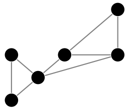
# Tight Bounds for Vertex Connectivity in Dynamic Streams

Sepehr Assadi & Vihan Shah

Rutgers University

# Vertex Connectivity

- Undirected Graph $G = (V, E)$
- Vertex Connectivity: Minimum number of vertices that need to be deleted to disconnect $G$



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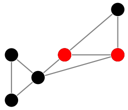

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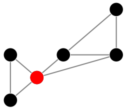

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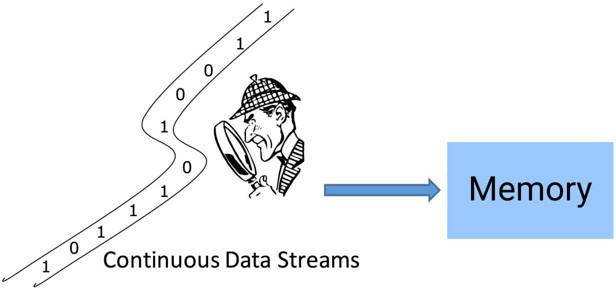
# Classical Setting

- Can find vertex connectivity in $\mathrm{polylog}\, m$ max flow time [LNP+21]
- Recent breakthrough for max flow: $m^{1+o(1)}$ time [CKL+22]
- Thus, finding vertex connectivity also takes $m^{1+o(1)}$ time

# Graph Streaming

- $G = (V, E)$
- Edges of $G$ appear in a stream
- Trivial Solution: Store all edges ($\Omega(n^2)$ space)
- Goal: Minimize Memory ($o(n^2)$ space)



# Streaming Models

**Insertion-Only**

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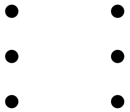
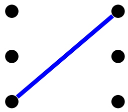


# Streaming Models

**Insertion-Only**

$\boxed{e_1}$



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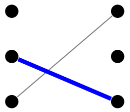
| | |
|---|---|
| $e_1$ | $e_2$ |



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| | | |
|---|---|---|
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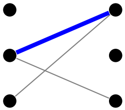
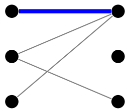


# Streaming Models

**Insertion-Only**

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|



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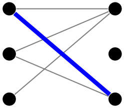
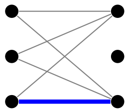
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|



# Streaming Models

**Insertion-Only**

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|---|---|---|---|---|---|



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| | | | | | | |
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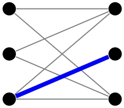
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |



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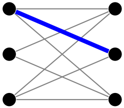

# Streaming Models

**Insertion-Only** (finite stream)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
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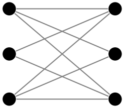

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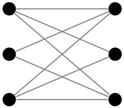

**Dynamic**

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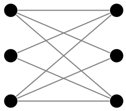

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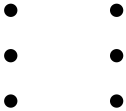

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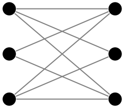

**Dynamic**

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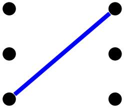
|---|



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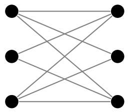
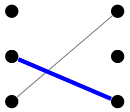


**Dynamic**

| | |
|---|---|
| $e_1$ | $e_2$ |



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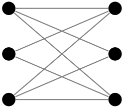

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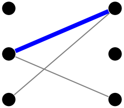

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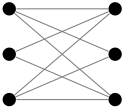
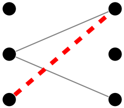


**Dynamic**

| | | | |
|---|---|---|---|
| $e_1$ | $e_2$ | $e_3$ | $\overline{e_1}$ |



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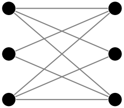

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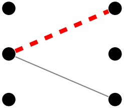

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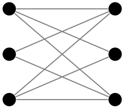

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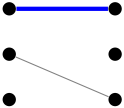

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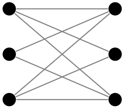

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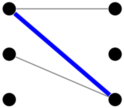

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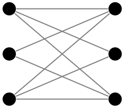

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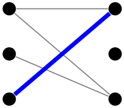

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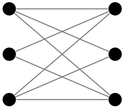

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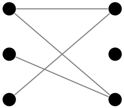

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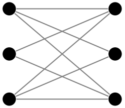

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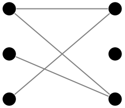

We want to solve the problem after a single pass of the stream

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- Finding exact vertex connectivity needs $\Omega(n^2)$ space in the worst case [SW15]

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- Finding exact vertex connectivity needs $\Omega(n^2)$ space in the worst case [SW15]
- We want to solve the $k$-vertex connectivity problem in streaming (is the vertex connectivity of the input graph $G < k$ or $\geq k$)
- We also want to output a certificate of connectivity (If $G$ is $k$-vertex connected, output a subgraph $H$ (certificate) that is also $k$-vertex connected)

# Previous Work

**Insertion-Only**

1. Upper bound: $\widetilde{O}(kn)$ [FKM+05]
2. Lower bound: $\Omega(kn)$ [SW15]

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**Dynamic**

1. Upper bound: $\widetilde{O}(k^2 n)$ [GMT15]
2. Lower bound: $\Omega(kn)$ [SW15]

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**Dynamic**

1. Upper bound: $\widetilde{O}(k^2 n)$ [GMT15]
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The lower bounds hold even when a certificate is not needed

# Previous Work

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**Dynamic**

1. Upper bound: $\widetilde{O}(k^2 n)$ [GMT15]
2. Lower bound: $\Omega(kn)$ [SW15]

There is a gap of factor $k$ between the best known upper and lower bound in dynamic streams

# Insertion-Only vs Dynamic Streams

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- Most graph problems studied in insertion-only streams, have similar guarantees in dynamic streams
- Examples: Connectivity [AGM12a], Cut Sparsifiers [AGM12b], Subgraph Counting [AGM12b], $(\Delta + 1)$-Vertex Coloring [ACK19]

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- However for Matching and Vertex Cover a 2-approximation in insertion-only streams takes space $\widetilde{O}(n)$ but an $O(1)$-approximation in dynamic streams needs $\Omega(n^2)$ space

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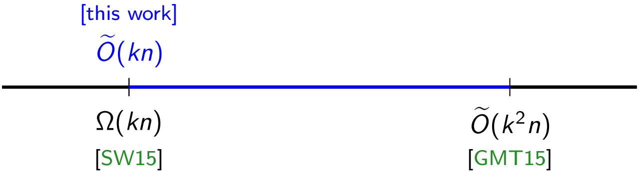
- However for Matching and Vertex Cover a $2$-approximation in insertion-only streams takes space $\widetilde{O}(n)$ but an $O(1)$-approximation in dynamic streams needs $\Omega(n^2)$ space
- It was unresolved which category vertex connectivity belonged to

# Our Results

We bridge the gap between the upper and lower bound in dynamic streams

**Theorem**

*There exists a randomized dynamic graph streaming algorithm for $k$-vertex connectivity that succeeds with high probability and uses $\widetilde{O}(kn)$ space.*



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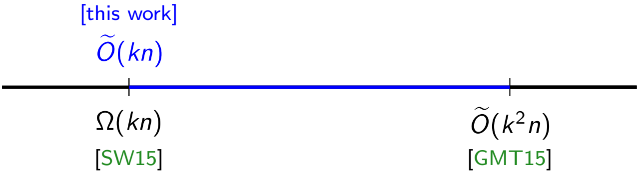


Note: We also output a certificate of vertex connectivity

# Our Results

We also extend the lower bound of [SW15] to multiple pass streams:

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*Any randomized $p$-pass insertion-only streaming algorithm that solves the $k$-vertex connectivity problem with probability at least $2/3$ needs $\Omega(kn/p)$ bits of space.*

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**Theorem**

*Any randomized $p$-pass insertion-only streaming algorithm that solves the $k$-vertex connectivity problem with probability at least $2/3$ needs $\Omega(kn/p)$ bits of space.*

Note: This lower bound is for multi-graphs (also the case for [SW15])

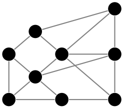
The upper bound also works for multi-graphs

# Algorithm of [GMT15]

For $i = 1$ to $r = O(k^2 \log n)$:

1. Sample every vertex in $V_i$ independently with probability $1/k$
2. Store a spanning forest $H_i$ on $G[V_i]$

Output $H = \cup_i H_i$ as the certificate



$G$

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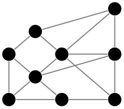
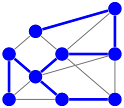


$G$



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# Guarantee of [GMT15]

[GMT15] proved the following:

- If $G$ is not $k$-connected then $H$ will not be $k$-connected
- If $G$ is $2k$-connected $H$ will be $k$-connected

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So, this was an approximation algorithm for $k$-vertex connectivity.

We give a better analysis of the same algorithm and show that it works for exact $k$-vertex connectivity.

# Space

- Each sample has $n/k$ vertices in expectation

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- Concentration bounds to get $\widetilde{O}(kn)$ space whp

# Key Properties

We have the following two key properties:

**Lemma (Property 1)**

*Every edge whose endpoints are less than $2k$ connected in $G$ exists in $H$ whp.*

**Lemma (Property 2)**

*Every pair of vertices that is at least $2k$ connected in $G$ is at least $k$ connected in $H$ whp. [GMT15]*

# Correctness

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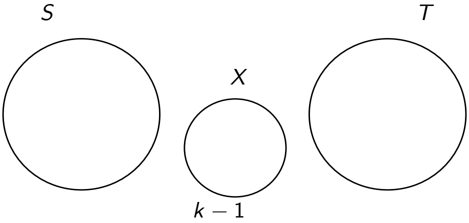

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- Assume $G$ is $k$-vertex-connected but $H$ is not $k$-vertex-connected
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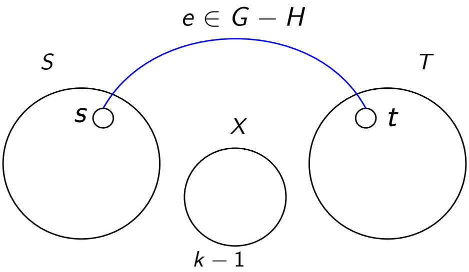
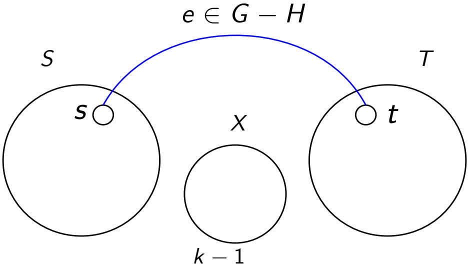
- There is an edge between $S$ and $T$ in $G$ but not in $H$



# Correctness

- Case 1: $s$ and $t$ have $< 2k$ vertex-disjoint paths between them in $G$



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- Case 1: $s$ and $t$ have $< 2k$ vertex-disjoint paths between them in $G$
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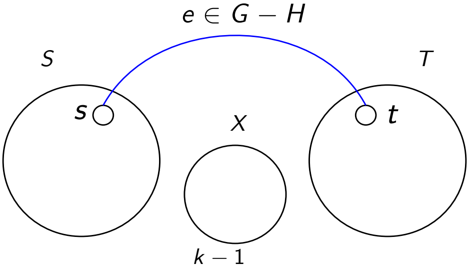

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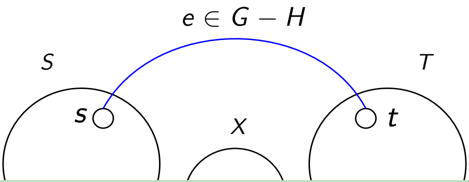


**Lemma (Property 1)**

*Every edge whose endpoints are less than $2k$ connected in $G$ exists in $H$ whp.*

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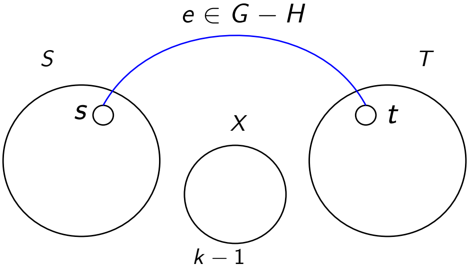
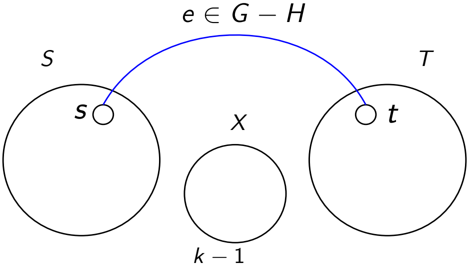


# Correctness

- Case 2: $s, t$ have $\geq 2k$ vertex-disjoint paths between them in $G$



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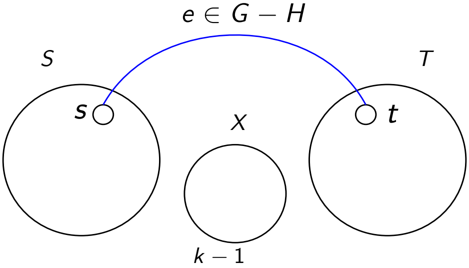
- Case 2: $s, t$ have $\geq 2k$ vertex-disjoint paths between them in $G$
- This means $s, t$ have $\geq k$ vertex-disjoint paths between them in $H$



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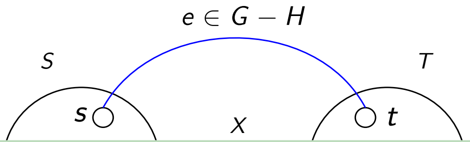


**Lemma (Property 2)**

*Every pair of vertices that is at least $2k$ connected in $G$ is at least $k$ connected in $H$ whp.*

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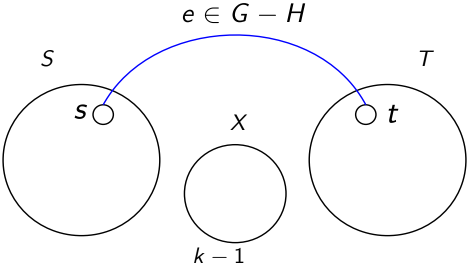
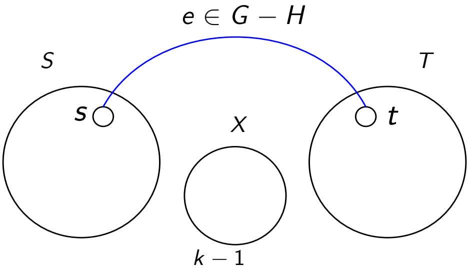


# Correctness

- Case 2: $s, t$ have $\geq 2k$ vertex-disjoint paths between them in $G$
- This means $s, t$ have $\geq k$ vertex-disjoint paths between them in $H$
- Thus, deleting $k-1$ vertices $(X)$ should not disconnect $s$ and $t$ in $H$



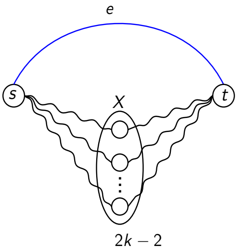
# Property 1

**Lemma**

*Every edge whose endpoints are less than $2k$ connected in $G$ exists in $H$ whp.*

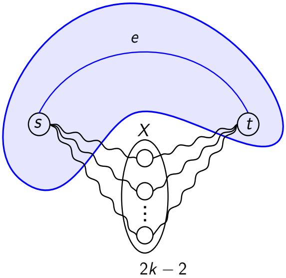
# Property 1

- Consider an edge $(s, t)$ whose endpoints are less than $2k$ connected
- If $s, t$ are sampled and $X$ is not then edge $(s, t)$ is in $H$
- Every spanning forest will contain the edge $(s, t)$



# Property 1

- $\Pr(s \text{ and } t \text{ sampled}) = 1/k^2$



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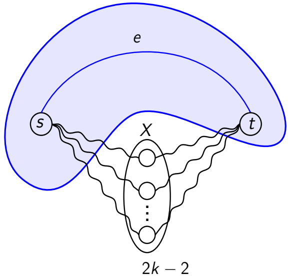

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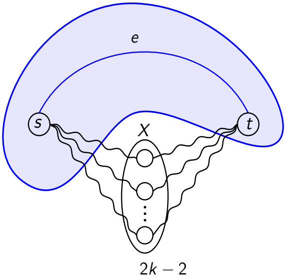

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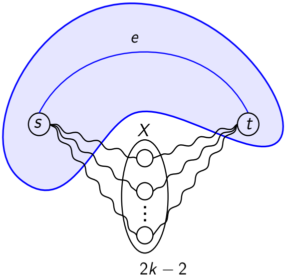

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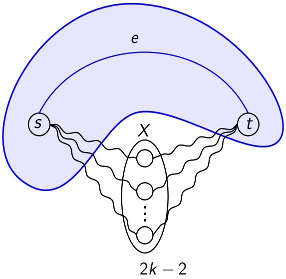
- Thus, whp in some iteration $s, t$ are sampled and $X$ is not



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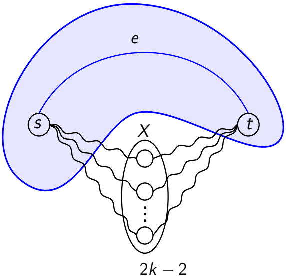
- The spanning tree in this iteration contains the edge $(s, t)$



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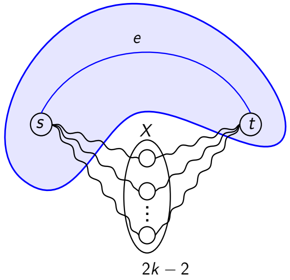

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- Thus, whp in some iteration $s, t$ are sampled and $X$ is not
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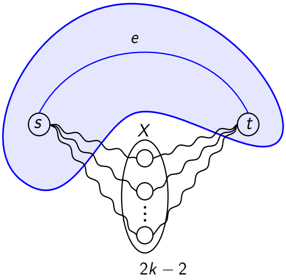
- Union bound over all such pairs
- Property 1 holds whp



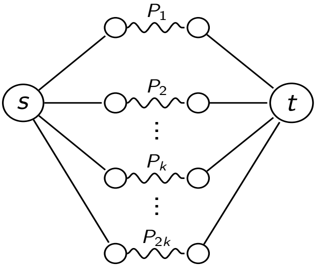
# Property 2

**Lemma**

*Every pair of vertices that is at least $2k$ connected in $G$ is at least $k$ connected in $H$ whp [GMT15].*

# Property 2

- Consider pair $s, t$ that is at least $2k$ connected



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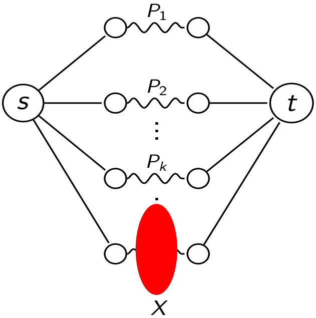
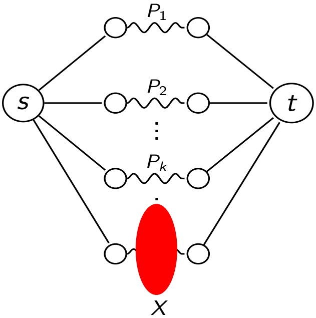
- Consider an arbitrary set $X$ of size $k-1$ (not containing $s, t$)



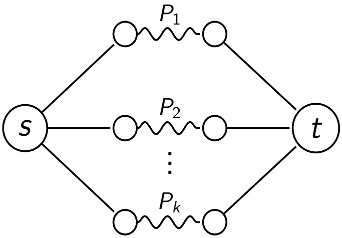
# Property 2

- Consider pair $s, t$ that is at least $2k$ connected
- Consider an arbitrary set $X$ of size $k - 1$ (not containing $s, t$)
- We will show that with very high probability $s, t$ are connected in the certificate $H$ even when $X$ is deleted



# Property 2

- We only focus on paths $P_1$ to $P_k$



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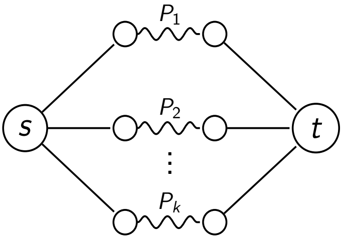
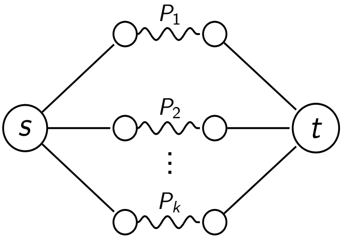
- $\Pr(X \text{ not sampled}) = (1 - 1/k)^{k-1} = \Theta(1)$



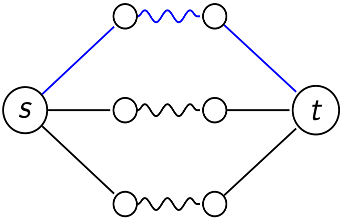
# Property 2

- We only focus on paths $P_1$ to $P_k$
- $\Pr(X \text{ not sampled}) = (1 - 1/k)^{k-1} = \Theta(1)$
- Consider only iterations where $X$ is not sampled (const fraction)



# Spanning Forest

- Need to sample at least one entire path



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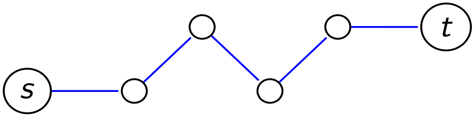
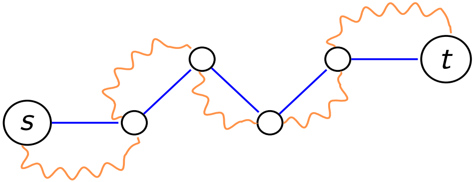


# Spanning Forest

- Need to sample at least one entire path
- Sample each edge in some iteration (i.e. sampling both endpoints)
- Spanning forest in each iteration will ensure that the endpoints are connected in the certificate $H$



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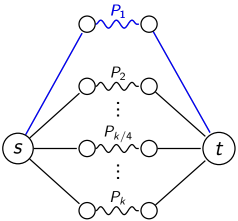

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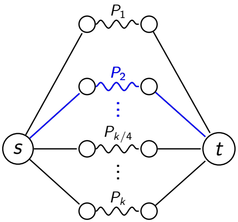

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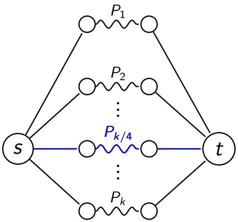

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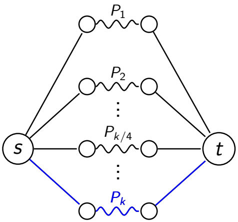

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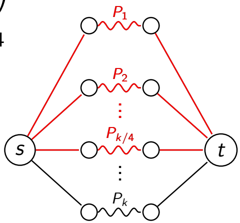
- An entire $P_i$ is sampled whp
- $$\Pr\left(\geq k/4\ P_i\text{'s not sampled}\right) \leq \binom{k}{k/4}\left(\frac{1}{poly(n)}\right)^{k/4}$$
$$\leq 2^k \cdot \left(\frac{1}{poly(n)}\right)^{k/4}$$
$$\leq \left(\frac{1}{poly(n)}\right)^{k}$$



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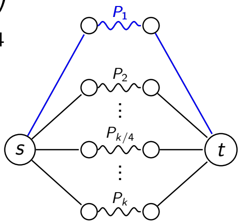


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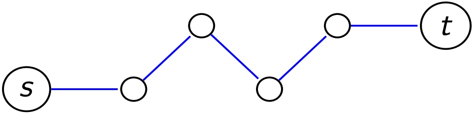
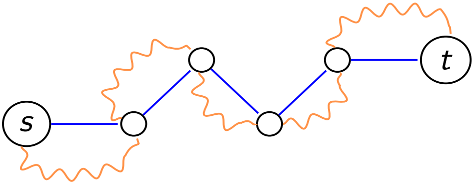


# Property 2

- At least one entire path is sampled with very high probability
- $s$ and $t$ are connected in the certificate $H$ with very high probability $(1 - 1/\mathrm{poly}(n^k))$



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- At least one entire path is sampled with very high probability

- $s$ and $t$ are connected in the certificate $H$ with very high probability $(1 - 1/\mathrm{poly}(n^k))$

- Union bound over all pairs $s, t$ and sets $X$: $n^2 \cdot n^k$ choices

# Property 2

- At least one entire path is sampled with very high probability
- $s$ and $t$ are connected in the certificate $H$ with very high probability $(1 - 1/\mathrm{poly}(n^k))$
- Union bound over all pairs $s, t$ and sets $X$: $n^2 \cdot n^k$ choices
- Property 2 holds whp

# Summary

We have two key properties:

**Lemma (Property 1)**

*Every edge whose endpoints are less than $2k$ connected in $G$ exists in $H$ whp.*

**Lemma (Property 2)**

*Every pair of vertices that is at least $2k$ connected in $G$ is at least $k$ connected in $H$ whp. [GMT15]*

# Summary

- There is a dynamic streaming algorithm that whp outputs whether the input graph $G$ is $k$-vertex connected or not using $\widetilde{O}(kn)$ space.

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- There is a dynamic streaming algorithm that whp outputs whether the input graph $G$ is $k$-vertex connected or not using $\widetilde{O}(kn)$ space.
- It does so by outputting a certificate of $k$-vertex connectivity.
- The lower bound of [SW15] is $\Omega(kn)$, making our algorithm optimal (up to polylog factors)
- The lower bound also holds when outputting a certificate is not required

# Summary

- There is a dynamic streaming algorithm that whp outputs whether the input graph $G$ is $k$-vertex connected or not using $\widetilde{O}(kn)$ space.
- It does so by outputting a certificate of $k$-vertex connectivity.
- The lower bound of [SW15] is $\Omega(kn)$, making our algorithm optimal (up to polylog factors)
- The lower bound also holds when outputting a certificate is not required
- We extend this lower bound to multiple passes and give a lower bound of $\Omega(kn/p)$ for $p$-pass insertion-only streaming algorithms.

# Open Problems

- We have settled the space of the $k$-vertex connectivity problem only up to polylog factors. So the question of optimal space bounds (up to constant factors) is still open.

- Our lower bound and those of Sun and Woodruff [SW15] use duplicate edges. Obtaining lower bounds for simple graphs is an open problem.

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**Thank you!**

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