointwise then f is continuous on a (residual,) *dense*, G_δ subset of $[a,b]$.

Proof. Use notation as in the above proof. Then the discontinuities of f are $\bigcup_{n=1}^{\infty} C_n$ where C_n is closed and $\text{Int}(C_n) = \emptyset$. So f is continuous on $\bigcap_{n=1}^{\infty} [a,b] \setminus C_n$. We see immediately that f is continuous on a G_δ set. Now notice $\overline{[a,b] \setminus C_n} = [a,b] \setminus \text{Int}(C_n) = [a,b]$ is clearly dense in $[a,b]$. But then by Baire Category Theorem $\bigcap [a,b] \setminus C_n$ is dense in $[a,b]$, as desired. □

Corollary: Suppose $f : [a,b] \to \mathbb{R}$ is differentiable. Then f' is continuous a residual, dense, G_δ set of $[a,b]$.

Proof. Consider $f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} \to f'(x)$ so that $f_n \to f'$ point wise. This then follows by our above corollary. □

Example: Consider $(-1,1)$ and $\mathbb{R}$. These are homeomorphic. To see this consider $\frac{2}{\pi}\arctan(x)$. However, $\mathbb{R}$ is complete and $(-1,1)$ is not. In particular homeomorphisms do not preserve completeness as this is a metric space property. However, isometric isomorphisms will preserve completeness.

10.3 Uniform Boundedness Principle

Remark: Recall if X, Y are normed vector spaces and $T : X \to Y$ is linear. Then T is continuous if and only if T is bounded if and only if

$$\|T\|_{op} := \sup\{\|T(x)\| : \|x\| = 1\} < \infty$$

Therefore, the space $B(X,Y)$ of bounded linear transformations from X to Y is a normed vector space with $\|\cdot\|_{op}$ as the norm. Note this is also the space of all continuous linear transformations. Also we see then that for $T \in B(X,Y)$

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \|T\|_{op} \qquad \iff \qquad \|T(x)\| \leq \|T\|_{op} \cdot \|x\|$$

Theorem. Uniform Boundedness Principle: Abbr. UBP. Let X, Y be normed vector spaces and suppose X is complete. Let $F \subseteq B(X,Y)$. If for all $x \in X$ we have $\sup\{\|T(x)\| : T \in F\} < \infty$ then $F \subseteq B(X,Y)$ is bounded. That is, $\sup\{\|T\|_{op} : T \in F\} < \infty$.

Proof. For $n \in \mathbb{N}$ let $C_n = \{x \in X : \forall T \in F, \|T(x)\| \leq n\}$. Then by our assumption $X = \bigcup_{n=1}^{\infty} C_n$. We will show each C_n is closed. Let $(x_k) \subseteq C_n$ such that $x_k \to x \in X$. Then for all $T \in F$, we have $T(x_k) \to T(x)$ (since each $T \in F$ is bounded and linear and therefore continuous). We also have $\|T(x_k)\| \leq n$ which implies that $\|T(x)\| \leq n$ since limits preserve continuity. By Baire Category Theorem, X is of second category, therefore there is an $n \in \mathbb{N}$ such that $\text{Int}(C_n) \neq \emptyset$ (otherwise $X = \bigcup_{n=1}^{\infty} C_n$ is a union of nowhere dense sets, and so X is of first category, a contradiction). There is then an $x_0 \in X$ and $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq C_n$. We see then that since C_n is closed that $\overline{B_\varepsilon(x_0)} \subseteq C_n$. Let $T \in F$ and $x \in X$ with $\|x\| = 1$. Therefore

$$\|T(x)\| = \frac{1}{\varepsilon}\|T(\underbrace{x_0 + \varepsilon x}_{\|x_0+\varepsilon x - x_0\| = \varepsilon}) - T(x_0)\| \leq \frac{1}{\varepsilon}\left(\|T(\underbrace{x_0 + \varepsilon x}_{\in C_n})\| + \|T(\underbrace{x_0}_{\in C_n})\|\right) \leq \frac{2n}{\varepsilon}$$

Since T and x were arbitrary, we see $\|T\|_{op} \leq \frac{2n}{\varepsilon} < \infty$ for all $T \in F$, and so F is bounded. □

Remark: We can also show the contrapositive is true. In particular, if $F \subseteq B(X, Y)$ is bounded, then

$$\|T(x)\| \leq \|T\|_{op}\|x\| \leq M\|x\|$$

where M is bound on F.

Example: Let $(f_n) \subseteq C(\mathbb{R})$. Suppose for all $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $f_n(x) \in \mathbb{Q}$. Prove that for all $a < b$ there are $a < c < d < b$ and an $n \in \mathbb{N}$ such that $f_n|_{(c,d)}$ is constant. $f_n|_{(a,b)}$ bounded by EVT.

Proof. Consider $g_n : [a, b] \to \mathbb{R}$ given by $g_n = f_n|_{[a,b]}$. Enumerate $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Define $C_{n,m} = g_n^{-1}(\{q_m\})$. Since g is continuous and $\{q_m\}$ is closed, $C_{n,m}$ is closed. So $[a, b] = \bigcup_{n,m} C_{n,m}$. Then there are $n, m \in \mathbb{N}$ so that $\text{Int}(C_{n,m}) \neq \emptyset$. In particular there is $(c, d) \subseteq C_{n,m}$. Then $g_n((c, d)) = \{q_m\}$, as desired. □

Week 11 Polynomial Approximation

11.1 Weierstrass Approximation

Remark: A classic result we will see is: "For all $\varepsilon > 0$ and for all $f \in C([a, b])$ there is a polynomial p such that $\|f - p\|_\infty < \varepsilon$. That is, the polynomials are dense in $C([a, b])$."

Remark: These are simplifications we will make throughout this section.

1. Let $\varphi : [a, b] \to [0, 1]$ be given by $\varphi(x) = \frac{x-a}{b-a}$. Then φ is a continuous, increasing bijection. More over, $\psi : C([0, 1]) \to C([a, b])$ given by $\psi(f) = f \circ \varphi^{-1}$ is an isometric isomorphism. Thus, to prove a result on $C([a, b])$, it suffices to prove it on $C([0, 1])$.

2. Let $f \in C([0, 1])$. Let $g(x) = f(x) - [(f(1) - f(0))x + f(0)]$. Notice $g(0) = g(1) = 0$. Moreover, if we can approximate g by a polynomial, we can do the same for f. This is since their difference is a polynomial. Thus, we may assume $f(0) = f(1) = 0$.

Lemma. Bernoulli's Inequality: For all $n \in \mathbb{N}$ and $x \in [0,1]$, then $(1-x^2)^n > 1 - nx^2$.

Proposition. Leibniz's Rule: Let $f : [a \times b] \to [c,d] \to \mathbb{R}$ be continuous. Further, suppose f_y is continuous. Then

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b f_y(x,y)dx$$

Proof. By Fubini's theorem, for $y \in [c,d]$ we have

$$\begin{aligned}
\int_c^y \int_a^b f_z(x,z)dxdz &= \int_a^b \int_c^y f_z(x,z)dzdx \\
\frac{d}{dy}\int_c^y \int_a^b f_z(x,z)dxdz &= \frac{d}{dy}\int_a^b \int_c^y f_z(x,z)dzdx \\
\int_a^b f_y(x,y)dxd &= \frac{d}{dy}\int_a^b f(x,y) - f(x,c)dx \\
\int_a^b f_y(x,y)dxd &= \frac{d}{dy}\int_a^b f(x,y)dx - \frac{d}{dy}\int_a^b f(x,c)dx \\
\int_a^b f_y(x,y)dxd &= \frac{d}{dy}\int_a^b f(x,y)dx
\end{aligned}$$

□

Theorem. Weierstrass Approximation Theorem: The set of polynomials (with domain $[a,b]$) is dense in $C([a,b])$.

Proof. We assume $a = 0$ and $b = 1$ by simplification 1. Let $f \in C([0,1])$ be fixed. Let $\varepsilon > 0$ be given. We may assume $f(0) = f(1) = 0$ by simplification 2. In particular, f may be extended to a uniformly continuous function on $\mathbb{R}$ by setting $f = 0$ for $(-\infty, 0) \cup (1, \infty)$ (f is continuous on compact $[0,1]$ so f is uniformly continuous on $[0,1]$, extending by a constant function preserves uniform continuity).

For $n \in \mathbb{N}$ let $Q_n(x) = c_n(1-x^2)^n$ where $c_n > 0$ is chosen such that $\int_{-1}^1 Q_n(x)dx = 1$. We may choose this as $\int_{-1}^1 (1-x^2)^n = \ell$ for $\ell > 0$ since $(1-x^2)^n$ is non-negative on $[-1,1]$, so we may pick $c_n = \frac{1}{\ell}$. Now

$$\begin{aligned}
\int_{-1}^1 (1-x^2)^n dx &= 2\int_0^1 (1-x^2)^n dx \\
&\overset{BI}{\geq} 2\int_0^1 1 - nx^2 dx \\
&\geq 2\int_0^{1/\sqrt{n}} 1 - nx^2 dx \\
&= \frac{4\sqrt{n}}{3} > \frac{1}{\sqrt{n}}
\end{aligned}$$

Then, multiplying both sides by c_n we have

$$1 = \int_{-1}^{1} Q_n(x)dx > \frac{c_n}{\sqrt{n}}$$

We see then that $c_n < \sqrt{n}$. Define $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$ (notice by our extension of f this is valid). However, since f is zero outside of $[0,1]$, we have that

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$

Then performing the substitution $u = x+t$, we have $\int_0^1 f(u)Q_n(u-x)du$.

$$\begin{aligned} P_n(x) &= \int_{-1}^{1} f(x+t)Q_n(t)dt && \text{valid by extension of } f \\ &= \int_{-x}^{1-x} f(x+t)Q_n(t)dt && \text{since } f = 0 \text{ for } x \notin [0,1] \\ &= \int_0^1 f(u)Q_n(u-x)du && \text{substituion for } u = x+t \end{aligned}$$

Then by Leibniz's Rule, since P_n is a polynomial of degree $2n$

$$\frac{d^{2n+1}}{dx^{2n+1}} P_n(x) = \int_0^1 \frac{\partial^{2n+1}}{\partial x^{2n+1}} Q_n(u-x)du = \int_0^1 0du = 0$$

Then by Taylor's approximation theorem, we have that $P_n(x)$ is a polynomial of degree at most $2n$. Let $M = \|f\|_\infty$. Let $0 < \delta < 1$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$. We then see

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t)Q_n(t)dt - \int_{-1}^{1} f(x)Q_n(t)dt \right| && \text{since } x \text{ fixed and } \int_{-1}^{1} Q_n(t)dt = 1 \\ &= \left| \int_{-1}^{1} (f(x+t) - f(x))Q_n(t)dt \right| \\ &\leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt && \text{since } Q_n > 0 \\ &= \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt \\ &\quad + \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt \\ &\quad + \int_{\delta}^{1} |f(x+t) - f(x)|Q_n(t)dt \\ &\leq \int_{-1}^{-\delta} 2MQ_n(t)dt \\ &\quad + \int_{-\delta}^{\delta} \varepsilon Q_n(t)dt \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\delta}^{1} 2MQ_n(t)dt \\
\leq &\int_{-1}^{-\delta} 2Mc_n(1-t^2)^n dt \\
&+ \int_{-1}^{1} \varepsilon Q_n(t)dt \qquad \text{integrating over larger region} \\
&+ \int_{\delta}^{1} 2Mc_n(1-t^2)^n dt \\
\leq &\int_{-1}^{-\delta} 2Mc_n(1-\delta^2)^n dt \\
&+ \varepsilon \int_{-1}^{1} Q_n(t)dt \\
&+ \int_{\delta}^{1} 2Mc_n(1-\delta^2)^n dt \\
\leq &\, 4M\sqrt{n}(1-\delta)(1-\delta^2)^n + \varepsilon
\end{aligned}
$$

Notice that since $\delta > 0$, as $n \to \infty$ we have $4M\sqrt{n}(1-\delta)(1-\delta^2)^n \to 0$ and so there is a polynomial $P_n(x)$ such that $|P_n(x) - f(x)| < \varepsilon$. □

Corollary: $C([a,b])$ is separable.

Proof. This holds since the rationals are dense in $\mathbb{R}$, and so the rational polynomials are dense in the polynomials and in turn are dense in $C([a,b])$. □

Example: Let $f \in C([0,1])$ be such that $\int_0^1 x^n f(x) = 0$ for all $n \geq 0$. Prove $f = 0$.

Proof. Since the polynomials are dense in $C([a,b])$, there is a sequence $(P_n) \subseteq C([0,1])$ such that $P_n \to f$ uniformly. We obviously see then that $P_n f \to f^2$ uniformly. Now notice for any polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ we have

$$\int_0^1 p(x)f(x)dx = a_0 \int_0^1 x^0 f(x)dx + a_1 \int_0^1 x^1 f(x)dx + \cdots + a_n \int_0^1 x^n f(x)dx = 0 + 0 + \cdots + 0 = 0$$

In particular, we see that

$$\underbrace{\int_0^1 P_n(x)f(x)dx}_{=0} \to \int_0^1 f^2(x)dx$$

and so since $f^2(x)$ is continuous and $f^2(x) \geq 0$, we see that

$$\int_0^1 f^2(x)dx = 0 \quad \implies \quad f^2(x) = 0 \quad \implies \quad f(x) = 0$$

as desired. □

11.2 Stone-Weierstrass-Lattice Version

Remark: Let (X, d) be a compact metric space. Let $x, y \in X$ such that $x \neq y$. Let $D \subseteq C(X)$ be dense. Let $f \in C(X)$ be given by $f(t) = d(t, x)$ so that $f(x) \neq f(y)$. Then there is a sequence $(g_n) \subseteq D$ such that $g_n \to f$ uniformly. In particular, we see that $g_n(x) \to f(x)$ and $g_n(y) \to f(y)$. This means there is an $n \in \mathbb{N}$ such that $g_n(x) \neq g_n(y)$.

Definition. Separating Points: Let (X, d) be a metric space. We say $A \subseteq C(X)$ separates points if for all $x, y \in X$ with $x \neq y$ we have that there is an $f \in A$ such that $f(x) \neq f(y)$.

Remark: As seen above, whenever (X, d) is a compact metric space and $D \subseteq C(X)$ is dense (with respect to $\|\cdot\|_\infty$), then D separates points. Notice we need (X, d) to be compact so that $C_b(X) = C(X)$ and therefore so that the uniform norm $\|\cdot\|_\infty$ may in fact be a norm on $C(X)$. Then convergence with respect to $\|\cdot\|_\infty$ implies uniform convergence on $C(X)$, as desired.

Lemma: Let (X, d) be a compact metric space. Let $V \subseteq C(X)$ be a linear subspace which separates points and such that $1 \in V$ (constant function). Then for all $a, b \in X$ with $a \neq b$ and $\alpha, \beta \in \mathbb{R}$, there is an $f \in V$ such that $f(a) = \alpha$ and $f(b) = \beta$.

Proof. Let $a, b \in X$ with $a \neq b$. Then since V separates points, there is a $g \in V$ such that $g(a) \neq g(b)$. Then by linearity, consider

$$f(x) = \alpha + (\beta - \alpha)\frac{g(x) - g(a)}{g(b) - g(a)} \in V$$

Then $f(a) = \alpha$ and $f(b) = \beta$. Notice if $\alpha = \beta$ we can use the function $f(x) = \alpha \cdot 1(x) = \alpha$. □

Definition. Lattice: Let (X, d) be a metric space. A linear subspace $V \subseteq C(X)$ is a lattice if for all $f, g \in V$

$$f \vee g := \max\{f, g\} \in V \qquad \text{and} \qquad f \wedge g := \min\{f, g\} \in V$$

Remark: Notice that $C(X)$ is a lattice since linear combinations preserves continuity and absolute values preserve continuity and so we see

$$f \vee g = \frac{f + g + |f - g|}{2} \in C(X)$$
$$f \wedge g = \frac{f + g - |f - g|}{2} \in C(X)$$

Remark: Assume $V \subseteq C(X)$ is a linear subspace such that $f \vee g \in V$ whenever $f, g \in V$. Therefore,

$$f \wedge g = -((-f) \vee (-g)) \in V$$

by the linearity of V. So if V is closed under maximums, it is also closed under minimums, and vice-versa.

Theorem. Stone-Weierstrass-Lattice Version: Let (X, d) be compact. If $V \subseteq C(X)$ is a lattice such that V separates points and such that $1 \in V$, then V is dense in $C(X)$.

Proof. Let $f \in C(X)$. Let $\varepsilon > 0$ be given. Fix $x \in X$. By our lemma, for all $y \in X$ there is a $\varphi_{x,y} \in V$ such that $\varphi_{x,y}(x) = f(x)$ and $\varphi_{x,y}(y) = f(y)$. Notice this is true even if $x = y$ as we can use the constant $\varphi_{x,y}(z) = f(x)$ function. Note that $\varphi_{x,y} - f$ is continuous at y because $\varphi_{x,y}$ and f are both continuous. By continuity, for all $y \in X$ there is a $\delta_y > 0$ such that

$$d(t, y) < \delta_y \qquad \Longrightarrow \qquad |\varphi_{x,y}(t) - f(t) - (\varphi_{x,y}(y) - f(y))| = |\varphi_{x,y}(t) - f(t)| < \varepsilon$$

By the compactness of X, there are $y_1, \ldots, y_n \in X$ such that

$$X = B_{\delta_{y_1}}(y_1) \cup \cdots \cup B_{\delta_{y_n}}(y_n)$$

Define

$$\varphi_x = \varphi_{x,y_1} \vee \cdots \vee \varphi_{x,y_n} \in V$$

(we can prove $\varphi_x \in V$ by induction since $f \vee g \in V$). For $z \in X$, there is a δ_{y_i} such that $z \in B_{\delta_{y_i}}(y_i)$ so that

$$-\varepsilon < \varphi_{x,y_i}(z) - f(z) < \varepsilon \qquad \Longrightarrow \qquad f(z) - \varepsilon < \varphi_{x,y_i}(z) \leq \varphi_x(z) \tag{1}$$

Where the latter inequality holds since $\varphi_x(z)$ is the maximum over all φ_{x,y_i}. We see then that since x was arbitrary (and no terms depend on x), that $f - \varepsilon < \varphi_x$. By the continuity of $\varphi_x - f$, for all $x \in X$, there is a $\delta_x > 0$ such that

$$d(t, x) < \delta_x \qquad \Longrightarrow \qquad |\varphi_x(t) - f(t) - \varphi_x(x) + f(x)| = |\varphi_x(t) - f(t)| < \varepsilon \tag{2}$$

by continuity (note $\varphi_x(x) = \varphi_{x,y_1}(x) \vee \cdots \vee \varphi_{x,y_n}(x) = \max\{f(x), \ldots, f(x)\} = f(x)$). As before, by compactness of X we may find $x_1, \ldots, x_m \in X$ such

$$X = B_{\delta_{x_1}}(x_1) \cup \cdots \cup B_{\delta_{x_m}}(x_m)$$

Define

$$\varphi = \varphi_{x_1} \wedge \cdots \wedge \varphi_{x_m} \in V$$

For $z \in X$, there is a δ_{x_j} such that $z \in B_{\delta_{x_j}}(x_j)$

$$f(z) - \varepsilon < \varphi(z) \leq \varphi_{x_j}(z) < f(z) + \varepsilon$$

Notice the first inequality holds since for all $1 \leq j \leq m$ we have $f(z) - \varepsilon < \varphi_{x_j}(z)$, so since this holds for all j, we have $f(z) - \varepsilon < \varphi(z)$, as desired (taking minimums). The last inequality holds by (2). We see then that since $z \in X$ is arbitrary, $\|f - \varphi\|_\infty < \varepsilon$. □

Example: Let V be the space of piecewise-linear continuous functions such that $V \subseteq C([a, b])$. That each $f \in V$ is a piecewise function defined on finitely many intervals, where f is linear on each. We can show that V is in fact a lattice and V separates points (with $1 \in V$), so that we see from above V is dense in $C(X)$.

11.3 Stone-Weierstrass-Subalgebra Version

Definition. Subalgebra: Let (X, d) be a metric space. A linear subspace $V \subseteq C(X)$ is a subalgebra if $f \cdot g \in V$ for all $f, g \in V$ where $(f \cdot g)(x) = f(x) \cdot g(x)$.

Remark: Let (X, d) be a compact metric space. We can show that if $V \subseteq C(X)$ is a subalgebra then $\overline{V}$ is a subalgebra.

Theorem. Stone-Weierstrass-Subalgebra Version: Let (X, d) be a compact metric space. If $V \subseteq C(X)$ is a subalgebra such that V separates points and $1 \in V$ (this is called a unital subalgebra) then V is dense in $C(X)$.

Proof. Without loss of generality, assume V is closed. Let $f \in V$. Since f is continuous and X is compact, we know $f(X)$ is bounded, in particular, there is an $M > 0$ such that $f(X) \subseteq [-M, M]$. Let $\varepsilon > 0$ be given. By the Weierstrass Approximation Theorem, there is a polynomial $P : \mathbb{R} \to \mathbb{R}$ such that $|P(x) - |x|| < \varepsilon$ for all $x \in [-M, M]$. Then for all $x \in X$ we have $\|P \circ f - |f|\|_\infty = |\underbrace{P(f(x))}_{\in V} - |f(x)|| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, and V is closed, we may conclude $|f| \in V$. Then for all $f, g \in V$, we have $f \vee g \in V$, so V is dense by the Stone-Weierstrass-Lattice Version. □

Remark: Big idea of the proof: Note if $\overline{V}$ is dense in $C(X)$, then we have that $\overline{V} = \overline{\overline{V}} = X$ (the double closure) and so V is dense in X. Note also that if p is a polynomial and $f \in V$, then $p(f) \in V$ since V is a subalgebra (note being closed under powers is equivalent to being closed under function multiplication). Recall that $f \vee g = \frac{f+g+|f-g|}{2}$, so we will show that $f \in V \implies |f| \in V$.

Example: Notice that where V are the polynomials, we see that V is a unital subalgebra, and so we see this quickly subsumes the Weierstrass Approximation Theorem (though this theorem still requires the Weierstrass Approximation Theorem).

Example: Let $V = \text{span}\{1, x^2, x^4, \ldots\}$, we can see then that V is dense in $C([0, 1])$, but V is not dense in $C([-1, 1])$ (it can't separate a from $-a$).

Definition. Infinity Norm on Complex Functions: Let (X, d) be compact. Then $C(X, \mathbb{C}) = \{f : X \to \mathbb{C} | f \text{ is continuous}\}$ and we define $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ where $|\cdot|$ is the complex modulus. Then $(C(X, \mathbb{C}), \|\cdot\|_\infty)$ is a Banach space.

Remark: If $f \in C(X, \mathbb{C})$, then $f(x) = (f)(x) + i(f)(x)$ where $(x + yi) = x$ and $(x + yi) = y$. Notice also if we $f_n \to f$ in $C(X, \mathbb{C})$, then $(f_n) \to (f)$ and $(f_n) \to (f)$. Moreover, $\text{Re}(f), \text{Im}(f) \in C(X)$. Further, $\text{Re}(f) = \frac{f+\overline{f}}{2}$ and $\text{Im}(f) = \frac{f-\overline{f}}{2i}$ where $\overline{f}(x) = \overline{f(x)}$ where $\overline{x + yi} = x - yi$. Notice finally $\text{Im}(f) = \text{Re}(-if)$.

Theorem. Stone-Weierstrass-Complex Version: Let (X, d) be a compact metric space. If $V \subseteq C(X, \mathbb{C})$ is a subalgebra such that $1 \in V$, V separates points, and if for all $f \in V$ then $\overline{f} \in V$ (this is call a self-conjugate unital subalgebra), then V is dense in $C(X, \mathbb{C})$.

Proof. Let $V \in C(X, \mathbb{C})$ be a subalgebra as above. Define $W = \{\text{Re}(f) : f \in V\} \subseteq C(X)$.

Notice if $f \in V$, then $\operatorname{Im}(f) = \operatorname{Re}(\underbrace{-if}_{\in V}) \in W$. Then W is a subalgebra of $C(X)$ such that $1 \in W$, and W separates points. By Stone-Weierstrass-Lattice Version, W is dense in $C(X)$. Take $f \in C(X, \mathbb{C})$. We know there are $(g_n), (h_n) \subseteq W$ such that $g_n \to \operatorname{Re}(f)$ and $h_n \to \operatorname{Im}(f)$. Then we see that $g_n + ih_n \in V$ is such that $g_n + ih_n \to \operatorname{Re}(f) + i\operatorname{Im}(f) = f$. □

Remark: If $T = \{z \in \mathbb{C} : |z| = 1\}$, then the polynomials on T are not dense in $C(T, \mathbb{C})$.

Example: Let $T = \{z \in \mathbb{C} : |z| = 1\}$ and consider $C(T, \mathbb{C})$. Let

$$X = \{f \in C(T, \mathbb{C}) : f(-\pi) = f(\pi)\}$$

Let $\varphi : X \to C(T, \mathbb{C})$ such that $\varphi(f)(e^{i\theta}) = f(\theta)$. We can check that φ is a homeomorphism. Let

$$\operatorname{Trig}([-\pi, \pi]) := \operatorname{span}_{\mathbb{C}}\{e^{inx} : n \in \mathbb{Z}\} \subseteq X$$

be the trigonometric polynomials. We claim $\overline{\operatorname{Trig}([-\pi, \pi])} = X$. Since φ is a homeomorphism it suffices to show $\varphi(\operatorname{Trig}([-\pi, \pi]))$ is dense in $\varphi(X) = C(T, \mathbb{C})$. Notice we know that T is compact by Heine-Borel Theorem, so $C(T, \mathbb{C})$ is compact. Let $f_n(x) = e^{inx} \in X$. Then $\varphi(f_n)(e^{ix}) = f_n(x) = e^{inx}$, or if $y = e^{ix}$, then $\varphi(f_n)(y) = y^n$. In particular $\{f_n\}_{n \in \mathbb{Z}}$ can be viewed as the polynomials (plus negative powers). That is

$$\varphi(\operatorname{Trig}([-\pi, \pi])) = \operatorname{span}_{\mathbb{C}}\{e^{ix} \mapsto e^{inx} : n \in \mathbb{Z}\}$$

Easily enough, by this knowledge we know $\varphi(\operatorname{Trig}([-\pi, \pi]))$ is a subalgebra of $C(T, \mathbb{C})$ which contains 1, separates points, and is closed under conjugation. By Stone-Weierstrass-Complex Version, we see that $\overline{\varphi(\operatorname{Trig}([-\pi, \pi]))} = C(T, \mathbb{C})$. This is the basis of Fourier Analysis, that (Lebesgue integrable) functions can be approximated by trigonometric polynomials.

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