Space Optimal Vertex Cover in Dynamic Streams

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Vertex Cover

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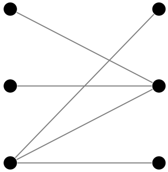

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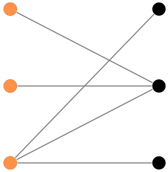

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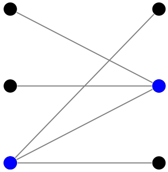

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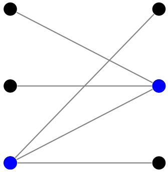
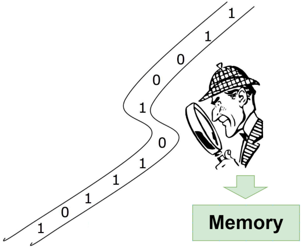


Graph Streaming:

- G arrives as a stream of edges



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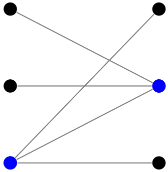

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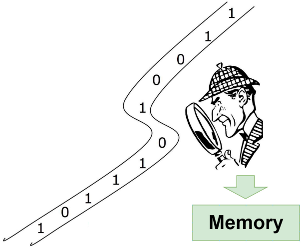

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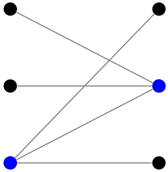

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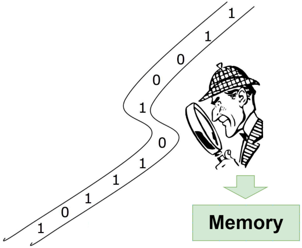
- Goal: Minimize memory ($o(n^2)$ space)



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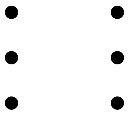

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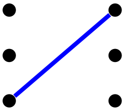
e_1



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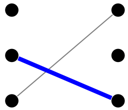
e_1	e_2



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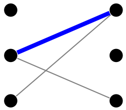

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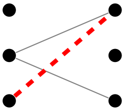

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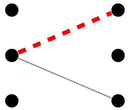

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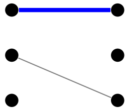

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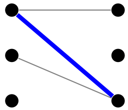

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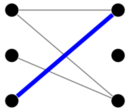

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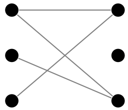

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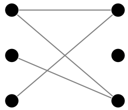

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- $O(1)$-approximation requires $\Omega(n^2)$ space
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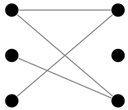

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- $O(1)$-approximation requires $\Omega(n^2)$ space
- α-approximation algorithms ($1 \leq \alpha \ll n$):
  - Lower bound: $\Omega(\frac{n^2}{\alpha^2})$ [DK20]
  - Upper bound: $O(\frac{n^2}{\alpha^2} \cdot \log \alpha)$ [DK20]

Understanding polylog factors

These type of polylog gaps appear frequently in the literature

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- [SW15] showed that for several problems (Bipartiteness, Approximate Minimum Cut, etc.) the lower bounds can be improved to $\Omega(n \log n)$
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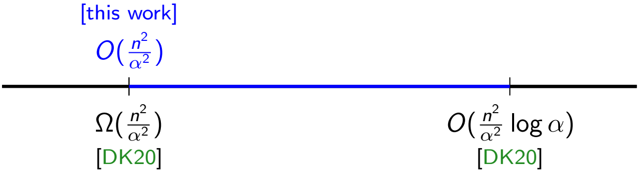
- [NY19] showed that Connectivity has a lower bound of $\Omega(n \log^3 n)$
- [AS22] gave the first result showing that the polylog factors can be removed by giving an algorithm for α-approximate Maximum Matching using $O(n^2/\alpha^3)$ bits, matching the lower bound [DK20]

Our Work

Theorem

There exists a randomised dynamic graph streaming algorithm for α-approximate minimum vertex cover that succeeds with high probability and uses $O(\frac{n^2}{\alpha^2})$ bits of space for any $\alpha \leq n^{1-\delta}$ where $\delta > 0$.



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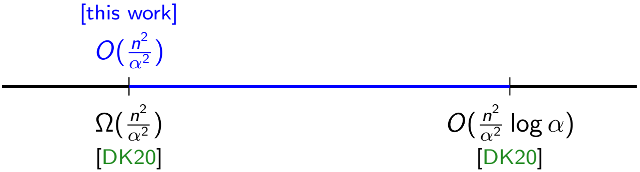
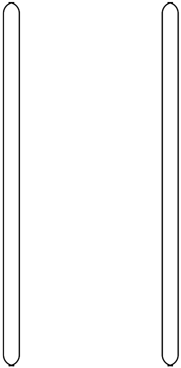


An algorithm that uses optimal space up to constant factors!

α-Approx Det. Dynamic Vertex Cover [DK20]



Simplifying Assumption (for the talk):

- The input graph is bipartite

It is easily lifted!

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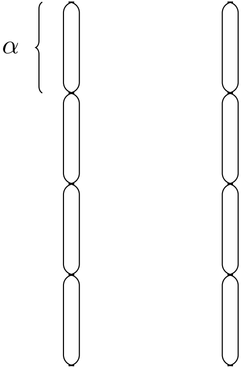

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   - about $\frac{n}{\alpha}$ groups

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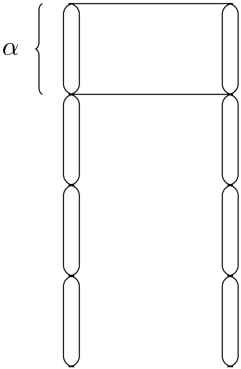

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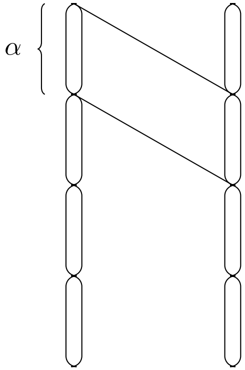

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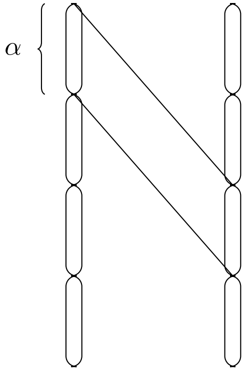

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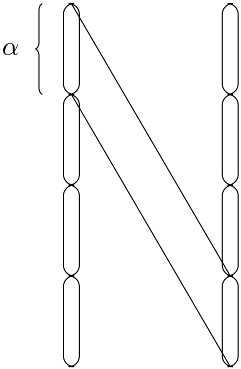

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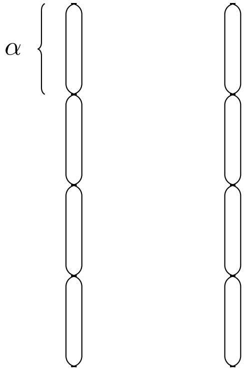

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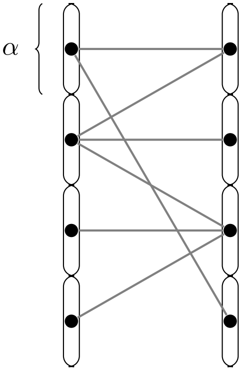

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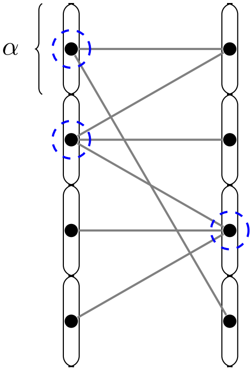

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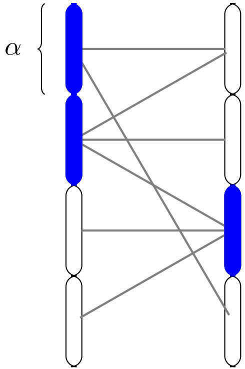

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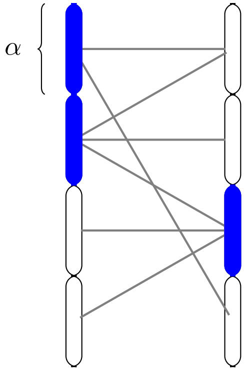

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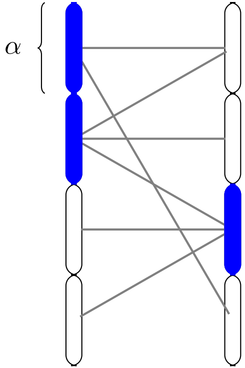

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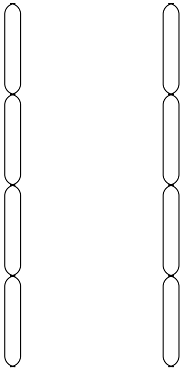
Space: $O(\frac{n^2}{\alpha^2})$ counters, each using $O(\log \alpha)$ bits. Hence, $O(\frac{n^2}{\alpha^2} \log \alpha)$ bits.

What's the issue?



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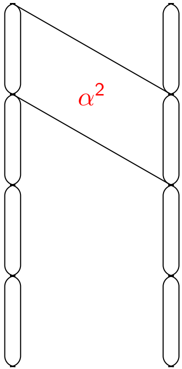

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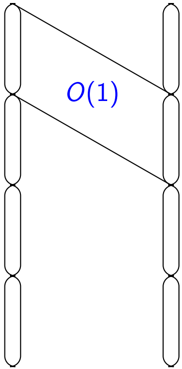
Goal:

- Counters to count upto $O(1)$ edges
- Counters to use $O(1)$ bits.



Solving the issue for sparse graphs

For G with $\approx \frac{n^2}{\alpha^2}$ edges

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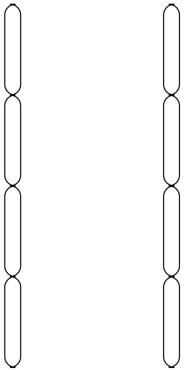
- Randomly partition into groups of size α



Solving the issue for sparse graphs

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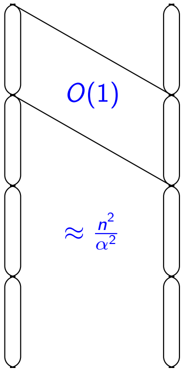

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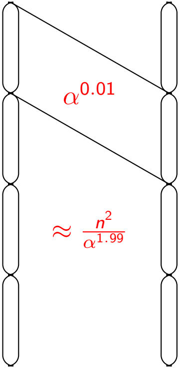
For G with $\approx \frac{n^2}{\alpha^{1.99}}$ edges or more:

- Counters use $\Theta(\log \alpha)$ bits



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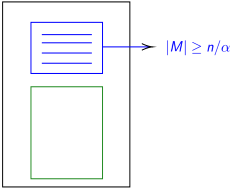

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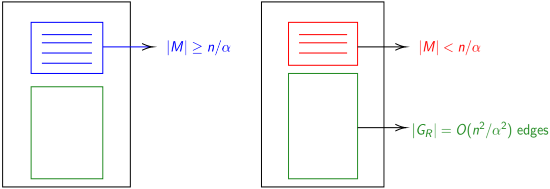
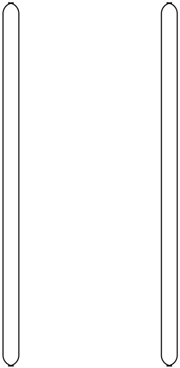
Match-or-Sparsify Lemma:

- **either** $|M| \geq \frac{n}{\alpha}$ then $|\mathrm{OPT}| \geq \frac{n}{\alpha}$
  $\implies$ V is an α-approx
- **or** $|G_R| = O(\frac{n^2}{\alpha^2})$
  $\implies$ counters use $O(1)$ bits (in expectation)

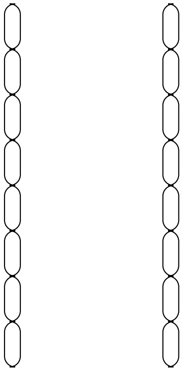


Space Optimal Algorithm



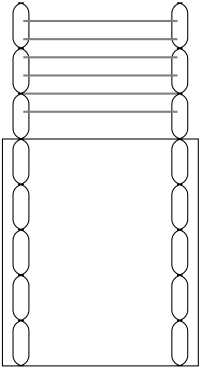
Space Optimal Algorithm

1. Randomly partition vertices ($\frac{n}{\alpha}$ groups)



Space Optimal Algorithm

1. Randomly partition vertices ($\frac{n}{\alpha}$ groups)
2. Run Match-or-Sparsify lemma
   - if $|M|$ is large, return V



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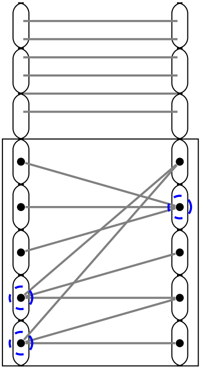
3. Check if an edge is present between pairs and compute group-level vertex cover



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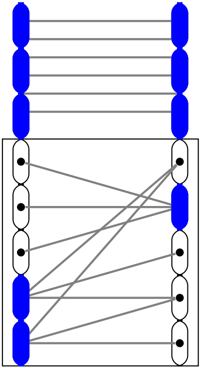
4. Return vertices of the covering groups including those with matched vertices



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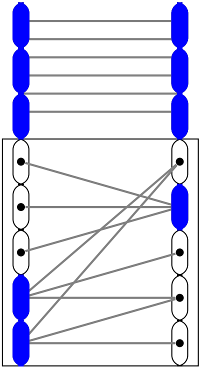

How to prove the Match-or-Sparsify lemma? Checkout the long talk!

Summary

1. There is a dynamic streaming algorithm that whp outputs an α-approximation to minimum vertex cover using $O(n^2/\alpha^2)$ bits of space
   - Match or Sparsify in $O(n^2/\alpha^2)$ bits of space
   - The ideas from [DK20] along with random partitioning solve the sparse case in $O(n^2/\alpha^2)$ bits of space
   - We run both algorithms in parallel and get the final algorithm

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2. The lower bound of $\Omega(n^2/\alpha^2)$ [DK20] makes our algorithm optimal
3. The $\mathrm{polylog}(n)$ overhead is not always necessary (Like [AS22])

Open Problems

- Could similar techniques to this work and [AS22] be used to bypass the $\mathrm{polylog}(n)$ overheads of other problems?
    - E.g. Dominating Set, Spectral Sparsification

- Can we get a deterministic algorithm for this problem that uses only $O(\frac{n^2}{\alpha^2})$ bits of space or improve the lower bound?
    - The current best deterministic algorithm is that of [DK20] which uses $O(\frac{n^2}{\alpha^2}\log\alpha)$ bits of space

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Thank you!

References I

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