Tight Bounds for Vertex Connectivity in Dynamic Streams

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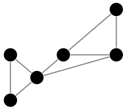
Rutgers University

Vertex Connectivity

- Undirected Graph $G = (V, E)$
- Vertex Connectivity: Minimum number of vertices that need to be deleted to disconnect G



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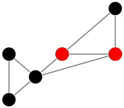

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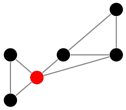
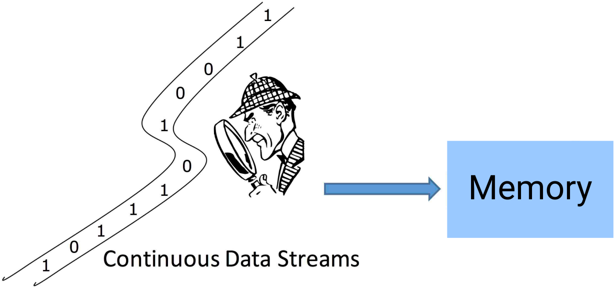


1

Graph Streaming

- $G = (V, E)$
- Edges of G appear in a stream
- Trivial Solution: Store all edges ($\Omega(n^2)$ space)
- Goal: Minimize Memory ($o(n^2)$ space)



Streaming Models

Insertion-Only

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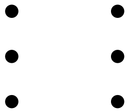
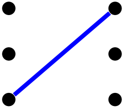


Streaming Models

Insertion-Only

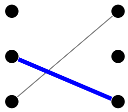
e_1



Streaming Models

Insertion-Only

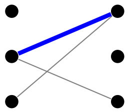
e_1	e_2



Streaming Models

Insertion-Only

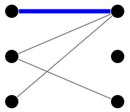
e_1	e_2	e_3



Streaming Models

Insertion-Only

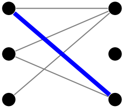
e_1	e_2	e_3	e_4



Streaming Models

Insertion-Only

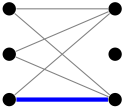
e_1	e_2	e_3	e_4	e_5



Streaming Models

Insertion-Only

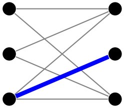
e_1	e_2	e_3	e_4	e_5	e_6



Streaming Models

Insertion-Only

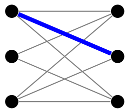
e_1	e_2	e_3	e_4	e_5	e_6	e_7



Streaming Models

Insertion-Only

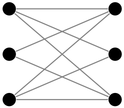
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8



Streaming Models

Insertion-Only (finite stream)

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8



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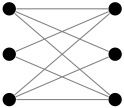

Dynamic

Streaming Models

Insertion-Only (finite stream)

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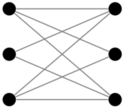

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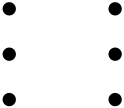

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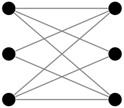
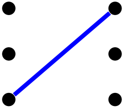


Dynamic

e_1



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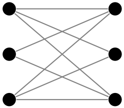
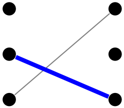


Dynamic

e_1	e_2



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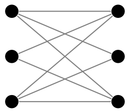

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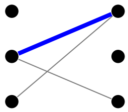
e_1	e_2	e_3



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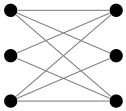
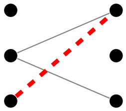


Dynamic

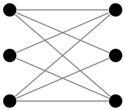
e_1	e_2	e_3	$\overline{e_1}$



Streaming Models

Insertion-Only (finite stream)

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8



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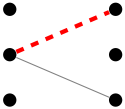
e_1	e_2	e_3	$\overline{e_1}$	$\overline{e_3}$



Streaming Models

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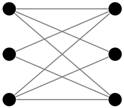
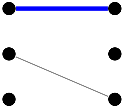


Dynamic

e_1	e_2	e_3	$\overline{e_1}$	$\overline{e_3}$	e_4



Streaming Models

Insertion-Only (finite stream)

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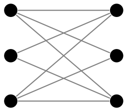

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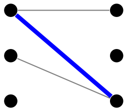

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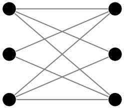

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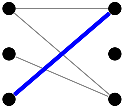

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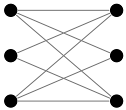

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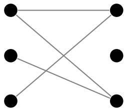

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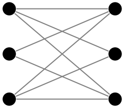

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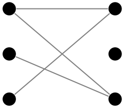

We want to solve the problem after a single pass of the stream

Our Problem

- Finding exact vertex connectivity needs $\Omega(n^2)$ space in the worst case [SW15]

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- Finding exact vertex connectivity needs $\Omega(n^2)$ space in the worst case [SW15]
- We want to solve the k-vertex connectivity problem in streaming (is the vertex connectivity of the input graph $G < k$ or $\geq k$)
- We also want to output a certificate of connectivity (If G is k-vertex connected, output a subgraph H (certificate) that is also k-vertex connected)

Previous Work

Insertion-Only

1. Upper bound: $\widetilde{O}(kn)$ [FKM+05]
2. Lower bound: $\Omega(kn)$ [SW15]

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Dynamic

1. Upper bound: $\widetilde{O}(k^2 n)$ [GMT15]
2. Lower bound: $\Omega(kn)$ [SW15]

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1. Upper bound: $\widetilde{O}(kn)$ [FKM+05]
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Dynamic

1. Upper bound: $\widetilde{O}(k^2 n)$ [GMT15]
2. Lower bound: $\Omega(kn)$ [SW15]

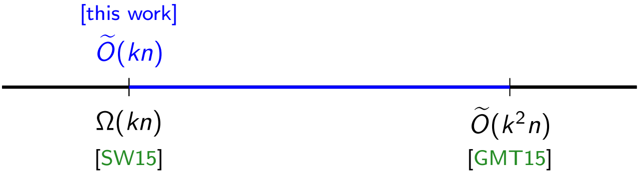
There is a gap of factor k between the best known upper and lower bound in dynamic streams

Our Results

We bridge the gap between the upper and lower bound in dynamic streams

Theorem

There exists a randomized dynamic graph streaming algorithm for k-vertex connectivity that succeeds with high probability and uses $\widetilde{O}(kn)$ space.



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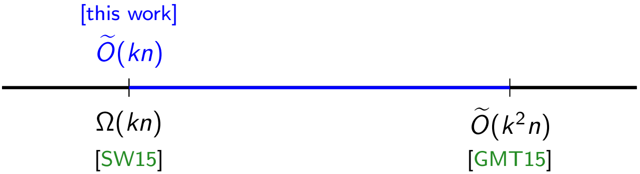


Note: We also output a certificate of k-vertex connectivity

Our Results

We also extend the lower bound of [SW15] to multiple pass streams:

Theorem

Any randomized p-pass insertion-only streaming algorithm that solves the k-vertex connectivity problem with probability at least $2/3$ *needs* $\Omega(kn/p)$ *bits of space.*

Our Results

We also extend the lower bound of [SW15] to multiple pass streams:

Theorem

Any randomized p-pass insertion-only streaming algorithm that solves the k-vertex connectivity problem with probability at least $2/3$ needs $\Omega(kn/p)$ bits of space.

Note: This lower bound is for multi-graphs (also the case for [SW15])

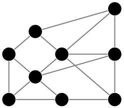
The upper bound also works for multi-graphs

Algorithm of [GMT15]

For $i = 1$ to $r = O(k^2 \log n)$:

1. Sample every vertex in V_i independently with probability $1/k$
2. Store a spanning forest H_i on $G[V_i]$

Output $H = \cup_i H_i$ as the certificate



G

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G H_1

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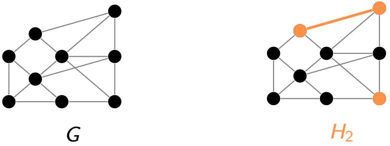

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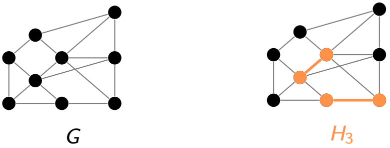

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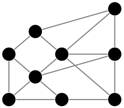

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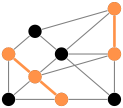

H_4

Algorithm of [GMT15]

For $i = 1$ to $r = O(k^2 \log n)$:

1. Sample every vertex in V_i independently with probability $1/k$
2. Store a spanning forest H_i on $G[V_i]$

Output $H = \cup_i H_i$ as the certificate

Open Problems

- We have settled the space of the k-vertex connectivity problem only up to polylog factors. So the question of optimal space bounds (up to constant factors) is still open.

- Our lower bound and those of Sun and Woodruff [SW15] use duplicate edges. Obtaining lower bounds for simple graphs is an open problem.

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You can visit my poster!

Thank you!

References I

Joan Feigenbaum, Sampath Kannan, Andrew McGregor, Siddharth Suri, and Jian Zhang, *On graph problems in a semi-streaming model*, Theor. Comput. Sci. **348** (2005), no. 2-3, 207–216.

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