An Asymptotically Optimal Algorithm for Maximum Matching in Dynamic Streams

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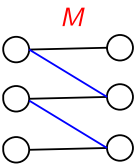
Joint work with Sepehr Assadi

Matching Problem

- Graph $G = (V, E)$
- Matching: $M \subseteq E$, (V, M) has max degree 1
- Maximum matching: Matching M^* of the largest size



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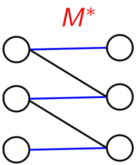

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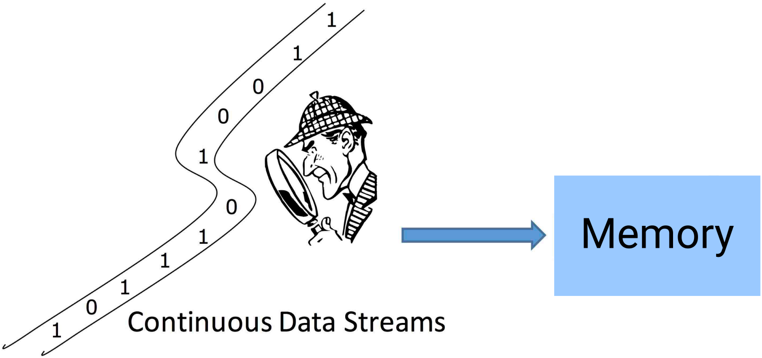
Streaming Setting



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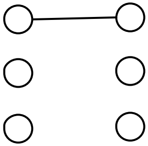

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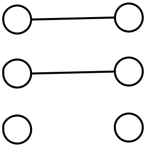

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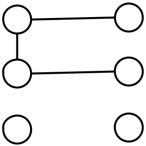

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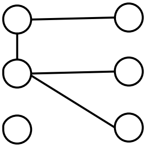

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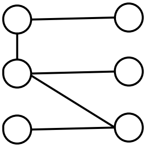

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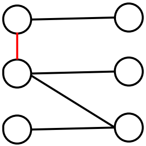

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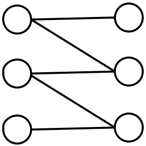


Streaming Setting

- $G = (V, E)$
- Edges of G appear in a stream
- Dynamic Stream: Insertions or Deletions
- Output a solution at the end of the stream
- Goal: Minimize Memory

Lower Bound

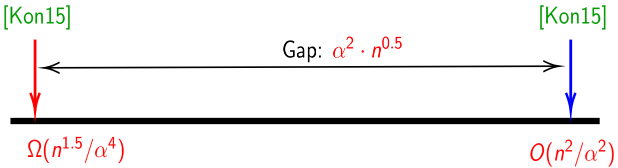
- Maximum Matching Lower bound: $\Omega(n^2)$ bits [FKM+05]
- Store the input: $O(n^2)$ bits
- No non-trivial solution

Approximation

- Question: What about an α approximation?
- Return a matching M of size at least $\frac{|M^*|}{\alpha}$
- Can we get $o(n^2)$ space?
- What is the trade off between α and the space?

Previous Work

Result	Upper Bound	Lower Bound
[Kon15]	$O(n^2/\alpha^2)$	$\Omega(n^{1.5}/\alpha^4)$



Space-Approximation Tradeoff

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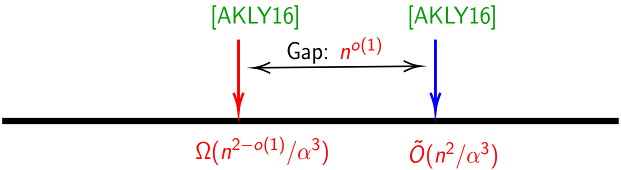

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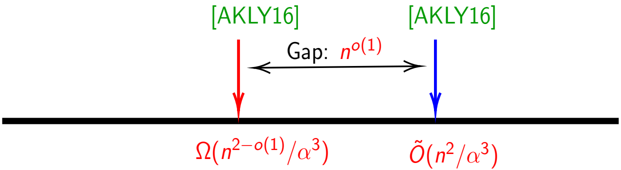

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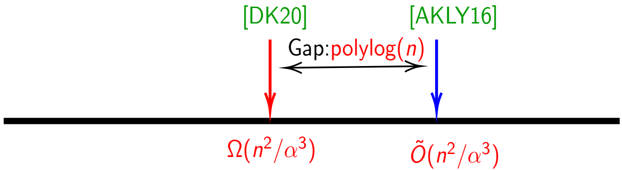
[DK20]		$\Omega(n^2/\alpha^3)$



Space-Approximation Tradeoff

Previous work

- Best known upper bound: $\tilde{O}(n^2/\alpha^3)$ bits ([AKLY16])
- Best known lower bound: $\Omega(n^2/\alpha^3)$ bits ([DK20])
- Gap of polylog(n) bits
- These types of polylog(n) gaps appear frequently in dynamic streams
- One key reason is a main technique for finding edges in a dynamic streams

Previous work

L_0-Samplers:

- It is non-trivial to find even one edge in a dynamic stream
- L_0-Samplers are a key tool to solve this problem
- They can sample an edge uniformly at random from a set of pairs of vertices undergoing edge insertions and deletions

Previous work

- L_0-Samplers can be implemented in $O(\log^3 n)$ bits of space ([JST11])
- $\Omega(\log^3 n)$ bits are also necessary ([Kap+17])
- Many problems in streaming have the $\text{polylog}(n)$ overhead because of the use of L_0-samplers
- Connectivity has a lower bound of $\Omega(n \log^3 n)$ ([NY19])

Our Result

We prove asymptotically optimal bounds on the space-approximation tradeoff:

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This closes the gap up to constant factors

Some problems do not need the polylog(n) overhead

If $\alpha > n^{1/2}$ then there is not enough space to output the answer:

$$\frac{n}{\alpha} > \frac{n^2}{\alpha^3}$$

Algorithm

We will now give a proof sketch

Assumptions

Simplifying Assumptions for this talk:

- The input graph is bipartite
- The maximum matching has size $\Omega(n)$
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All these assumptions can be lifted!

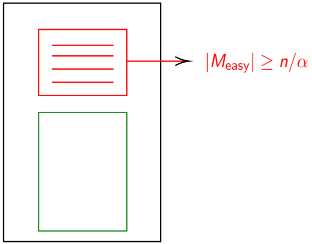
Approach

1. Match or Sparsify:
    - Either find a large matching
    - Or identify hard instances
2. Solve the hard instances

Note: We run these algorithms in parallel

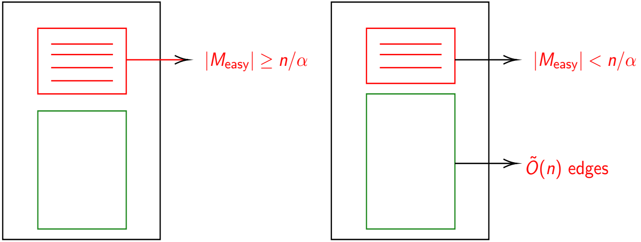
Match Or Sparsify

1. Find a matching M_{easy} in space $O(n^2/\alpha^3)$ bits such that:
   - Either $|M_{\text{easy}}| = \Omega(n/\alpha)$



Match Or Sparsify

1. Find a matching M_{easy} in space $O(n^2/\alpha^3)$ bits such that:
   - Either $|M_{\text{easy}}| = \Omega(n/\alpha)$
   - Or Subgraph induced on unmatched vertices has $\tilde{O}(n)$ edges and a matching of size $\Omega(n)$



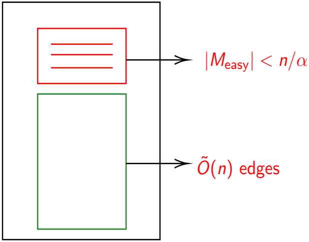
Match Or Sparsify

Idea:

- Sample $O(n^2/\alpha^3 \text{polylog}(n))$ random edges
- L_0-samplers take space $\text{polylog}(n)$
- M_{easy} is a greedy matching over the sampled edges
- Similar to residual greedy property of matching (used in [Ahn+18, Kon18])
- Different proof but along the same lines

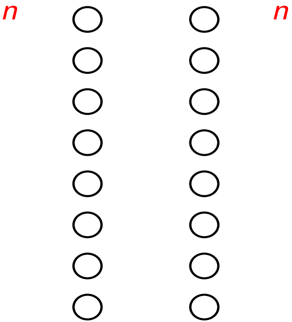
Solving Hard Instances

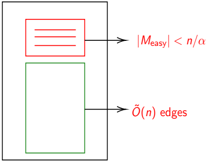
We know the partition at the end of the stream from Match Or Sparsify step



Grouping

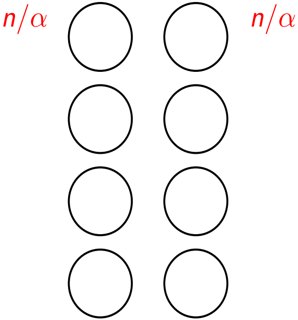
Consider the bipartite graph

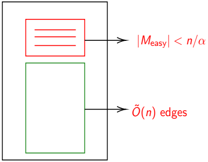




Grouping

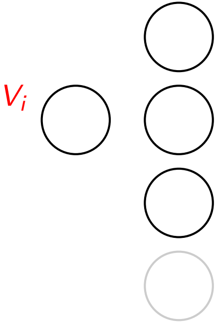
Random grouping on both sides

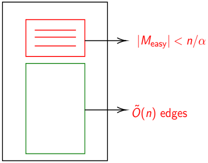




Grouping

$1/\alpha$ fraction of groups on right are in the neighborhood of V_i

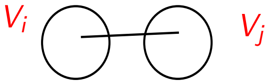


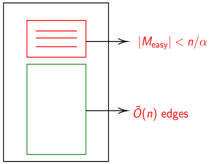


Done to reduce the neighbors of V_i

Recovery

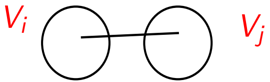
- There are $\Omega(n/\alpha)$ pairs of groups with exactly one edge between them
- V_i, V_j do not contain any vertices of M_{easy}

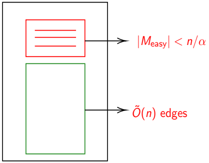




Recovery

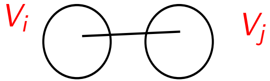
Want to recover the edge between V_i and V_j

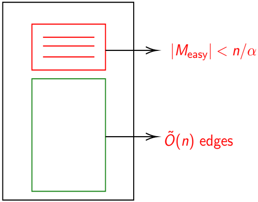




Recovery

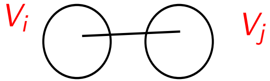
- V_i does not contain any vertices of M_{easy}
- Neighbors of V_i: $O(n/\alpha^2)$
- Trivial solution: $O((n/\alpha^2) \cdot \log n)$ bits

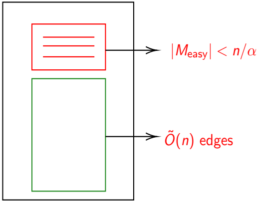




Recovery

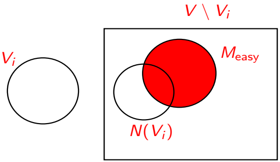
- Goal: $O(n/\alpha^2)$ bits
- So n/α groups will imply space of $O(n^2/\alpha^3)$ bits
- V_j does not contain any vertices of M_{easy}
- Recover $N(V_i) - M_{\text{easy}}$

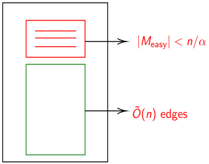




Sparse neighborhood recovery sketch

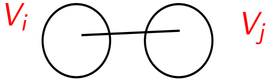
- Given V_i at the beginning
- Given M_{easy} at the end
- Output: $N(V_i) - M_{\text{easy}}$
- Space: $O(n/\alpha^2)$ bits

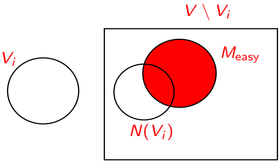




Grouping

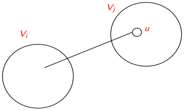
V_j lies completely within $N(V_i) - M_{\text{easy}}$

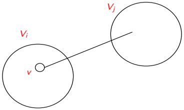




Recovery

- We know u is a neighbor of V_i (from Neighborhood sketch of V_i)
- We know v is a neighbor of V_j (from Neighborhood sketch of V_j)
- Thus, (u, v) must be an edge





Summary

Concluding Remarks

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- The lower bound of [DK20] is $\Omega(n^2/\alpha^3)$ bits making our algorithm optimal

Summary

- There is a dynamic streaming algorithm that whp outputs an α-approximation to maximum matching using $O(n^2/\alpha^3)$ bits of space
- The lower bound of [DK20] is $\Omega(n^2/\alpha^3)$ bits making our algorithm optimal
- $\text{polylog}(n)$ overhead of L_0-samplers is not always necessary (Unlike [NY19])

Open Problems

- These polylog(n) overheads due to use of L_0-samplers are prevalent in dynamic stream literature
- Can our techniques be used to bypass polylog(n) overheads for other problems:
  - E.g. Vertex Cover, Dominating Set, Vertex Connectivity

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Thank you!