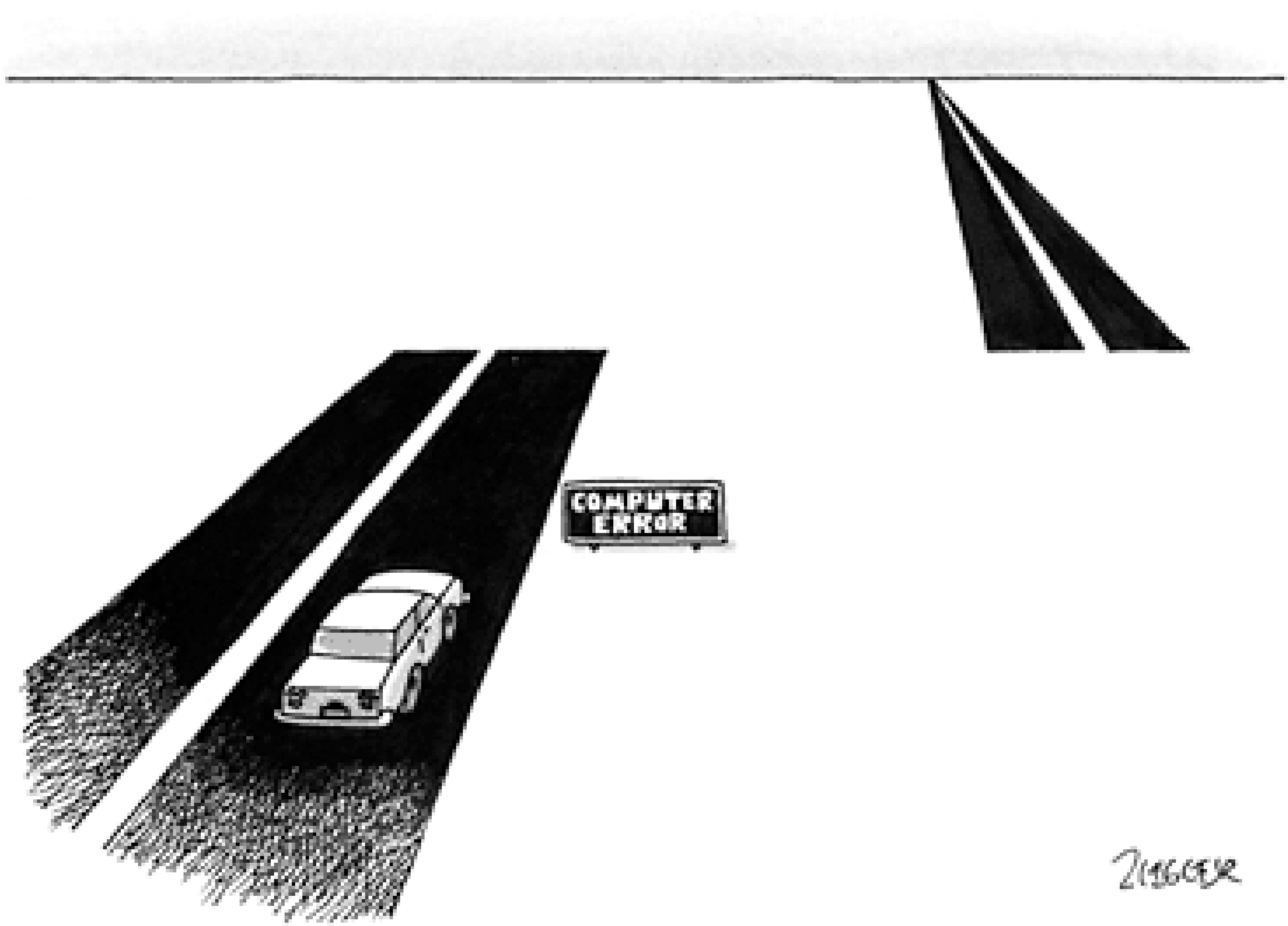
Error in Calculations (Part 2)



Today's Lecture

1. Intro to Software Engineering
2. Inexact quantities
3. Error propagation
4. Floating-point numbers
5. Design process
6. Teamwork
7. Project planning
8. Decision making
9. Professional Engineering
10. Software quality
11. Software safety
12. Intellectual property

Agenda

Goal: Given a computation $f(x_0 \pm \Delta x, y_0 \pm \Delta y)$, we want to determine the uncertainty Δf

Method 1: Calculate the exact range of values of $f(x_0 \pm \Delta x, y_0 \pm \Delta y)$

Method 2: Approximate the worst-case range of Δf

Method 3: Approximate the standard deviation of Δf

Method 1: Calculate the exact range

Calculate the *minimum* and *maximum* values of a function result, given the ranges of the arguments' values

Example: A Lego robot rotates at a rate $r=d/t$.
If the robot rotates 30 ± 5 degrees in 6 ± 1 s, then what is the value range of r?

r_{MAX} =

r_{MIN} =

Exact Range - Assumptions

- Must know the exact range of values for each function argument.

- Must know how each argument affects the value of the function (i.e., need to know which of the arguments' errors maximize the function value and which minimize the function value).

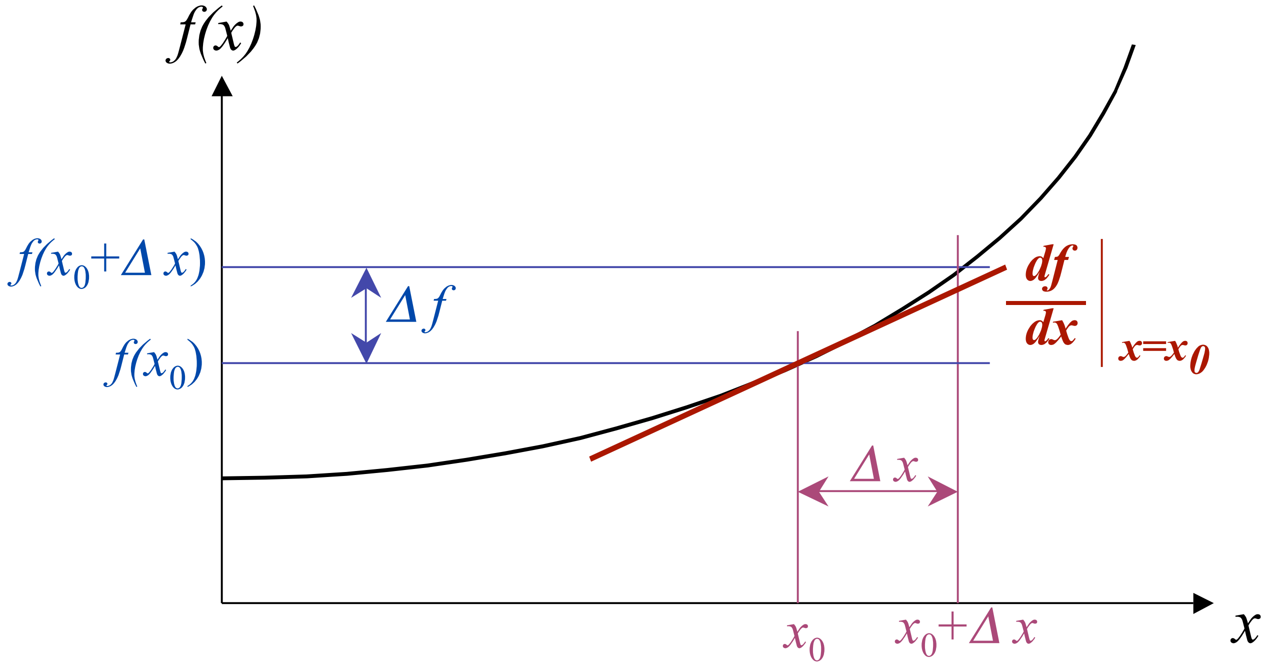
$$f(x,y) = x^3 - 24x^2y + 5y^2x - 361xy + 50$$

Andrews, Aplevich, Fraswer, Ratz, *Introduction to Professional Engineering in Canada,* Pearson, 2002.

Method 2: Linear Approximation

A fundamental approximation of calculus asserts that, for any function *f(x)* and for any sufficiently small increment Δx,

$$\Delta f = \left| \frac{df}{dx} \right| \cdot \Delta x$$



Method 2: Linear Approximation

Example: A prospective home buyer determines the depth of a well by dropping a stone into the well and timing its fall. The time it takes for the stone to hit water is 3.0 ± 0.5 sec. How deep is the well?

Method 2: Linear Approximation

Change in a function with one variable $f(x) \approx$ "approximately"

$$(\text{rate of change w.r.t. } x) \times \Delta x$$

Similarly, change in $f(x,y,z,\ldots) \approx$

$$(\text{rate of change w.r.t. } x) \times \Delta x$$
$$+ (\text{rate of change w.r.t. } y) \times \Delta y$$
$$+ (\text{rate of change w.r.t. } z) \times \Delta z$$
$$+ \ldots$$

Method 2: Linear Approximation

This is expressed as

$$\Delta f \approx \left|\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta x| + \left|\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta y| + \left|\frac{\partial f}{\partial z}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta z| + \ldots$$

Partial derivative - the derivative of *f* with respect to *z*, keeping other variables (i.e., *x*, *y*, …) constant.

Partial Derivatives

Partial derivative - the derivative of *g* with respect to one variable, keeping other variables constant.

Example: $g(l,T) = 4\pi^2 \, l \,/\, T^2$

$$\frac{\partial g}{\partial l} =$$

$$\frac{\partial g}{\partial T} =$$

Method 2: Linear Approximation

This is expressed as

$$\Delta f \approx \left|\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta x| + \left|\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta y| + \left|\frac{\partial f}{\partial z}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} |\Delta z| + \ldots$$

Sensitivity S_z - reflects how sensitive Δf is to the value of Δz.

Partial derivative - the derivative of *f* with respect to *z*, keeping other variables (i.e., *x*, *y*, …) constant.

Method 2: Linear Approximation

Example: As part of a physics lab, students are to measure g, the acceleration of gravity, using a pendulum and an equation relating g to the pendulum's period:

$$g = 4\pi^2 l \,/\, T^2$$

where l is the length of the pendulum and T is the pendulum's period. The length of the pendulum is (1.000 ± 0.003) m, and the pendulum period is (2.000 ± 0.004) s. What is the value of g?

Method 2: Linear Approximation

Example: A Lego robot rotates at a rate $r=d/t$.

If the robot rotates 30 ± 5 degrees in 6 ± 1 s, then what is the *approximate* error of r?

Linear Approximation - Assumptions

- An approximate calculation of error is OK.
- The function *f* is differentiable at the measured values of its arguments.
- Deviations of the measured values are independent of each other.
- The range of values for each function argument is known, although it need not be exact.

Andrews, Aplevich, Fraswer, Ratz, *Introduction to Professional Engineering in Canada,* Pearson, 2002.

Method 3: Standard Deviation

Estimate the *expected range* of values of a function when the extreme values in the full range are unlikely.

Example: Find the efficiency of a D.C. electric motor by using it to lift a mass m through a height h. The work accomplished is mgh and the electric energy delivered to the motor is VIt, where V is the applied voltage, I the current, and t the time the motor runs.

$$\text{efficiency } e = \frac{\text{work done by motor}}{\text{energy delivered to motor}} = \frac{mgh}{VIt}$$

Uncertainty for m, h, V, I is 1%

Uncertainty for t is 5%

Method 3: Standard Deviation

Estimate the *expected range* of values of a function result by calculating the *standard deviation* of the function.

$$\Delta f \approx \sqrt{\left(\left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0 \\ z=z_0}} \cdot \Delta x \right)^2 + \left(\left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_0 \\ y=y_0 \\ z=z_0}} \cdot \Delta y \right)^2 + \left(\left. \frac{\partial f}{\partial z} \right|_{\substack{x=x_0 \\ y=y_0 \\ z=z_0}} \cdot \Delta z \right)^2}$$

Method 3: Standard Deviation

Example: Efficiency of D.C. electric motor

$$\Delta f \approx +\sqrt{\left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta h}{h}\right)^2 + \left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2 \left(\frac{\Delta t}{t}\right)^2}$$

$$= \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 5^2}\ \% \qquad = \sqrt{29}\ \% \approx 5\%$$

- Large input errors dominate final errors
- There is a 68% probability that the actual value of *f(x,y,z)* is within the standard deviation *f(x,y,z)* ± Δf

Method 3: Standard Deviation

Example: A Lego robot rotates at a rate $r=d/t$.

If the robot rotates 30 ± 5 degrees in 6 ± 1 s, then what is the *standard* error in *r*?

Standard Deviation - Assumptions

- Want standard deviation of function *f*, rather than maximum deviation

- The function *f* is differentiable at the measured values of its arguments.

- Deviations of the measured values are independent random quantities symmetrically distributed about a mean of zero of each other.

- The standard deviation of each function argument is known.

Andrews, Aplevich, Fraswer, Ratz, *Introduction to Professional Engineering in Canada,* Pearson, 2002.

Summary of Uncertainty in Calculations

Method 1: Calculate the exact range of $f(x_0,y_0)$ by calculating the minimum and maximum values of $f(x_0 \pm \Delta x,\ y_0 \pm \Delta y)$

Method 2: Approximate the worst-case range of Δf

as

$$\left|\left.\frac{\partial f}{\partial x}\right|_{x=x_0} \cdot \Delta x\right| + \left|\left.\frac{\partial f}{\partial y}\right|_{y=y_0} \cdot \Delta y\right|$$

Method 3: Approximate the standard deviation of Δf

as

$$\sqrt{\left(\left.\frac{\partial f}{\partial x}\right|_{x=x_0} \cdot \Delta x\right)^2 + \left(\left.\frac{\partial f}{\partial y}\right|_{y=y_0} \cdot \Delta y\right)^2}$$

Announcements

- **Bring hard copy of LegoRocker Robot code to lab**

  Example program accessible from course web site

  http://www.student.cs.uwaterloo.ca/~se101/ZincRocker.java

- **Bring laptops to lab (if you have one)**

Readings

- **For next week's web review**
  - **Error propagation** IPE 12
  - **Report formatting** Dupré 21,26,43,96,118,126
  - **References** http://www.computer.org/author/style/refer.htm

- **For next week's lecture**
  - **Computer Error** Overton: Chapter 3
    - Buy from Pixel Planet (MC 2018)
    - Ask for SE 101 Notes
    - Should be available by midweek