Class Notes

线性代数

Math 146
Algebra II
Advanced Level
Winter 2020

By 李泽天

1.6

Chapter 1 : Vector Spaces

Vector Spaces

Def'n (Fields):

Roughly-Speaking: a field $\mathbb{F}$ is a set that

① Contains 2 distinct elements 0 and 1

② 4 operations : $+, -$, multiplication $\times$, inverse $\div$ of a non-zero term

of any two elements in the set is an element in that set

③ Satisfies "obvious" algebraic laws of additions and multiplication : ex: $\left.\begin{matrix}\mathbb{Q}\\ \mathbb{R}\\ \mathbb{C}\end{matrix}\right\}$ fields $\mathbb{Z}$ is not field

- Commutativity: $a+b=b+a$
- associativity: $(a+b)+c = a+(b+c)$
- distin : $a(b+c)= ab+ac$
- existence of Identities of addition and multiplication: $x+0=x$, $x\cdot 1=x$ $\forall x\in\mathbb{F}$
- Existence of the inverses of addition and multiplication : $\forall x\in\mathbb{F}$, $\exists y\in\mathbb{F}$ $x+y=0$, $\forall x\in\mathbb{F}\setminus\{0\}$ $\exists z\in\mathbb{F}$ s.t. $x\cdot z=1$

Def'n ② (Vector Spaces) : a vector space over a field $\mathbb{F}$ is a set V on which two operations

① addition: $V\times V\to V$
$(x,y)\mapsto x+y\in V$ (sum of x and y)

② Scalar Multiplication: $\mathbb{F}\times V\to V$
$(c,x)\mapsto cx\in V$

are defined and satisfies the following conditions (8 conditions)

Addition:

- VS1 : $a+b=b+a$ (Commutativity of addition)
- VS2 : $(a+b)+c = a+(b+c)$ (associativity of addition)
- VS3 : exist an element, 0 in V s.t. $x+0=x$
- VS4 $\forall x\in V$, $\exists y\in V$ s.t. $x+y=0$ additive inverse

Multiplication:

- VS5 $1\cdot x = x\in V$
- VS6 $(ab)x = a(bx)$ for $\forall a,b\in\mathbb{F}$, $x\in V$
- VS7 $a(x+y) = ax+ay$
- VS8 $(a+b)x = ax+bx$

To qualify as a vector space, the set V and the operations of addition and multiplication must adhere to a number of requirements called axioms.[1] In the list below, let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be arbitrary vectors in V, and a and b scalars in F.

Axiom	Meaning
Associativity of addition	$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
Commutativity of addition	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0}\in V$, called the *zero vector*, such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$.
Inverse elements of addition	For every $\mathbf{v}\in V$, there exists an element $-\mathbf{v}\in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v})=(ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v}=\mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$

These axioms generalize properties of the vectors introduced in the above examples. Indeed, the result of addition of two ordered pairs (as in the second example above) does not depend on the order of the summands:

$$(x_\mathbf{v},y_\mathbf{v})+(x_\mathbf{w},y_\mathbf{w})=(x_\mathbf{w},y_\mathbf{w})+(x_\mathbf{v},y_\mathbf{v}).$$

Elements in $\mathbb{F}$ are call <u>scalars</u>

Elements in V are call <u>vectors</u>

<u>Note</u> a vector space V = a vector space over a given field $\mathbb{F}$

<u>Remark</u> from (VS2) $(a+b)+c = a+(b+c)$ We can define the addition of any finite number of vectors in V

Does not depend on the parentheses the order from (VS1 and VS2)

let V be a vector space, and $x, y, z, w \in V$

$$((x+y)+z)+w \overset{VS2}{=} (x+(y+z))+w$$
$$\overset{VS1}{=} ((y+z)+x)+w$$
$$\overset{VS2}{=} (y+z)+(x+w)$$
$$\vdots$$
$$= x+y+z+w$$

Counter example: $1+2+3\cdots = \infty \notin \mathbb{R}$

4 Standard examples (Denote $\mathbb{F}$ is a field)

Ex① : $\mathbb{F}^n = \{(a_1, a_2, \cdots, a_n) \mid a_k \in \mathbb{F}, 1 \le k \le n\}$ n-tuples of entries in $\mathbb{F}$

Addition: let $a = (a_1, a_2, \cdots, a_n) \in \mathbb{F}^n$, $b = (b_1, b_2, \cdots, b_n) \mathbb{F}^n$

define $a+b = (a_1+b_1, \cdots, a_n+b_n)$

<u>Scalar multiplication</u> : Take $c \in F$, $a = (a_1, a_2, \cdots, a_n) \in \mathbb{F}^n$

$$ca = (ca_1, ca_2, \cdots ca_n) \in \mathbb{F}^n$$

<u>Claim</u>: $\mathbb{F}^n$ is a vector space over $\mathbb{F}$, with the above operations

<u>Proof</u>: Check all 8 conditions

For example (VS3) $\forall x \in V, \exists \vec{0} \in V$ s.t. $x + \vec{0} = x$

let $0 = (0, 0, \cdots, 0) \in \mathbb{F}^n$, take $x = (x_1, x_2, \cdots, x_n) \in \mathbb{F}^n$

by the defn of the addition : $x + 0 = (x_1+0, x_2+0, \cdots, x_n+0)$

$$= (x_1, x_2, \cdots, x_n)$$
$$= x$$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$

$\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$

$\mathbb{C}^n$ is a vector space over $\mathbb{C}$

$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

is $\mathbb{Q}^n$ a vector space over $\mathbb{R}$? no

is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

$\mathbb{R} \times \mathbb{Q}^n \to \mathbb{Q}^n$ No

$\pi \cdot \vec{q} \notin \mathbb{Q}^n$

1.8

Last time: 4 Standard examples (Denote $\mathbb{F}$ is a field)

Ex① : $\mathbb{F}^n = \{(a_1, a_2, \cdots, a_n) \mid a_k \in \mathbb{F}, 1 \le k \le n\}$ n-tuples of ent in $\mathbb{F}$

Notation: Denote elemenes in $\mathbb{F}^n$ as column vectors : $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ or $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

② $M_{m\times n}(\mathbb{F})$ = $\left\{ \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{is called an } m\times n \text{ matrix with entries in } \mathbb{F}} \;\middle|\; \begin{matrix} a_{ij} \in F \\ 1 \le j \le m \\ 1 \le j \le n \end{matrix} \right\}$

m rows n columns

rectangular arrays

Denote $m\times n$ matrix : A

A_{ij} = entry of A at row ith, column jth

Two $m\times n$ matries $A, B \in M_{m\times n}(\mathbb{F})$ are said to be equal, $A = B$ if $A_{ij} = B_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$

Define: (1) addition: component · addition

Given $A, B \in M_{m\times m}(\mathbb{F})$, define $A + B$ be a matrix : $(A+B)_{ij} = A_{ij} + B_{ij}$

(2) Scalar multiplication

Given $A \in M_{m\times n}(\mathbb{F})$, $c \in \mathbb{F}$, define cA be a matrix $(cA) = cA_{ij}$

row column

Claim: $M_{m\times m}(\mathbb{F})$ is a vector space over $\mathbb{F}$, under those operations

Ex③ Space of Functions $\mathbb{F}^D = \{f : D \to \mathbb{F}\}$ is a vector space over $\mathbb{F}$ under the following operations

function

(i) take $f, g \in \mathbb{F}^D$, we define $f+g : D \to \mathbb{F}$ $(f+g)(x) = f(x) + g(x)$, $x \in D$ addition in $\mathbb{F}$

(ii) take $f \in \mathbb{F}^D$, $c \in \mathbb{F}$, we define $cf : D \to \mathbb{F}$, $(cf)(x) = c \cdot f(x)$, $x \in D$ multiplication in $\mathbb{F}$

Proof: Check all 8 Conditions

Ex (4): (Space of Polynomials)

let $\mathbb{F}$ be a field, a Polynomial with Coefficients in $\mathbb{F}$ is a form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_1, a_2, \cdots a_n \in \mathbb{F}$, x: Variable

ex: $1+2x-3x^2$ is a Polynomial with Coefficient $\mathbb{R}$

$P_n(\mathbb{F}) = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_k \in \mathbb{F}\} = \{$ Set of all Polynomial of degree at most $n \}$

<u>Recall</u>: $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_k \in F$

$\deg f = \max\{k, 0 \le k \le n \mid a_k \neq 0\}$ largest index s.t. $a_k \neq 0$

$\deg(c) = 0$, $c \in \mathbb{F}$, $c \neq 0$

$\deg(c) = -1$ Convention

$\deg(3x^2 - x + 1) = 2$

<u>Denote</u> $\mathbb{F}[x] = \{$ Set of all Polynomials (of all finite degrees) with Coefficients in $\mathbb{F}$

Ex: $1 + \sqrt{2}x \in \mathbb{R}[x]$

Ex (4) (Space of Polynomials) :

let $f, g \in P_n(\mathbb{F})$, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_k \in \mathbb{F}$

$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$. $b_k \in \mathbb{F}$

① Denote $f+g$ a Polynomial $(f+g)(x) = (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_1 + b_1)$ addition

② take $c \in \mathbb{F}$, define $c \cdot f$ a new Polynomial $(c \cdot f)(x) = c a_n x^n + \cdots + c a_1 x + c a_0$ Multiplication

<u>Note</u> $f(x) = a_n x^n + \cdots + a_1 x + a_0$
$g(x) = b_m x^m + \cdots + b_1 x + b_0$
> we can define $f+g$ as follows

Wlog. we can assume $n \geqslant m$, we can set $b_j = 0$ for all $j > m$, then $g(x) = b_n x^n + \cdots + b_1 x + b_0$

$f+g$

Claim: $F[x]$ is a vector space under these operations (check all 8 conditions)

Counter examples (not Vector Space) ① $N[X]$ is not a vector space over R (Scalar multiplication is not well defined, additive inverse does not exist)

② $S=\{(a,b)\in R\}$

Define $(a_1,b_1)+(a_2,b_2)=(a_1+2a_2,\ b_2+b_1)$

$c(a_1,b_1)=(ca_1,cb_1)$

S is not a vector space since $\underbrace{(a_1,b_1)+(a_2,b_2)}_{(a_1+2a_2,\ b_1+b_2)} \neq \underbrace{(a_2+b_2)+(a_1+b_1)}_{(a_2+2a_1,\ b_1+b_2)}$

Basic Properties of vector spaces

Theorem 1 (Cancellation Law) Let V be a Vector space and $x,y,z\in V$, if $x+z=y+z$, then $x=y$

Proof: let $\check{y}\in V$, s.t. $z+\check{y}=0$

$$
\begin{aligned}
x &= x+0 \quad (VS3)\\
&= x+(z+\check{y}) \quad (VS4)\\
&= (x+z)+\check{y} \quad (VS2)\\
&= (y+z)+\check{y} \quad (\text{assumption}) \overset{VS2}{=} y+(z+\check{y}) = y+0 = y
\end{aligned}
$$

Note: If $z+x=z+y$, then $x=y$, because $z+x=x+z$, $x+y=y+z$

Corollary 1 let V be a Vector space, $x\in V$, Then

① There is a unique Vector 0: $x+0=x$

It means If 0_1 and $0_2\in V$ s.t. $z+0_1=z$, $z+0_2=z$, $\forall z\in V$, then $0_1=0_2$

Proof assume $0_1, 0_2$ be two Vector in V, s.t. $z+0_1=z$, $z+0_2=z$ $\forall z\in V$

Then $z+0_1=z+0_2$, by Cancellation law, $0_1=0_2$

② For $\forall x\in V$, $\exists y\in V$ s.t. $x+y=0$, Denote at y to be $(-x)$

Denote $x-z=x+(-z)$. - - - - -

Theorem 2 Same assumption as Theorem 1

① $\underset{F}{0}x=\underset{V}{0}$

② $(-a)x=-(ax)$
$=a(-x)$

③ $(-1)x=-x$

④ $\underset{F}{a}\,\underset{V}{0}=\underset{V}{0}$

Today: ① Subspace (differ from books)
② linear and system linear combination

1.10

Recall Theorem let V be a vector space

Then ① $\underset{\mathbb{F}}{0}\cdot\underset{V}{x}=\underset{V}{0}$ for $\forall x\in V$

② $\underset{\mathbb{F}}{c}\cdot\underset{V}{0}=\underset{V}{0}$ for $\forall c\in\mathbb{F}$

③ $(-a)x=-(ax)=a(-x)$ For $\forall a\in\mathbb{F}, x\in V$

Proof $(-a)x+ax \overset{VS8}{=} ((-a)+a)\cdot x = \underset{\mathbb{F}}{0}\cdot\underset{V}{x} \overset{①}{=} \underset{V}{0}$, So $(-a)x=-ax$

$ax+a(-x) \overset{VS7}{=} a[x+(-x)] = \underset{\mathbb{F}}{a}\cdot\underset{V}{0} \overset{②}{=} \underset{V}{0}$

Question let V be a vector space over field $\mathbb{F}$, let $\in\mathbb{F}$. Find all solutions of x of $c\cdot x=0_V$

$c=0,\ x\in\mathbb{F}$

$c\neq 0,\ c^{-1}(cx)=c^{-1}0_v=0$

$\quad\quad \| VS6$

$\quad (c^{-1}c)x = 1\cdot x \overset{VS5}{=} x$

Definition let W be a subset of a vector space over a field $\mathbb{F}$ then W is called a subspace of V if it satisfies the following conditions

S① $W\neq\emptyset$

S② Closed under addition: If $x,y\in W$, then $x+y\in W$

S③ Close under scalar multiplication: If $x\in W$ and $c\in F$, then $c\cdot x\in W$

Theorem let V be a vector space, then W under the operations of V restricted is a vector space over $\mathbb{F}$

Proof: Verify all 8 conditions of a vector space except VS3, VS4, satisfy for elements in V because V is a vector space since W is a subset of V, all those 6 conditions hold for elements in W

VS1 $x+y=y+x$

VS2 $(x+y)+z=x+(y+z)$

☆ VS3 $\exists\ 0\in V$ s.t. $x+0=x$

☆ VS4 $\exists\ u\in V$, s.t. $x+u=0$

VS5: $1\cdot x=x$

VS6: $a(bx)=(ab)x$

VS7 $a(x+y)=ax+ay$

VS8 $(a+b)x=ax+bx$

$\forall x,y,z\in V,\ a,b\in\mathbb{F}$

Verify VS3 Since $W\neq 0$, let $u\in W$ $0\in F$, then $0u\in W$ by S3
, on the other hand, scale mult in V. $0\cdot u \overset{\text{basic Property}}{=} 0_v$
} So $0_v\in W$

Verify VS3   let $x \in W$, Define $u = (-1)\cdot x \in W$ by S3 : $x + u = x + (-1)x \overset{VS5}{=} 1\cdot x + (-1)x \overset{VS8}{=} (1+(-1))x = 0x \overset{\text{Theorem ①}}{=} 0$

Recall : $-ax = a(-x) = (-a)x$ In Particular, $(-1)x = -x$

(have subspace)

Ex: ① let V be a vector space, then $\{\vec{0}\}$, V are two subspace of V

② $P_2(R)$ is a subspace of $R[x]$

Proof (i) obvious $P_2(R)$ is a subset of $\mathbb{R}[x]$

(ii) $0 \in P_2(R)$, so $P_2(R) \neq \emptyset$

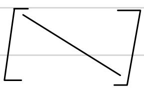
(iii) Take $f, g \in P_2(R)$ $f(x) = a_2x^2 + a_1x + a_0$, $g(x) = b_2x^2 + b_1x + b_0$, $a_k, b_k \in R$

$(f+g)(x) \overset{def'n}{=} (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$

(iv) take $f(x) = a_2x^2 + a_1x + a_0$. take $c \in R$, then $(cf)(x) = c\cdot a_2x^2 + c\cdot a_1x + c\cdot a_0$

By def'n of subspaces, $P_2(R)$ is a subspace of $\mathbb{R}[x]$

Ex③: $W = \{\overset{a_{ij}}{A} = M_{n\times n}(R) \mid \sum_{k=1}^{n} a_{kk} = 0\}$



$W \subset M_{n\times n}(R)$ is a subspace of $M_{n\times n}(R)$

Ex④: $W = \{A \in M_{n\times n}(R) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ is not subspace of $M_{n\times n}(R)$, Since it is not closed under addition

$C = A + B$ $\sum_{k=1}^{n} c_{kk} = \sum_{k=1}^{n} (a_{kk} + b_{kk}) = \sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = 1 + 1 = 2$

$C \notin W$

Remark : To check $W \neq \emptyset$ in the subspace def'n, we will check $0 \in W$

Basic question : Given a subset of a vector V, Verify W is a subspace (3 conditions)

Linear Combinations

Def'n : let V be a vector space and S be a subset of V, let $\vec{x} \in V$, The vector $\vec{x}$ is called a **linear combination of vector in S**

If there exist a finite number of vector in S, $\vec{u}_1, \cdots, \vec{u}_n \in S$ and $C_1, \cdots, C_n \in \mathbb{F}$

Such that $\vec{x} = C_1 \vec{u}_1 + \cdots\cdots + C_n \cdot \vec{u}_n$

Note: S could be an infinite set

Basic Question Check $\vec{x} \in W$, whether $\vec{x}$ is a linear combination of vectors in S

Example $V = M_{2\times 2}(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in R \right\}$

$S = \{ \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\vec{u}_2}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\vec{u}_3} \}$

is $\vec{x} = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ a linear Combination of vectors in S?

Answer: Consider $\vec{x} = C_1 \vec{u}_1 + C_2 \cdot \vec{u}_2 + C_3 \vec{u}_3$ for $C_1, C_2, C_3 \in R$

$$\begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = C_1 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + C_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1 + C_2 + C_3 & 2C_2 + C_3 \\ -C_1 + 3C_2 & 4C_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} C_1 + C_2 + C_3 & 2C_2 + C_3 \\ -C_1 + 3C_2 & 4C_2 \end{bmatrix} \Rightarrow \begin{cases} C_1 + C_2 + C_3 = 3 \\ 2C_2 + C_3 = 1 \\ -C_1 + 3C_2 = 4 \\ 4C_2 = 5 \end{cases}$$

System of linear equations.

1.13

Today: Span of a set S in a given vector space $\underline{V}$

Recall let V be a vector space and S be a non-empty subset of V

a vector $x \in V$ is a linear combination of vector in S if there exists a finite number of vector in S.

Why finite? $\mathbb{Q} \not\ni \sum_{n=1}^{\infty} m = 1+2+\cdots$

Vectors: $u_1, \cdots, u_n \in S$

Scaler $c_1, \cdots c_n \in \mathbb{F}$

Such that $x = c_1 u_1 + \cdots + c_n u_n$

Today, we also say x is a linear combination of $u_1, \cdots u_n$

$c_1, \cdots, c_n$ are called the coefficients of the linear combination

Remark ① S could be infinite

Set $\to$ non-empty $\to \neq \emptyset$

Set $\to$ finite / infinite

Ex. $V = \mathbb{R}[x] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_k \in \mathbb{R}, n \geq 0\}$, $S = \{1, x^2, x^4, \cdots\}$

A vector $f(x) \in V$ is a linear combination of vectors in S if $f(x) = c_0 + c_1 x^2 + \cdots + c_m \cdot x^{2m}$

② S is a finite set, $S = \{w_1, \cdots w_m\} \subset V$

Then vector x is a linear combination of vectors in S if x is a linear combination of $w_1, \cdots, w_n$

$x = c_1 w_1 + \cdots + c_m w_m$ for some $c_1, \cdots, c_m \in \mathbb{F}$

by defn. $x = c_1 w_{t_1} + \cdots + c_k w_{t_k}$. $\{w_{t_1}, \cdots w_{t_k}\} \subset S$

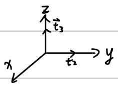
$= c_1 w_{t_1} + \cdots + c_k w_{t_k} + \underbrace{\sum_{u \in S \setminus J} 0\, u}_{}$ 系项

Definition: Let V be a Vector Space and S be a non-empty subset of V, Denote $\text{Span}(S) = \{$all linear Combination of vectors in $S\}$

For Convenience, $\text{Span}\{\emptyset\} = \{\vec{0}\}$

In Particular, if S is a finite set, $S = \{w_1, \cdots, w_m\} \subset V$, then $\text{Span}(S) = \{c_1 w_1 + \cdots + c_m w_m \mid c_1, \cdots, c_m \in \mathbb{F}\}$

Ex: ① $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset V$



$$\text{Span}(S) = \left\{ c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix} \middle| c_1, c_2 \in \mathbb{R} \right\}$$
$$= yz\text{-Plane}$$

② $V = \mathbb{R}[x]$, $S = \{x, x^2, x^3, \cdots\}$, $\text{Span}(S) = \{c_1 x + c_2 x^2 + \cdots + c_m x^m \mid c_1, \cdots, c_m \in \mathbb{R},\ m \geq 1, m \in \mathbb{N}\}$

$$= \{f(x) \in \mathbb{R}[x] \mid f(0) = 0\}$$

c_0 : free coefficient $- f(0)$

Theorem Let V be a Vector space and S be a subset of V, then $\text{Span}(S)$ is a subspace of V

More over, $\text{Span}(S)$ is the smallest subspace of V that contains S

Take W subspace of V and $W \supset S$, then $\text{Span}(S) \subset W$

Proof Step 1: Prove $\text{Span}(S)$ is subspace of V

If S is empty, $\text{Span}\,\emptyset = \{\vec{0}\}$ is a subspace of V

(i) If $S \neq \emptyset$, take $x \in S$, $0 \in \mathbb{F}$, $0 = 0 \cdot x \in \text{Span}(S)$, so $\text{Span}(S) \neq \emptyset$

(ii) $\text{Span}(S)$ closed under condition, take $u, v \in \text{Span}(S)$.

Therefore exists $u_1, \cdots, u_n \in S$ vector, $c_1, \cdots, c_n \in \mathbb{F}$ Scalar s.t. $u = c_1 u_1 + \cdots + c_n u_n$

$v_1, \cdots, v_n \in S$, $d_1, \cdots, d_n \in \mathbb{F}$ s.t. $v = d_1 v_1 + \cdots + d_n v_n$

$\Rightarrow u + v = c_1 u_1 + d_1 v_1 + \cdots + c_n u_n + d_n v_n$

$\{u_1, \cdots, u_n, v_1, \cdots, v_n\} \subset S$, by def'n $u + v \in \text{Span}(S)$

(iii) exercise: $\text{Span}(S)$ is closed under Scalar multiplication

Proof ② $\text{Span}\,S$ is the smallest subspace that contains S, take W a subspace of V and $S \subset W$, we will prove $\text{Span}(S) \subset W$

Let $x \in \text{Span}(S)$ by def'n, $\exists\, w_1, \cdots, w_k \in S$, $c_1, \cdots, c_k \in \mathbb{F}$ s.t. $x = c_1 w_1 + \cdots + c_k w_k$

Since $S \subset W$, we have $w_1, \cdots, w_k \in W$. Since W is a subspace, W is closed under Scalar multiplication $c_1 w_1, \cdots, c_k w_k \in W$

Since W is also closed under addition, $x = c_1 w_1 + \cdots + c_k w_k \in W$ — formally Prove by induction $k \to k+1$ Suppose $c_1 w_1 + \cdots + c_k w_k \in W$ Take $u_1, \cdots, u_{k+1} \in W$, $d_1, \cdots, d_{k+1} \in \mathbb{F}$ $\Rightarrow d_1 u_1 + \cdots + d_{k+1} u_{k+1} = (d_1 u_1 + \cdots + d_k u_k) + d_{k+1} u_{k+1} \in W$

Therefore $\text{Span}(S) \subset W$

$$\text{Span}(S) = \underset{\text{一个集合}}{\text{xx}} \cap W$$

W: subspace of V, $S \subset W$

Question: $\text{span}(S) = S$ if only if S is a subspace of V Prove it $S = V$, $\{\vec{0}\}$

Basic question: Given a vector space V, a subset S, $x \in V$, check $x \in \text{span}(S)$ or not, $\text{span}(S) = V$ or not

Tutorial: 1.13

Goal: Be able use the elimination to solve the system of linear equations

$$\begin{cases} a_1 + 2a_2 - a_3 + a_4 = 5 & (1) \\ a_1 + 4a_2 - 3a_3 - 3a_4 = 6 & (2) \\ 2a_1 + 3a_2 - a_3 + 4a_4 = 8 & (3) \end{cases}$$

If we get $C_1 = C_2$, but C_1, C_2 are distinct, then we stop and conclude the system

Step (1) eliminate two column of a_1

(1) add $-1 \times (1) + (2)$:

$$\begin{cases} a_1 + 2a_2 - a_3 + a_4 = 5 \\ 2a_2 - 2a_3 - 4a_4 = 1 \\ 2a_1 + 3a_2 - a_3 + 4a_4 = 8 \end{cases}$$

Step (2) Eliminate all but one occurrence of a_1: $(1) \times (-2) + (3)$

(2) $$\begin{cases} a_1 + 2a_1 - a_3 + a_4 = 5 & (1) \\ 2a_2 - 2a_3 - 4a_4 = 1 & (2) \\ -a_2 + a_3 + 2a_4 = -2 & (3) \end{cases}$$

Step (3) Eliminated all but one occurrence of a_2

a: $(3) \times 2 + (1)$: $$\begin{cases} a_1 + \underline{0} + a_3 + 3a_4 = 1 & (1) \\ 2a_2 - 2a_3 - 4a_4 = 1 & (2) \\ -a_2 + a_3 + 2a_4 = -2 & (3) \end{cases}$$

b: $(3) \times 2 + (2)$: $$\begin{cases} a_1 + 0 + a_3 + 5a_4 = 1 & (1) \\ 0 = -3 & (2) \\ -a_2 + a_3 + 2a_4 = -2 & (3) \end{cases}$$

By far we get contradiction, the system of equations has no solutions

Summary:
- Add a scholar multiple of one equation to another.
- multiply an equation by a scholar
- Swap two equations

$$\begin{cases} a_1 + 2a_2 - a_3 + a_4 = 5 & ① \\ a_1 + 4a_2 - 3a_3 - 3a_4 = 6 & ② \\ 2a_1 + 3a_2 - a_3 + \cancel{4}\,3\,a_4 = 8 & ③ \end{cases}$$

After changing "4" as "3" in equations, we get $\begin{cases} a_1 + \quad + a_3 + 3a_4 = 1 \\ \qquad -a_4 = -3 \\ a_2 + a_3 + a_4 = -2 \end{cases}$

Summary: Sometimes, we can not transform my system which every variable only occurs once!

When we should stop, eliminate "a_3" in the example, If when ever I try to eliminate one occuron of a_3, I will get one ocuron of "a_1" and "a_2", then we stop eliminate of a_3

Step ④: Continue eliminate the occurence of a_4:

② $\times \frac{3}{2}$ + ②: (a) $\begin{cases} a_1 + a_3 = -\frac{7}{2} \\ -2a_4 = -3 \\ -a_2 + a_3 + a_4 = -2 \end{cases}$ (b) $2 \times \frac{1}{2}$ + ③ $\begin{cases} a_1 + a_3 = -\frac{7}{2} \\ -2a_4 = -3 \\ -a_2 + a_3 = -\frac{7}{2} \end{cases}$

Step ⑤: $\begin{cases} a_1 = -a_3 - \frac{7}{2} \\ a_3 = a_3 \\ a_4 = \frac{3}{2} \\ a_2 = a_3 + \frac{7}{2} \end{cases} \Longrightarrow$ let $a_3 = t$ $\begin{cases} a_1 = t - \frac{7}{2} \\ a_2 = t + \frac{7}{2} \\ a_3 = t \\ a_4 = \frac{t}{2} \end{cases}$

1.15

Def'n Let V be a vector space and S be a subset of V, we say S generates (spans) V iff $\text{Span}(S) = V$

Question: Verify whether at set S spans a vector space V

Remark Since S is a subset of V, $\text{Span}(S)$ is a subspace V

To verify the set S spans V we only need to verify $V \overset{?}{\subseteq} \text{Span}(S)$

It means take $x \in V$ arbitrary show that x is a linear combination of vectors in S

Example ① show that $\{(1,0,0), (0,1,1), (1,0,1)\}$ generates $\mathbb{F}^3$, where $\mathbb{F}$ is a given field

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\\1\end{bmatrix}$

Answer: Take $\begin{bmatrix}a\\b\\c\end{bmatrix} \in F^3$ where $a,b,c \in \mathbb{F}$, Find c_1, c_2, c_3 in terms of a, b, c

Consider $\begin{bmatrix}a\\b\\c\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}c_1+c_3\\c_2\\c_2+c_3\end{bmatrix}$

We have $\begin{cases} c_1+c_2=a \\ c_2=b \\ c_2+c_3=c \end{cases} \Rightarrow \begin{cases} c_1=a+b-c \\ c_2=b \\ c_3=c-b \end{cases}$, So $\begin{bmatrix}a\\b\\c\end{bmatrix} = (a+b-c)\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + (c-b)\begin{bmatrix}1\\0\\1\end{bmatrix}$, which complete the proof

Example ② Consider $F^n = \left\{ \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} \middle| \ a_k \in \mathbb{F},\ 1 \le k \le n \right\}$ a vector space

$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, e_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \cdots\cdots, e_n = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \in \mathbb{F}^n$ (finite)

Since $\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in F^n$, with $a_1, a_2, \cdots, a_n \in \mathbb{F}$, So $\{e_1, e_2, \cdots, e_n\}$ spans $\mathbb{F}^n$

Practice, $M_{m\times n}(\mathbb{F})$

Find a spanning set for $M_{m\times n}(\mathbb{F})$

Quick question: Let $S = \{x_1, \cdots, x_k\}$ be a spanning set for a vector space V

finite set

① $S_1 = \{x_1, \cdots, x_k, x_{k+1}\}$ Does S_1 span V? Yes

$\vec{v} \in V$, $\vec{v} = c_1x_1 + \cdots\cdots + c_kx_k + c_{k+1}x_{k+1}$

② $S_2 = \{x_1, \cdots\cdots, x_{k-1}\}$ Does S_2 always? span V?

given $x_1, x_2, \cdots x_{k-1} \in V$

Linear Dependence and Linear Independence Sets

Def'n: Let V be a Vector Space and S a subset of V

① S is called **linearly dependence** if there exist a finite number of **distince** vector in S, $u_1, u_2, \cdots, u_k \in S$ and scalars $\underbrace{c_1, c_2, \cdots c_k}_{\text{not all zeros}} \in \mathbb{F}$

Such that $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$

② If S is **not linear dependens**, S is called **linearly independence**

Remark ① For $\{u_1, \cdots u_k\} \subset V$, we always have following representation $0_V = 0u_1 + 0u_2 + \cdots + 0u_k$ (the trival representation of 0_V in terms of $u_1, u_2, \cdots u_k$)

② **not all** $c_1, \cdots, c_k$ are zeros = at least one of $c_1, c_2, \cdots c_k$ must be non-zero

$\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq 0_v$ / $c_1, \cdots, c_k \in \mathbb{R}$, $c_1^2 + \cdots + c_k^2 \neq 0$

$1 \cdot 0 + (-1) 0 = 0 \quad \forall v \in V$

Remark ③ a Set is **linearly independence** if for every distince Vectors $u_1, u_2, \cdots, u_k \in S$

whenever, $c_1u_1 + \cdots + c_ku_k = 0$, then $c_1 = c_2 = \cdots = c_k = 0$

Remark ④ $S = \emptyset$ is **linear independence** because $\emptyset$ is not linearly dependence

Remark ⑤: $S = \{\vec{v}\} \subset V$, case # 1 : $\vec{v} = \vec{0} = \underset{c}{1} \cdot \underset{S}{\vec{0}}$ ($c_1 = 1$, $u_1 = \vec{0}$), So $\{\vec{0}\}$ is **always linear dependence**

Case # 2: $\vec{v} \neq 0$, claim: $\{\vec{v}\}$ is **linearly independence**, Consider $0 \cdot \vec{v} = 0$, if $c \neq 0$, $c^{-1}(c\vec{v}) = c^{-1}0 = 0$

$\vec{v} = 0$, a Contradiction

So $c = 0$, therefore $\{\vec{v}\}$ is linearly independent

Remark ⑥ a finite set $S = \{u_1, \cdots, u_k\} \subset V$ is called a linearly independence if when ever $c_1u_1 + \cdots + c_ku_k = 0$ with $c_1, \cdots, c_k \in \mathbb{F}$

the only solution $c_1 = c_2 = \cdots = c_k = 0$

Clear: linearly dependence if the equation $c_1u_1 + \cdots + c_ku_k = 0$, has a non-zero solution $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \neq 0$

Fun example: $S = \{v_1, \cdots, v_k\} \subset V$ linearly independence, every vector in S is a linear combination of $\{w_1, \cdots, w_\ell\}$ Compare with k, ℓ (3.3 System of linear equation)

1.17

Linear Independent and linear.

Basis → Existence
→ Dimension

Example : Determine whether $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix} \right\}$ is linearly independent or not

Answer Consider: $C_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{cases} C_1 - 2C_2 - C_3 = 0 \\ -C_1 + 3C_2 + 3C_3 = 0 \\ 2C_1 + C_2 + 8C_3 = 0 \end{cases} \quad \text{Solve by substitution and subtraction} \Rightarrow \begin{cases} C_1 = -3C_3 \\ C_2 = -2C_3 \end{cases}$$

$(C_1, C_2, C_3) = (-3C_3, -2C_3, C_3)$ for any $C_3 \in \mathbb{F}$.

For example $(-3, -2, 1)$ is a Solution, which is non-zero

Therefore, S is linearly dependent

Theorem Let S be a subset of a Vector space V. Then S is linearly dependent iff either $S = \{\vec{0}\}$ or some vector in S is a linear Combination of other Vectors in S

Proof "$\Leftarrow$" easy exercise

"$\Rightarrow$" Given S is linearly dependent

Case #1 : $S = \{\vec{0}\}$ Done

Case #2 : $S \neq \{\vec{0}\}$ By def'n, $\exists$ $u_1, \dots, u_n \in S$, $C_1, \dots, C_n \in \mathbb{F}$ (not all zeros) distinct

S.t. $C_1 u_1 + \dots + C_n u_n = 0$

Since $C_1, \dots, C_n$ not all zeros, WLOG we can assume $C_1 \neq 0$

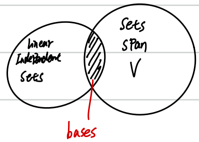
$C_1^{-1} \cdot (C_1 u_1 + C_2 u_2 + \dots + C_n u_n) = 0 \Rightarrow \underset{S}{\underset{\Uparrow}{u_1}} = -(C_1^{-1} C_2 u_2 + \dots + C_1^{-1} \cdot C_n u_n)$

It means $\overset{\in S}{u_1}$ is a linear combinations of other vectors $\{u_2, \dots, u_n\} \in S$

$C_1 u_1 + \dots + C_n u_n = 0$: Case ①: $n=1$, $C_1 u_1 = 0$, $C_1 \neq 0 \Rightarrow u_1 = 0 \in S$. Take $v \in S$, $v \neq 0$, $0 \in S$ is a linear combinations of $\underset{\neq 0}{v \in S}$

Case ② $|S| \geq 2$ $\quad \underset{\in S}{x} = C_1 x_1 + \dots + C_k x_k$, $x_k \in S$

Bases of Vector Spaces ;



Def'n: Let V be a vector space, a subset $S \subset V$ is called a **basis** of V it satisfies the 2 conditions

① S is linearly independent

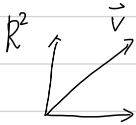
② S spans V, $V = \mathrm{Span}(S)$. (Spans $\supset S$, $\mathrm{span}(S) \supset S$, $\mathrm{span}(S)$ smallest subspaces)

$\forall x$ = element in $\mathrm{Span}(S)$

Questions: ① Given a vector space V, does a basis always exist? ←

② how to find a basis? ←

uniqueness of basis? No!



Relation between these basis
Same Cardinality
Dimensions of a vector
Why basis is so important?

Ex: ① F^n: $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots\cdots, e_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subset F^n$

S is linearly independent
S spans F^n
} S is a basis of F^n, called the standard basis of F^n

② $M_{m\times n}(F) = \{(a_{ij}) : 1 \le i \le m,\ 1 \le j \le n,\ a_{ij} \in F\}$

$S = \{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $M_{m\times n}(F)$

$E_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$ 1 is at row i^{th}, column j^{th}: standard basis

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

③ $P_n(F) = \{$all polynomials of degree at most $n\}$. a basis $\{1, x, \cdots, x^n\}$ an example of a finite set

$F[x] = \{$all polynomials$\}$

a basis $\{1, x, x^2, \cdots\}$ is an example a **countably infinite** set: there exist one to one mapping with N

Example of countable set: $\mathbb{Z}$, $\mathbb{Q}$

① finite set
or ② countable set (infinite)

Theorem Let V be a Vector space generate by a Countable subset S ($Span(S) = V$, S either finite or Countably finite)

Then, there exists a subset of S which is a basis for V

Idea : $S = \{u_1, ---, u_n\}$

$S = \{v_1, v_2, ----\}$, $v_{i_1} \in S$. $v_{i_1} \neq 0$, $v_{i_2} \notin Span\{v_{i_1}\}$, $v_{i_3} \notin Span\{v_{i_1}, v_{i_2}\}$ ....

1.20

Existance Theorem If a vector space V is generated by a Countable set S then some subset of S will be a basis for V

Provide Constructive Prove how to find a basis

Proof Sketch ① If $S=\emptyset$ or $S=\{0\}$ then $V=\{0\}$ and $\emptyset$ is a basis for V

② $Span(S)=V$, Since S is Countable, $S=\{v_1, v_2, \cdots, v_n\}$ or $S=\{v_1, \cdots, v_n, \cdots\}$

at least one non-zero element : i_1 Smallest index s.t. $v_{i_1} \neq 0$

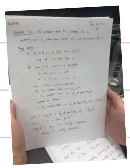
Ex. $S=\{0, 0, v_1, 2v_1, \frac{1}{3}v_1 v_2\}$

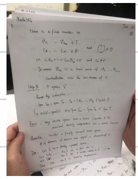
v_{i_1} v_{i_2}

i_2: Smallest index s.t. $v_{i_2} \notin Span(v_{i_1})$

Continuing this Process, $v_{i_j} \notin Span(\{v_{i_1}, \cdots, v_{i_{j-1}}\})$

$T=\{v_{i_k} \in S$ s.t. $v_{i_k} \notin Span(\{v_{i_1}, \cdots, v_{i_{k-1}}\})$





Claim: T is a basis for V

Step A show that T is linear independent (Unknown finite/infinite : use Contradiction)

Assume T is linear dependent

There exists a finite number, $v_{i_1}, \cdots, v_{i_m} \in T$, $c_1, \cdots, c_m \in \mathbb{F}$ $\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \neq 0$ s.t. $c_1 v_{i_1} + \cdots + c_m \cdot v_{i_m} = 0$ and $c_m \neq 0$

So $c_m^{-1} \cdot c_1 v_{i_1} + \cdots + c_m^{-1} \cdot c_{m-1} \cdot v_{i_{m-1}} + v_{i_m} = 0$, It means v_{i_m} is a linear combination of $v_{i_1}, \cdots, v_{i_{m-1}}$

Constradicts with the construction of T, So T is linear Independence

Step B T spans V, we know $span(S)=V$, Prove by induction. $Span(S_k) = Span(T_k)$ where $S_k=\{v_1, \cdots, v_k\}$, $T_k=\{v_{i_1}, \cdots, v_{i_q} \mid i_q \leq k\}$

Next step Prove $Span\, T = V$

$v \in V = Span(S)$, $v \in Span(S_m) = Span(T_m) \subset Span(T)$ for m big enough

Fact: any vector space has a basis (section 1.7) maximum linearly independence + Zorn's Lemma (always $Span(V)=V$)

Question: Consider a finitely spanned vector space

Relation between two bases of V. Answer: they have same number of element $\underline{\text{def}}$ dimension of V

Theorem: Let V be a finitely spanned vector space, let $\{v_1, \cdots, v_m\}$ be a basis for V. Let $\{w_1, \cdots, w_n\} \subset V$, and $n > m$

Then, $\{w_1, \cdots, w_n\}$ is linearly dependent

Proof: Assume $\{w_1, \cdots, w_n\}$ is linearly independent $w_1 = a_1v_1 + \cdots + a_m v_m$ $(a_1, \cdots, a_m) \neq 0$ Since if not, then $w_1 = 0$, contradicts the assumption

WLOG, we can assume $a_1 \neq 0$ so $a_1 v_1 = w_1 - a_2v_2 - \cdots - a_m v_m \Rightarrow v_1 = a_1^{-1} w_1 - a_1^{-1}a_2 v_2 - \cdots - a_1^{-1} a_m v_m$

v_1 is a linear combination $w_1, v_2, \cdots, v_m$, $v_1 \in \text{Span}\{w_1, v_2, \cdots, v_m\} \Rightarrow \underbrace{\text{Span}\{v_1, \cdots, v_m\}}_{=V} \subset \underbrace{\text{Span}\{w_1, v_2, \cdots, v_m\}}_{=V}$

Prove by induction — Statement: $\text{Span}\{v_1, v_2, \cdots, v_m\} = \text{Span}\{w_1, \cdots, w_n, v_{n+1}, \cdots v_m\} = V$ for $1 \leq x \leq m-1$

Base case $i=1$, we just proved. Assume the statement is true for n, we need to prove for $n+1$

That means we need to show $V = \text{Span}\{w_1, \cdots, w_{n+1}, v_{n+2}, \cdots, v_m\}$

$w_{n+1} \in V \overset{\text{assumption}}{=\!=} \text{Span}\{w_1, \cdots, w_1, v_{n+1}, \cdots, v_m\}$, so $w_{n+1} = a_1 w_1 + \cdots a_n w_n + b_{n+1} v_{n+1} + \cdots + b_m v_m$

If $b_{n+1} = \cdots = b_m = 0$ then $w_{n+1} = a_1 w_1 + \cdots + a_n w_n$ therefore, $\{w_1, \cdots, w_n\}$ is linearly dependent, a contradiction

Therefore, $(b_{n+1}, \cdots, b_m) \neq 0$ after numbering, we can assume WLOG $b_{n+1} \neq 0$

$b_{n+1} \cdot v_{n+1} = w_{n+1} - a_1 w_1 - \cdots - a_n w_n - b_{n+2} \cdot v_{n+2} - \cdots - b_m v_m \Longrightarrow v_{n+1} = b_{n+1}^{-1}(w_{n+1} - a_1 w_1 - \cdots - a_n w_n - b_{n+2} \cdot v_{n+2} - \cdots - b_m v_m)$

So $v_{n+1} \in \text{Span}\{\{w_1, \cdots, w_n, v_{n+1}, \cdots v_m\}$, $\text{Span}\{w_1, w_2, w_n, v_{n+1}, \cdots v_m\} \subset \underbrace{\text{Span}\{w_1, \cdots, w_n, v_{n+1}, v_{n+2}, \cdots, v_m\}}_{=V} \subset V$

Finally, we have $V = \text{Span}(\{w_1, \cdots, w_n\})$

$n > m$, $w_n \in V$, $w_n = c_1 w_1 + \cdots + c_m w_m$ so $\{w_1, \cdots, w_n\}$ linearly dependent a contradiction

The assumption is wrong, which means $\{w_1, \cdots, w_n\}$ is linearly dependent

Def'n Let V be a vector space, having basis of n elements, n is called dimension of V, otherwise V is called infinitely dimensional

$$\left.\begin{array}{l} \dim V = n \\ \dim_{\mathbb{F}} V = n \\ \dim\{0\} = 0 \end{array}\right\} \text{finitely dimensional} \quad \dim \mathbb{F}^n = n, \ \dim P_n(\mathbb{F}) = n+1, \ \dim M_{m\times n}(\mathbb{F}) = m \cdot n, \ \dim F[x] = \infty, \ \dim \mathbb{R}^{[0,1]} = \infty$$

1.20 Tutorial

Direct Sum

* <u>Recall</u>: if V is a vector space over F, then a set $S \subseteq V$ is <u>linear dependence</u> if $\exists v_1, \cdots, v_n \in S$ and $c_1, \cdots, c_n \in F \setminus \{0\}$ (i.e. $c_i \neq 0$)

s.t. $c_1 v_1 + \cdots + c_n v_n = 0$

- S is linearly independence if it's not linear dependence

Exercise: In R^4, define sets
$$S_1 = \left\{ \underset{x}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}}, \underset{y}{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}} \right\} \quad S_2 = \left\{ \underset{y-x}{\begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}}, \underset{z}{\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}} \right\} \quad S_3 = \left\{ \underset{x+y}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \right\}$$

(a) which are line 'ind': S_1 S_2 S_3 $S_1 \cup S_2$, $S_2 \cup S_2$, $S_1 \cup S_3$, $S_1 \cup S_2 \cup S_3$?

(b) Let $U_i = \mathrm{Span}(S_i)$: Find finite spanning sets for U_1, U_2, U_3, $U_1 + U_2$, $U_2 + U_3$, $U_1 + U_3$, $U_1 + U_2 + U_3$?

$\mathrm{Span}(S) + \mathrm{Span}(T) = \mathrm{Span}(S \cup T)$

(c) Find spanning set for: $U_1 \cap U_2$, $U_1 \cap U_3$, $U_2 \cap U_3$

* Subspace sum if V is a vector space and if $U, W \subseteq V$. $U + W = \{u + w : u \in U \text{ and } w \in W\}$

(a) S_1 — $\begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 - c_2 = 0 \\ c_2 = 0 \\ -c_1 = 0 \end{cases}$ ind

S_2 — $\begin{cases} c_1 = 0 \\ -3c_1 + 2c_2 = 0 \\ c_1 + c_2 = 0 \\ c_1 = 0 \end{cases}$ ind

S_3 $\begin{cases} 3c_1 = 0 \\ c_1 = 0 \\ c_1 = 0 \\ -c_1 = 0 \end{cases}$ ind

$S_1 \cup S_2$ dep $\quad$ $S_2 \cup S_3$ $\quad$ $S_1 \cup S_3$ dep

$S_1 \cup S_2 \cup S_3$ x dep

(b) U_1: $c_1 x + c_2 y$

U_2: $c_1(y-x) + c_2 z$

U_3: $c_1(x+y)$

$U_1 + U_2$: $c_1 x + c_2 y + c_3(y-x) + c_4 z \Rightarrow$ Not basis

$U_2 + U_3$: $c_1(y-x) + c_2 z + c_3(x+y) \Rightarrow \mathrm{Span}(S_2 \cup S_3)$ independence $\Rightarrow$ basis for $U_2 + U_3$

$U_1 + U_3$: $c_1 x + c_2 y + c_3(x+y)$

$B = S_1 \cup S_2 \setminus \{y-x\} \rightarrow$ $\mathrm{Span}(B) \rightarrow$ basis

(c) $U_1 \cap U_2 = \mathrm{Span}(y-x)$

$U_2 \cap U_3 = \{0\}$

$U_1 \cap U_3 = U_3$ (because $U_3 \subseteq U_1$)

Spanning Set: A spanning set for a subspace U is any set S s.t. $U = \mathrm{Span}(S)$

Isabelle Thm: Let S, T be linearly independent sets, then $S \cup T$ is linearly independent iff $\mathrm{Span}(S) \cap \mathrm{Span}(T) = \{0\}$

Proof: Suppose $S \cup T$ is dependant. Let $x \in \mathrm{Span}(S) \cap \mathrm{Span}(T)$; we want to show $x = 0$

know we can write $x = a_1 s_1 + \cdots + a_n s_m = b_1 t_1 + \cdots + b_n t_n$ since $a_i, b_i \in F$, $t_i \in T$, $s_i \in S$

$\Rightarrow 0 = x - x = \left(\sum_{i=1}^{m} a_i s_i\right) - \left(\sum_{j=1}^{n} b_j t_j\right) \Rightarrow$ all $a_i = 0$ and $b_j = 0 \Rightarrow x = 0$

"$\Leftarrow$": exercise for you! :)

- Let's say that two subspaces U, W form a direct sum if $U \cap W = \{0\}$

In this case write $U + W = U \overset{\text{oplus}}{\oplus} W$

- We just saw: if $U = \mathrm{Span}(S)$, $W = \mathrm{Span}(T)$, then $U + W$ is direct sum iff $S \cup T$ is linear independance

Exercise for you

1. For any $U \subseteq \mathbb{R}^n$, let $U^{\overset{垂直}{\perp}} = \{x \in \mathbb{R}^n : x \circ u = 0 \text{ for all } u \in U\}$

Then prove $U \oplus U^{\perp} = \mathbb{R}^n$. (at least show $U \cap U^{\perp} = \{0\}$)

2. Let $C(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$ (This is an $\mathbb{R}$-vector space) (Point wise)

$C_{odd}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f(x) = -f(-x) \ \forall x \in \mathbb{R}\}$

Claim: $C(\mathbb{R}) = C_{odd}(\mathbb{R}) \oplus W$ for some subspace $W \subseteq C(\mathbb{R})$

W is called "comp.... subspace" for $C_{odd}(\mathbb{R})$

Why? take $W = C_{even}(\mathbb{R})$ now check:

① $C(\mathbb{R}) = C_{odd}(\mathbb{R}) + C_{even}(\mathbb{R})$

② $C_{odd}(\mathbb{R}) \cap C_{even}(\mathbb{R}) = \{0\}$

1.22

Recall 1: a set is a basis of a vector space V if ① S linearly independent

② $\operatorname{Span} S = V$

Recall 2: any two bases of a finitely spanned vector space V have the same number of elements

Recall 3: V, $\dim V = n$, any subset of V that have more than n elements, must be linearly dependent

Why basis?

Theorem: Let V be a vector space, $\{v_1, \cdots, v_n\}$ is a basis of V, take $x \in V$, there is a unique $(a_1, \cdots, a_n) \in F^n$

Such that $x = a_1v_1 + \cdots + a_nv_n$

Uniqness: If $x = a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$ where $a_1, \cdots, a_n, b_1, \cdots, b_n \in F^n$, then $a_1 = b_1$, $a_2 = b_2$, $\cdots$, $a_n = b_n$

Proof: Assume $x = a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$ where $a_1, \cdots, a_n, b_1, \cdots, b_n \in F^n$

get $(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_2 = 0$

Since $\{v_1, \cdots, v_n\}$ is a basis of V, $\{v_1, \cdots, v_n\}$ is linearly independent So $a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$, So $a_k = b_k$ $\forall k \in N$

Existance: Trivial since $V = \operatorname{Span}(S)$

Definition: Given a set $\{v_1, \cdots, v_n\}$ of linearly independent elements of V. The set $\{v_1, \cdots, v_n\}$ is called a maximal set of linearly independent elements of V if for every $w \in V$, the set $\{w, v_1, \cdots, v_n\}$ is linearly dependent.

Corollary: ① Let V be a vector space

① If $\{v_1, \cdots, v_n\}$ is a maximal set of linearly independent element of V, then $\{v_1, \cdots, v_n\}$ is a basis of V

Proof: It sufficies to prove $\operatorname{Span}(\{v_1, \cdots, v_n\}) = V$, take $w \in V$, by assumption $\{w, v_1, v_2, \cdots, v_n\}$ is linearly dependent

There exists $a_0, a_1, \cdots, a_n \in F$ not all zero s.t. $a_0w + a_1v_1 + \cdots + a_nv_n = 0$

Claim: $a_0 \neq 0$. Indeed if $a_0 = 0$, then $0 + a_1v_1 + \cdots + a_nv_n = 0$. Since $\{v_1, \cdots, v_n\}$ is linearly independent, $a_1 = \cdots = a_n = 0$ So $a_0 = a_1 = \cdots = a_n = 0$. Contradiction

So $a_0 \neq 0$

Getting: $w = -a_0^{-1}a_1v_1 - \cdots - a_0^{-1}a_nv_n$ so $V = \operatorname{Span}\{v_1, \cdots, v_n\}$. Which complete the proof

Every basis is a maximal set of linearly independent elements

Corollary ②: $\dim V = n$ and $\{v_1, \dots, v_n\}$ a set of linearly independent elements of V, Hence $\{v_1, \dots, v_n\}$ is a basis for V

Proof: Take $w \in V$. The set $\{w, v_1, \dots, v_n\}$ have $n+1 > n = \dim V$. So $\{w, v_1, \dots, v_n\}$ is linearly dependent, So $\{v_1, \dots, v_n\}$ is a maximal set of linearly independent element of V. By key lemma, $\{v_1, \dots, v_n\}$ is basis

Corollary ③: V, $\dim V = n$ and $\{v_1, \dots, v_k\}$ are linearly independent, $k < n$

Then there exists $v_{k+1}, \dots, v_n \in V$ so that $\{v_{k+1}, \dots, v_n\}$ is a basis of V

Proof: Since $k < n$, the set $\{v_1, \dots, v_k\}$ is not basis of V. So $\{v_1, \dots, v_k\}$ is not maximal set of linearly independent elements of V (by Part 1) Corollary ①

By def'n. $\exists v_{k+1} \in V$ s.t. $\{v_1, \dots, v_k, v_{k+1}\}$ is linearly Independent

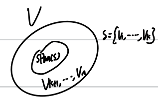
Repeat this Process --- $\{v_1, \dots, v_{n-1}\} \xrightarrow[v_n]{\text{add}} \{v_1, \dots, v_n\}$ linearly independent. by Part ② Corollary ② $\{v_1, \dots, v_n\}$ is a basis of V

Summary V, $\dim V = n$, a subset $\{v_1, \dots, v_k\} \subset V$

① If $k < n$, and $\{v_1, \dots, v_k\}$ is linearly independent. then we can add more vectors so $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis

② If $k = n$, and $\{v_1, \dots, v_k\}$ linearly independent, then $\{v_1, \dots, v_k\}$ is basis

③ If $k > n$, the set $\{v_1, \dots, v_k\}$ is linearly dependent



④ Let V be a vector space and W is a subspace of V. Then $\dim W \le \dim V$. If $\dim W = \dim V$, then $W = V$

Proof: If $\{w_1, \dots, w_k\}$ is a basis for W. then there exists $\{w_1, \dots, w_k\}$ is a basis for W. then there exists $v_{k+1}, \dots, v_n \in W/V$

s.t. $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ is a basis for V Produced that $\dim V < \infty$

Observation: If $w_1, \dots, w_k$ are linearly independent element of W (W: subspace of V) are $w_1, \dots, w_k$ linearly independent element of V? Yes

☺ Consider $c_1w_1 + \dots + c_kw_k = 0$ in V where $c_1, \dots, c_k \in \mathbb{F}$

Since $\{w_1, \dots, w_k\}$ linearly independent in W. $c_1 = \dots = c_k = 0$

Proof: Case#1 $W = \{0\}$

Case#2 W Contains a non-zero vector $w_1 \in W$. $\{w_1\}$ linearly Independent. We can add $w_2, \dots$ s.t. $\{w_1, w_2, \dots, w_m\}$ is a maximal set of linearly independent element of W

This Process stops at some $m \le n$ Since $\{w_1, \dots, w_m\}$ is linearly independent in V.

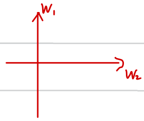
1.24

Vector spaces → basis → linearly ind' / spaning ; dimantion

| finite / infinite sets

Vector spaces → a set, 2 operations, 8 VS conditions

Sub spaces: W_1, W_2 subspaces of vector space V : W_1+W_2 , $W_1 \cap W_2$, ~~$W_1 \cup W_2$~~



Vector spaces → F^n , $(a, a_2, \dots, a_n)$

→ $M_{m\times n}(F)$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(F)$,

→ $F[x]$ a Polynomial

→ $R^{[0,1]} \Rightarrow \{f: R \to [0,1]\}$, a function

→ New one, vector space = a set of sets

← generate from V, and the subspace W

Ex: Set of sets : ① $S = \{1,2,3\}$ The set of all subset of S $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

② $Z_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, $\bar{2} = \{n : n \bmod 5 = 2\}$

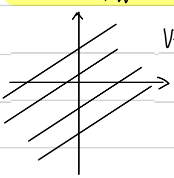
Quotient Spaces

Definition: Let V be a vector space and W be a subspace of V

Define $V/W \overset{def}{=} \{x+W \mid x \in V\}$ where $x+W = \{x+y \mid y \in W\}$

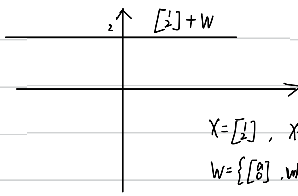
$X+W = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1+w_1 \\ x_2+w_2 \end{bmatrix}$

ex.①



$V = R^2$

$V/W = \{$all lines that parallel to $W\}$



$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $X+W = \left\{\begin{bmatrix} a+1 \\ 2 \end{bmatrix}, a \in R\right\}$

$W = \left\{\begin{bmatrix} a \\ 0 \end{bmatrix}\right.$, where $a \in R\}$

ex. ② $L[0,1] = \{$all integrable functions: $f: [0,1] \to R\}$, We need $L[0,1]/W$ identify $W = \{f$ integrable s.t. $\int_0^1 f dx = 0\}$

Lemma Let V be a Vector space, W is a subspace, $x_1, x_2 \in V$. Then $x_1 + W = x_2 + W$ iff $x_1 - x_2 \in W$

Proof: $\Rightarrow$ Suppose $x_1 + W = x_2 + W$. $x_1 = x_1 + 0 \in x_1 + W$ so $x_1 \in x_2 + W$. Therefore $x_1 = x_2 + \vec{w}$ for some $\vec{w} \in W$, $x_1 - x_2 = \vec{w} \in W$

$\Leftarrow$ If $x_1 - x_2 = \vec{w} \in W$, then we can prove $x_1 + W = x_2 + W$, take $\vec{v} \in W$. $x_1 + \vec{v} \in x_1 + W$. $x_1 + \vec{v} = x_2 + \vec{w} + \vec{v} \in x_2 + W$

$\in W$, since W is a subspace

So $x_1 + W \subset x_2 + W$. Similarly $x_2 + W$

Call x_1 is a representation of $x_1 + W$

Next task: Define 2 operations on V/W to make V/W is a vector space. take $x + W, y + W \in V/W$ (V mod W)

$$x + W + y + W \stackrel{def}{=} x + y + W$$

$$c \cdot (x + W) \stackrel{def}{=} cx + W$$

New terminology It's a Well-defined definition

$x + W = x' + W$ $\quad x + y + W \stackrel{show}{=} x' + y' + W$ $\quad$ Does not depend on the representation that you chose

$y + W = y' + W$

Recall If W is a subspace of V, then $x_1 + W = x_2 + W$ iff $x_1 - x_2 \in W$

Since $x + W = x' + W$, $x - x' \in W$

Since $y + W = y' + W$, $y - y' \in W$

Since W is a subspace, $x - x' + y - y' \in W$

$(x + y) - (x' + y') \in W$

So $x + y + W = x' + y' + W$

The addition operator is well-defined

Theorem V/W with those operators is a vector space, V/W is call the quotient space of V over W

Proof Verify all 8 conditions

VS3 There exist a zero element in V/W, $0 + W = W$ and if $x + W \in V/W$, then $(x + W) + (0 + W) \stackrel{def}{=} (x + 0) + W = x + W$

Theorem: ① Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V where $\{v_1, v_2, \cdots, v_k\}$, $k \le n$ be a basis for the subspace W $\quad v_1 + W = W$

Then $\{v_{k+1} + W, \cdots, v_n + W\}$ is a basis for V/W $\quad v_1 + W = W$

② If W is a subspace of V and $\dim V < \infty$, then $\dim V/W = \dim V - \dim W$

Benefit. $\dim V/W$ = Codimension of V over W. Some times $\dim V = \infty$, $\dim W = \infty$ but $\dim V/W < \infty$

Ex. $V = \mathbb{F}^\infty = \{(a_1, a_2, \cdots) \mid a_k \in \mathbb{F}\}$, $W = \{(0, a_2, \cdots) \mid a_k \in \mathbb{F}\}$, $\dim V/W = 1$

1.27

Quotient Spaces

Definition: Let V be a vector space and W be a subspace of V

Define $V/W \overset{\text{def}}{=} \{x+W \mid x \in V\}$ where $x+W = \{x+y \mid y \in W\}$

$x+W$ is called a coset of W in V

V/W new terminology "well-defined"

Linear Transformations (Preserve the linearity)

mapping between two vector spaces

Def'n Let V and W be vector space over the same field F, a mapping $T: V \to W$ is called a linear transformation from V to W

If it satisfies the following ① $T(x+y) = T(x) + T(y)$ for every $x, y \in V$

② $T(cx) = c\,T(x)$ for every $c \in F$, $x \in V$

For simplicity, we say $T: V \to W$ is linear

Ex. ① $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1, x_2) = (x_1, -x_2)$ reflection about the x-axis

Claim T is linear

Proof: Take $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$

① $LHS = T((x_1, x_2) + (y_1, y_2)) = T(x_1+y_1, x_2+y_2) \overset{\text{def}}{=} (x_1+y_1, -x_2-y_2)$

$RHS = T(x_1, x_2) + T(y_1, y_2) = (x_1, -x_2) + (y_1, -y_2) = (x_1+y_1, -x_2-y_2)$

② $T(c\cdot(x_1, x_2)) = T(cx_1, cx_2) = (cx_1, -cx_2) = c(x_1, -x_2) = c\,T(x_1, x_2)$

Therefore, it's linear

Properties Let $T: V \to W$ be linear, Then

① $T(0) = 0$ Proof $T(0) = T(x-x) = T(x) - T(x) = 0$ by ②

② $T(x-y) = T(x) - T(y)$

③ $T(a_1x_1 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$ where $a_1, \cdots, a_n \in F$, $x_1, \cdots, x_n \in V$

Proof ③ Prove by induction on n, Base case $n=1$, $T(a_1x_1) = a_1T(x_1)$ by def'n

Assume ...... $T(\sum_{k=1}^{n} a_kx_k) = \sum_{k=1}^{n} a_kT(x_k)$

Prove $n+1$, take $a_1, \cdots a_{n+1} \in F$, $x_1, \cdots, x_{n+1} \in V$

$T(a_1x_1 + \cdots + a_{n+1}x_{n+1}) \overset{\text{def } T}{=} T(a_1x_1 + \cdots + a_nx_n) + T(a_{n+1}x_{n+1}) = \sum_{k=1}^{n} a_kT(x_k) + a_{n+1}T(x_{n+1}) = \sum_{k}^{n+1} a_kT(x_k)$ $\square$

More examples: ① $T_0: V \to W$, given V, W vector spaces

$\cdot\ T_0(x) = 0$ is linear, called the zero transformation

② Given V a vector space. Define $I_V: V \to V$. $I_V(x) = x$ is linear, called the identity transformation

③ $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$, $T(p(x)) = p'(x)$ (derivative)
is linear

④ $T: M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$. $T(A) := A^t$ (, transpose of A. row of A = column of A^t, column of A = row of A^t)

$$\underset{M_{2\times 3}(\mathbb{R})}{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \underset{M_{3\times 2}(\mathbb{R})}{A^T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad A_{ij} = A^t_{ji}$$

Claim T is linear : Prove $(A+B)_{ij} = A^t_{ji} + B^t_{ji}$

$T(A+B) = (A+B)^t_{ij} = [A^t + B^t]_{ij} = A^t_{ij} + B^t_{ij} = T(A)_{ij} + T(B)_{ij}$

So $T(A+B) = T(A) + T(B)$

⑥ $T: V \to \mathbb{R}^n$ where V is an n-dimensional vector space over $\mathbb{R}$

Let $\{v_1, \dots, v_n\}$ be a basis of V, Define $T: V \to \mathbb{F}^n$ as follows

Let $x \in V$, $x = a_1v_1 + \dots + a_nv_n$ where $a_1 \dots a_n \in \mathbb{R}$

$T(a_1v_1 + \dots a_nv_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$

Claim T is linear

⑦ $T: \mathbb{R}^n \to \mathbb{R}^k$ $1 \le k \le n$ $T(x_1, \dots x_n) = (x_1, \dots, x_k)$ a projection mapping, T is also linear

Counter example $T: \mathbb{R}^2 \to \mathbb{R}$ not linear. $T(x) = \|x\|$ $T(-x) = \|-x\| = \|x\| \neq -T(x)$ for any $x \neq 0$

Proposition Let $T: V \to W$ be a mapping between V and W, then T is linear iff $T(cx+y) = cT(x) + T(y)$ for $\forall\ x, y \in V$, $c \in \mathbb{F}$

Proof: $\Rightarrow$ Assume T is linear, then $T(cx+y) = T(cx) + T(y) = cT(x) + T(y)$ done

$\Leftarrow$ $c=1$, addition $T(x+y) = T(x) + T(y)$

Pick $c=-1$, $y=x$ $\underset{T(0)}{T(-x+y)} = (-1)T(x) + T(x) = 0$

Pick $y=0$ $\underset{T(cx)}{T(cx+0)} = cT(x) + T(0) = cT(x)$ Done

Common: in order to verify a mapping $T: V \to W$ is linear just verify $T(cx+y) = cT(x) + T(y)$ $\forall\ x, y \in V$ $\forall\ c \in \mathbb{F}$

Recall $\{v_1, \cdots, v_n\}$ a basis of a vector space V. $T: V \to \mathbb{F}^n$ $T(\sum a_k x_k) = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$

Generalization W a vector space, $\{w_1, \cdots, w_n\}$ arbitrary in W. $T: V \to W$

There exist an unique linear mapping $T(v_1) = w_1$

$\vdots$

$T(v_n) = w_n$

Tutorial 1.27

Definition: Let V be a vector space and W be a subspace of V

Define $V/W \overset{def}{=} \{x + W \mid x \in V\}$ where $x + W = \{x + y \mid y \in W\}$

$$\begin{cases} (x+W) + (y+W) = (x+y) + W \\ c(x+W) = cx + W \end{cases}$$

ex. $V = \mathbb{R}^3$, $W = \text{Span}\{(0,0,1)\}$

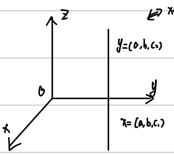
$\frac{\mathbb{R}^3}{W}$ — quotient space

Question: What are the elements in $\mathbb{R}^3/W$?

A: $P + W$ $P \in \mathbb{R}^3$

B: $[P+W] = \{x \in \mathbb{R}^3 \mid x - P \in W\}$

All lines that parallel to z axis (why?) $\Leftrightarrow x+w$ is the line $\begin{cases} s = a \\ t = b \\ z = \mathbb{R} \end{cases}$



$W = z$ axis

From B: $[x+W] = [y+W]$ $\quad x + W = y + W \Leftrightarrow x - y \in W$

$x = (a, b, c_1) \Leftrightarrow x - y = k(0,0,1) = (0,0,k)$

$\Rightarrow y = (a, b, c_2)$

Remark "vectors" in $\mathbb{R}^3/W$ are in 1-1 correspondence with points in $\mathbb{R}^2$ (s-t plane)

$\{\frac{\mathbb{R}^3}{W}\} \longleftrightarrow [\mathbb{R}^2]$

$x + W \longleftrightarrow (a, b)$

$x = (a, b, c)$

$(e, f, 0) + W \longleftrightarrow (e, f)$

$\equiv (e, f, k) + W$

Question: dimentions of $\mathbb{R}^3/W$?

① Guess: 2

② try to find a basis of $\mathbb{R}^3/W$

$e_1 = (1,0,0) \quad e_2 = (0,1,0) \in \mathbb{R}^2$

$L_1 = e_1 + W \qquad L_2 = e_2 + W$

$\{L_1, L_2\}$ is linear inde?

Assume $k_1 L_1 + k_2 L_2 = \bar{0} = 0 + W = W$

$(ke_1 + W) + (k_2 e_2 + W)$

$= (ke_1 + ke_2) + W$

$= (k_1, k_2, 0) + W$

$= (0,0,0) + W$

$\Rightarrow (k_1, k_2, 0) - (0,0,0) \in W$

$\Rightarrow k_1 = k_2 = 0$

$\{L_1, L_2\}$ is linearly inde.

Goal $\{L_1, L_2\}$ spans $\mathbb{R}^3/W$, take an arbitrary $M = (a,b,c) + \mathbb{R}^3/W$. Claim: $M = aL_1 + bL_2$

$aL_1 + bL_2 = (ae_1 + W) + (be_2 + W)$

$= (ae_1 + be_2) + W$

$= (a,b,0) + W$

$(a,b,0) - (a,b,c) \in W$

$\Rightarrow \{L_1, L_2\}$ is basis of $\mathbb{R}^3/W \Rightarrow \dim(\mathbb{R}^3/W) = 2$

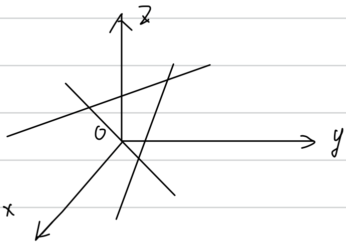
$W = \{ P = (a,b,c) \in R^3 \mid a+b+c=0 \}$

R^3/W - quotient

Question: tell me the element in R^3/W

tell me the dim of ..

find a basis of R^3/W



① $(a,b,c)+W$

② $[(a,b,c)] = \{ (x,y,z) \in R^3 \quad (x,y,z-(a,b,c) \in W \}$

$\Longleftrightarrow (x-a)+(y-b)+(z-c)=0$

$\Leftrightarrow x+y+z = a+b+c = 3$

every element of R^3/W is a plane

$R^3/W \longleftrightarrow R$

$(x,y,z)+W$

$\Downarrow$

$x+y+z=a_0 \longleftrightarrow a_0$

Conjecture

$\dim (V/W) = \dim V - \dim W$

$\underline{\infty - 3}$? not defined

here we use $\dim V < \infty$

1.29

Theorem Let V and W be vector space over $\mathbb{F}$ and $\{v_1, \dots, v_n\}$ be a basis of V. Suppose $w_1, \dots, w_n$ a subset of W

Claim There exists a unique linear transformation $T: V \to W$. Such that $T(v_k) = w_k$ for all $1 \le k \le n$

In words, to know a linear transformation T, we only need to know the value of that linear transformation T on a basis of the input space, Provided $\dim V < \infty$

Proof **Existence** (Construct $T: V \to W$ $\begin{cases} \text{linear} \\ T(v_k) = w_k, \ \forall 1 \le k \le n \end{cases}$)

Take $x \in V$. Since $\{v_1, \dots, v_n\}$ is a basis of V, then there exists $a_1, \dots, a_n \in \mathbb{F}$ s.t. $x = a_1 v_1 + \dots + a_n v_n$

Define $T(x) := a_1 w_1 + \dots + a_1 w_n$

$T: V \to W$ a mapping

$T(a_1 v_1 + \dots + a_n v_n) = a_1 w_1 + \dots + a_n w_n \quad \forall a_1, a_2, \dots, a_n \in \mathbb{F}$

Clearly $T(v_k) = w_k$. $\forall 1 \le k \le n$ by construction of T.

Verify T is linear. Take $c \in \mathbb{F}$, $x, y \in V$. $x = a_1 v_1 + \dots + a_n v_n$ $y = b_1 v_1 + \dots + b_n v_n$, $a_n, b_n \in \mathbb{F}$

$$
\begin{aligned}
T(cx+y) &= T[c(a_1 v_1 + \dots + a_n v_n) + b_1 v_1 + \dots + b_n v_n] \\
&= T[(ca_1 + b_1) v_1 + \dots + (ca_n + b_n) v_n] \\
&= (ca_1 + b_1) w_1 + \dots + (ca_n + b_n) w_n \\
&= (ca_1 w_1 + \dots + ca_n w_n) + (b_1 w_1 + \dots + b_n w_n) \\
&= c\,T(a_1 v_1 + \dots + a_n v_n) + T(b_1 v_1 + \dots + b_n v_n) \\
&= cT(x) + T(y)
\end{aligned}
$$

Done

Proof **Uniqueness** $T(a_1 v_1 + \dots + a_n v_n) = a_1 w_1 + \dots + a_n w_n$

Suppose there exist $L_1, L_2 : V \to W$ s.t. $L_{1,2}(v_k) = w_k \quad \forall k$

$L_{1,2}$ linear

We will show $L_1 = L_2$

Take $x \in V$, $x = a_1 v_1 + \dots + a_n v_n$.

$$
\begin{aligned}
L_1(x) &= L_1(a_1 v_1 + \dots + a_n v_n) \\
&\overset{\text{Since } L \text{ linear}}{=} a_1 L_1(v_1) + \dots + a_n L_1(v_n) \\
&\overset{\text{by assumption}}{=} a_1 w_1 + \dots + a_n w_n \\
&\overset{\text{Since } L_2 \text{ linear}}{=} a_1 L_2(v_1) + \dots + a_n L_2(v_n) \\
&= L_2(a_1 v_1 + \dots + a_n v_n) = L_2(x)
\end{aligned}
$$

So $L_1 = L_2$

Def'n Let $T: V \to W$ be a linear transformation (V, W are both Vector Space over $\mathbb{F}$)

Define the following sets :

① $N(T) = \{x \in V \mid T(x) = 0_W\}$

$N(T)$ is called the Null Space of T

② $R(T) = \{T(x) \mid x \in V\}$

$R(T)$ is called the range space of T

Theorem ① $N(T)$ is a subspace of V

② $R(T)$ is a subspace of W

Proof ① $N(T)$ is a subspace of V

(a) Since T is linear, $T(0_V) = 0_W$, so $0_V \in N(T)$

(b) Take $x, y \in N(T)$, then $T(x) = 0$, $T(y) = 0$

$T(x+y) \overset{def}{=} T(x) + T(y) = 0 + 0 = 0$, So $x + y \in N(T)$

(c) Take $x \in N(T)$ and $c \in \mathbb{F}$ Then $T(cx) \overset{T \text{ linear}}{=} c\,T(x) = c \cdot 0 = 0$, So $cx \in N(T)$

Therefore $N(T)$ is a subspace of V

Ex: $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$, $D_n(P(x)) = P'(x)$. What are $N(D_n)$ and $R(D_n)$?

Answer: $N(D_n) = \{P(x) \in P_n(\mathbb{R})$ s.t. $P'(x) = 0\}$

$= \{P(x) = c \mid c \in \mathbb{R}\}$

$\{1\}$ is a basis for $N(D_n)$, $\dim N(D_n) = 1$

$R(D_n) = \{P'(x) \mid P(x) \in P_n(\mathbb{R})\}$

$= P_{n-1}(\mathbb{R})$ $\quad \dim R(D_n) = n$

$\boxed{\text{Observation : } \dim N(T) + \dim R(T) = \dim V}$

Dimension Therem $\{1, x, \cdots, x^{n-1}\}$

Rank Nullity Theorem, Provided that $\dim V < \infty$

Recall: $T: V \to W$ is one-to-one (injective) if $x \neq y$ then $T(x) \neq T(y)$, if $T(x) = T(y)$ then $x = y$

Theorem Let $T: V \to W$ be linear, Then T is injective iff $N(T) = \{0\}$

Proof $\Rightarrow$: Suppose T is injective, We need to show $N(T) = \{0\}$

Take $x \in N(T)$, $T(x) = 0$. also, Since T is linear, $T(0) = 0$

So $N(T) = \{0\}$, Since T is injective $T(x) = T(0)$, $x = 0$

$\Leftarrow$ Suppose $N(T) = \{0\}$ we need to verify T is injective

Let $x, y \in V$ s.t. $T(x) = T(y)$, $T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$ Since T is linear, $x - y \in N(T) \overset{\text{assumption}}{=} \{0\}$ So $x - y = 0$, $x = y$ $\square$

Recall $T: V \to W$ is onto iff for every $z \in W$, there exist $x \in V$ s.t. $T(x) = z$

Lemma: Given $T: V \to W$, then T is onto iff $R(T) = W$

Observation ① If $T: V \to W$ linear and $\dim W < \infty$, then T is surjective (onto) iff $\dim R(T) = \dim W$

Proof: Since $R(T)$ is a subspace of W and $\dim R(T) = \dim W$ then $R(T) = W$

② $T: V \to W$ linear $V = \text{span}(\{v_1, \cdots, v_n\})$, $R(T) = \text{Span}(\{T(v_1), \cdots, T(v_n)\})$

Proof $R(T) = \{T(x) \mid x \in V\}$

$\overset{T \text{ linear}}{=} \{T(a_1 v_1 + \cdots a_m v_m) \mid a_1, \cdots, a_m \in F\}$

$= \{a_1 T(v_1) + \cdots + a_m T(v_m) \mid a_1, \cdots, a_m \in F\}$

$= \text{Span}(\{T(v_1) + \cdots T(v_m)\})$

1.31

Mid term Contents

Beginning to today 75 min

4-5 easy-mid level } question
1 hard level

Review ① Lecture notes
② Assignment 1, 2
③ Practice Problems will be posted.

Def'n Let $T: V \to W$ be linear. If $\dim N(T) < \infty$, $\text{nullity}(T) = \dim N(T)$

$\dim R(T) < \infty$, $\text{Rank}(T) = \dim R(T)$

Rank-Nullity Theorem (Dimension Theorem)

Let $T: V \to W$ be linear and $\dim V < \infty$, then $\text{nullity}(T) + \text{Rank}(T) = \dim V$

$\dim N(T) + \dim R(T) = \dim V$

Idea Build a basis for $N(T) \subset V$
basis for $R(T) \subset W$

Proof Since $N(T)$ is a subspace of V, $\dim N(T) \leq \dim V$. Let $\{v_1, \dots, v_k\}$ be a basis for $N(T)$ $0 \leq k \leq \dim V$

Extend $\{v_1, \dots, v_k\}$ to get a basis of V. $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$, $n = \dim V$

Claim: $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$

Spanning: $V = \text{Span}(\{v_1, \dots, v_n\})$, $R(T) = \text{Span}(\{T(v_1), \dots, T(v_n)\})$

Note that $T(v_1) = \dots = T(v_k) = 0$ Since $v_1, \dots, v_k \in N(T)$ by Construction

$R(T) = \text{Span}(\{0, \dots, 0, T(v_{k+1}), \dots, T(v_n)\})$

$= \text{Span}(\{T(v_{k+1}), \dots, T(v_n)\})$

Linearly Independence: Consider $c_{k+1}T(v_{k+1}) + \dots + c_n T(v_n) = 0$ (*)

Since T is linear $T(c_{k+1}v_{k+1} + \dots + c_n v_n) = 0$

So $c_{k+1}v_{k+1} + \dots + c_n v_n \in N(T) = d_1 v_1 + \dots + d_k v_k$ for some $d_1, \dots, d_k \in \mathbb{F}$

$c_{k+1}v_{k+1} + \dots + c_n v_n = d_1 v_1 + \dots + d_k v_k \Rightarrow -d_1 v_1 - \dots - d_k v_k + c_{k+1}v_{k+1} + \dots + c_n v_n = 0$

Since $\{v_1, \dots, v_n\}$ is a basis of V, $\{v_1, \dots, v_n\}$ is linearly independent

So $d_1 = \dots = d_k = c_{k+1} = \dots = c_n = 0$

From (*), We Conclude $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly Independence

---

Question: Given $W \subset V$, $\{v_1, \dots, v_n\}$ a basis of V. It does not mean $v_i \in W$

Extend a basis of W to a basis of V. Assignment 2 P2!

What's it for

Recall $T: V \to W$ linear, Then T is injective iff $N(T) = \{0\}$

Then T is surjective iff $R(T) = W$

T linear + injective + surjective = isomorphism

Theorem Let $T: V \to W$ be an isomorphism and $\dim V < \infty$

Then ① $\dim W = \dim V$

② T sends any basis in V to a basis in W

Proof ① T isomorphism $\to$ linear

injective $N(T) = 0$

surjective $R(T) = W$

Since $\dim V < \infty$, we can apply Rank-Nullity Theorem : $\dim V = \dim N(T) + \dim R(T)$

$= 0 + \dim W$

② Recall Previous Proof $\{v_1, \cdots, v_k\}$ is a basis of $N(T) = \{0\}$ $k=0$

$\{T(v_{k+1}), \cdots, T(v_n)\}$ a basis of $R(T) = W$, So $\{T(v_1), \cdots, T(v_n)\}$ is a basis of W

Question : $T: V \to W$ is given , $\dim V = \dim W$

T might not be an isomorphism For example $T: V \to W$, $T(x) = 0$

Then T is injective iff T is surjective

iff T is an isomorphism

Proof If T is injective , we need to prove T is surjective , If T is injective , we will prove T is injective

$N(T) = \{0\}$, $\underset{\substack{\| \text{ by assumption} \\ \dim W}}{\dim V} = \dim N(T) + \dim R(T) = \dim R(T)$

Since $R(T)$ is a subspace of W, $\dim R(T) = \dim W$ then , $R(T) = W$ so T is surjective

Exercises

① Let $\{v_1, \cdots, v_k\}$ vectors in V. $T: V \to W$ linear

$W = \text{Span}(\{T(v_1) \cdots, T(v_k)\})$

Prove

Claim $\dim V \geq \dim W$ $W = \text{Span}\{\{T(v_1), \cdots, T(v_k)\}$ So $\dim W \leq k$

Solution: Case #1 $\dim V = \infty$ done

Case #2 $\dim V < \infty$, Apply Rank-Nullity Theorem, $\dim V = \dim N(T) + \dim R(T) \Rightarrow \overset{(**)}{\dim V \geq 0 + \dim R(T)}$

$R(T) = W$ on the other hand, $T(v_1), \cdots, T(v_k) \in R(T)$

$W = \text{Span}(T(v_1), \cdots, T(v_k)) \subseteq R(T) \subseteq W$

So $R(T) = W$, so $**$ $\dim V \geq \dim W$

② $T_1: V \to U$, $T_2: U \to W$ linear, $\dim V$, $\dim U$, $\dim W < \infty$

Then $T_2 \circ T(\cdot): V \to W$ is linear

Claim: ① $\text{Rank}(T_2 \circ T_1) \leq \min\{\text{Rank}(T_2), \text{Rank}(T_1)\}$

② $\text{Nullity}(T_2 \circ T_1)$, Compare with $\text{Nullity}(T_2)$ $\text{Nullity}(T_1)$

2.3

A2 Q3
$\dim(W_1 \cap W_2)$ not need to prove $W_1 \cap W_2$ is a subspace

Matrix Representation of linear Transformation

① **Def'n an ordered basis**

Let V be a finite dimensional vector space, an order basis for V is a basis $\{v_1, \cdots, v_n\}$ empowered with a specific order

Explain: $\{v_1, \cdots, v_n\}$ a basis $\to v_k \neq v_j$ for $k \neq j$

$\{v_1, v_2, \cdots, v_n\}$ order basis $\neq \{v_2, v_1, v_3, \cdots, v_n\}$

Example ① F^3 $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\beta = \{e_1, e_2, e_3\}$ is an order basis for F^3

$\gamma = \{e_2, e_3, e_1\}$ is another order basis for F^3 $\quad \beta \neq \gamma$

② In F^n, $\{e_1, e_2, \cdots, e_n\}$ is called the Standard order basis for F^n $\quad e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$

③ $P_n(F)$. $\{1, x, x^2, \cdots, x^n\}$ is called the standard ordered basis for $P_n(F)$

Def'n 2 Coordinate Vectors

Let V be an n-dim vector space, $\beta = \{v_1, \cdots, v_n\}$ be an ordered basis for V

Let $x \in V$, there exist unique scalars $a_1, a_2, \cdots, a_n \in F$ s.t. $x = a_1 v_1 + \cdots + a_n v_n$

Define $[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ is called the coordinate vector space of x relative to β

Theorem Let V be an n-dim vector space and $\beta = \{v_1, \cdots, v_n\}$ be an ordered basis for V

Then, $[\cdot]: V \to F^n$ is an isomorphism, $x \mapsto [x]_\beta$

Proof (linear, one to one, on to) take $c \in F$, $x, y \in W$, we will verify $[cx+y]_\beta = c[x]_\beta + [y]_\beta$

suppose $x = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \cdots, a_n, b_1, \cdots, b_n \in F$

$y = b_1 v_1 + \cdots + b_n v_n$ $\quad [x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $[y]_\beta = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$cx + y = (ca_1 + b_1) v_1 + \cdots + (ca_n + b_n) v_n$ $\quad$ So $[cx+y]_\beta = \begin{bmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = c\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = c[x]_\beta + [y]_\beta$

Verify $[\,]_\beta$ is one to one $(N([\cdot]_\beta) = \{0\})$: take $x \in N([\cdot]_\beta)$ so $[x]_\beta = 0$, $x = 0v_1 + \cdots + 0v_n = 0_V$ So $N([\cdot]_\beta) = \{0\}$ Hence $[\cdot]_\beta$ is one to one

Verify onto: Take $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$, then $x = c_1 v_1 + \cdots + c_n v_n \in V$, clearly $[x]_\beta = c$

Hence $[\cdot]_\beta$ $V \to F^n$ is an isomorphism $\quad$ We also say V is isomorphic to F^n

Ex. ① $V = P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$, $P(x) = 3 - x + 4x^2$ so $[P(x)]_\beta = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

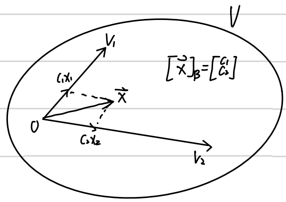
② $V = M_{2\times 2}(\mathbb{R})$, $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

③ $V = \mathbb{R}^2$, $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 8 \\ -1 \end{bmatrix}_\beta = ?$

Answer? $\begin{bmatrix} 8 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ so $\begin{bmatrix} 8 \\ -1 \end{bmatrix}_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^{2 = \dim V}$

Geometric Interpretation of Coordinate Vectors

$\dim V = 2$, $\beta = \{v_1, v_2\}$ an (ordered) basis $\quad \vec{x} \in V$



Def'n Matrix Representation of a linear Transformation

Given V, $\dim V = n$, $\beta = \{v_1, \dots, v_n\}$ an ordered basis of V

W, $\dim W = m$, $\gamma = \{w_1, \dots, w_m\}$ an ordered basis of W

$T: V \to W$ linear

Define $[T]_\beta^\gamma := \left[[T(v_1)]_\gamma, [T(v_2)]_\gamma, \dots, [T(v_n)]_\gamma \right]$ $\quad$ Called the matrix representation of T in ordered basis β and γ

$m \times n$ matrix

$\dim W \times \dim V$

\# rows $\quad$ \# col

$[T]_\beta^\gamma \in M_{m\times n}(\mathbb{F})$ where $T: V \to W$ $\quad n = \dim V \quad m = \dim W$

Ex ③ $T: P_2(\mathbb{R}) \to \mathbb{R}^2$, $T(a + bx + cx^2) = \begin{bmatrix} a \\ b+4c \end{bmatrix}$

① Verify T is linear $\quad$ ② $\beta = \{1, x+1, (x+1)^2\}$, $\gamma = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ find $[T]_\beta^\gamma \in M_{2\times 3}(\mathbb{R})$

$T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[T(1)]_\gamma = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$T(x+1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[T(x+1)]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T((x+1)^2) = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-5}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad [T((x+1)^2)]_\gamma = \begin{bmatrix} \frac{7}{2} \\ -\frac{5}{2} \end{bmatrix}$

$\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} \frac{1}{2} & 1 & \frac{7}{2} \\ -\frac{1}{2} & 0 & -\frac{5}{2} \end{bmatrix}$

Next Connection between $[T]_\beta^\gamma$ and composition of linear transformation

Question: $T: V^n \to W^m$, $L: W^m \to U^p$ $\dim V = n$, $\dim W = m$, $\dim U = p$

$\beta = \{v_1, \cdots, v_n\}$ $\gamma = \{w_1, \cdots, w_m\}$ $\alpha = \{u_1, \cdots, u_p\}$

$[T]_\beta^\gamma$ $[L]_\gamma^\alpha$

Verify $L \circ T: V^n \to U^p$ is linear $[L \circ T]_\beta^\alpha$

$[L \circ T]_\beta^\alpha \overset{\text{Want}}{=} [L]_\gamma^\alpha \circledast [T]_\beta^\gamma$ Matrix multiplication

$[p \times n]$ $[p \times m]$ $[m \times n]$

In general
$A \in M_{p \times m}(\mathbb{F})$
$B \in M_{m \times n}(\mathbb{F})$
$AB \in M_{p \times n}$

Tutorial 2.3

Let V be a v.s over $\mathbb{F}$.

A linear functional on V is a linear map $f: V \to F$

The collection of all linear functions is denoted V^*, and is called the dual space of V

ex: Let $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ $f(x) = f(x \cdot 1) = x \cdot f(1)$, $x \in \mathbb{R}$

So the linear maps $f: \mathbb{R} \to \mathbb{R}$ are given by $f(x) = ax$ for some $a \in \mathbb{R}$

ex. $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$ Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ $f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 a + x_2 b + x_3 c = [a\ b\ c]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The $f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$ is linear

Let $f \in (\mathbb{R}^3)^*$, Recall that a linear map f is determined by its values on a basis

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ So $x = x_1 e_1 + x_2 e_2 + x_3 e_3$, the standard unit basis

$f(x) = f(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3)$

The values of f on the basis vectors determine f

Let $a_i = f(e_i)$, then $f(x_1 e_1 + x_2 e_2 + x_3 e_3) = (a_1, a_2, a_3)^T \cdot (x_1, x_2, x_3)^T$

So $f(x) = f_{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}(x)$ where $f_{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}$ is as before

So all linear functional on $\mathbb{R}^3$ are of the form $f(x_1, x_2, x_3) = x_1 a + x_2 b + x_3 c$ for some $a, b, c \in \mathbb{R}$

Let, $f_1, f_2 \in V^*$

addition in $\mathbb{F}$

$f_1 + f_2$ is the function $x \mapsto f_1(x) + f_2(x)$, $x \in V$

If $\lambda \in \mathbb{F}$, λf_1 is the function $x \mapsto \lambda f_1(x)$ scalar mulitiplication in $\mathbb{F}$

ex. $f: R^3 \to R$ $\quad f(x,y,z) = x+y+z$

$g: R^3 \to R$, $g(x,y,z) = x$

$(f+g)(x,y,z) = (x+y+z) + x = 2x+y+z$

$z \cdot f(x,y,z) = z(x+y+z)$

Thm : If V is a V.S over $\mathbb{F}$, the so is V^*

Suppose that $\dim V < \infty$, what if any is the relationship between $\dim V$ and $\dim V^*$

Let $P_1(x_1, x_2, x_3) = x_1$, $P_2(x_1, x_2, x_3) = x_2$, $P_3(x_1, x_2, x_3) = x_3$

$\text{Span}\{P_1, P_2, P_3\} = \{aP_1(x_1, x_2, x_3) + bP_2(x_1, x_2, x_3) + c \cdot P_3(x_1, x_2, x_3)\ a, b, c \in R\}$

$= \{ax_1 + bx_2 + cx_3 : a, b, c \in R^3\}$

Since all linear maps $f: R^3 \to R$ have the form $f(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$, $\text{Span}\{P_1, P_2, P_3\} = R^{3*}$

Suppose that P_1, P_2, P_3 were L.D. then $\exists\, a, b, c \in R$, not all 0 s.t. $aP_1 + bP_2 + cP_3 = 0$ which means $\forall (x_1, x_2, x_3) \in R^3$, $ax_1 + bx_2 + cx_3 = 0$

Suppose that $a \neq 0$, then $(aP_1 + bP_2 + cP_3)(1,0,0) = a = 0$, by $ax_1 + bx_2 + cx_3 = 0$, we have contradiction

It follows that P_1, P_2, P_3 are L.I and spanning so they are a basis

$\{P_1, P_2, P_3\}$ is a basis of R^{3*} So $\dim (R^3)^* = \dim R^3$

In general, if $P_i : \mathbb{F}^n \to \mathbb{F}$ is defined by $P_i(\sum_{k=1}^{n} x_k e_k) = x_i$, then $\{P_1, \cdots, P_n\}$ is a basis of $(\mathbb{F}^n)^*$ So $\dim (\mathbb{F}^n)^* = n$

Remark (i) If $\dim V < \infty$, then $\dim V = \dim V^*$

Let V be a V.S over $\mathbb{F}$ and let

$\beta = \{b_1, \cdots, b_n\}$ be a basis, Recall that if $x \in V$, then $[x]_\beta = (x_1, \cdots, x_n)$ where $x = \sum_{i=1}^{n} x_i b_i$

We may now define $P_i^\beta : V \to \mathbb{F}$ as follows

If $x = \sum_{k=1}^{n} x_k b_k \in V$, $(x_1, \cdots x_n) = [x]_\beta$ set $P_i^\beta(x) = x_i$ if $x = x_1 b_1 + \cdots + x_n b_n$ then $P_i^\beta(x) = x_i$

ex: If $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$, $B = (e_1, e_2, e_3)$

Then $P_1^B(x_1, x_2, x_3) = x_1$, $P_1^B = P_1$, $P_2^B = P_2$, $P_3^B = P_3$

Theom: The function $P_i^\beta : V \to \mathbb{F}$ are linearly are form a basis for V^* call the dual basis of B. we call this B^*

Note that the coordinate mapping: $[\ ]_B : V \to \mathbb{F}^n$ is a linear mapping, then $P_i^B = P_i \circ [\ ]_B$

Definition. Suppose V is a vector space over $\mathbb{F}$.

(1) A *linear functional on* V is any linear transformation $g : V \to \mathbb{F}$.
(2) The set of all linear functionals on V is denoted V^*.

Thus $V^* \subseteq \mathbb{F}^V$ (the set of *all* functions $V \to \mathbb{F}$). Recall that $\mathbb{F}^V$ is a vector space w.r.t. pointwise addition and scalar multiplication. It's easy to check that V^* is closed under addition and scalar multiplication (and $V^* \neq \varnothing$), so V^* is a subspace of $\mathbb{F}^V$.

V^* is called the **dual space** of V.

Example. Let $V = \mathbb{R}^3$ as a vector space over $\mathbb{R}$. $(\mathbb{R}^3)^*$ is the set of all linear transformations $g : \mathbb{R}^3 \to \mathbb{R}$. Suppose $g \in (\mathbb{R}^3)^*$; so $g : \mathbb{R}^3 \to \mathbb{R}$ and for any $(x, y, z) \in \mathbb{R}^3$,

$$g((x, y, z)) = g(xe_1 + ye_2 + ze_3) = xg(e_1) + yg(e_2) + zg(e_3) = ax + by + cz$$

where $a = g(e_1)$, $b = g(e_2)$, and $c = g(e_3)$. Note also that

$$L_{[a\ b\ c]}((x, y, z)) = [a\ b\ c] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz.$$

(Here $[a\ b\ c]$ is the 1×3 matrix.) So $g = L_{[a\ b\ c]}$. Every linear functional of $\mathbb{R}^3$ has this form.

Based on this, it's easy to guess that $\dim((\mathbb{R}^3)^*) = 3$.

Now what would be a good <u>basis</u> for $(\mathbb{R}^3)^*$?

In general, if $\dim(V)$ is finite, there is a standard way to convert any ordered basis $\beta = (v_1, \ldots, v_n)$ for V into an ordered basis $\beta^* = (f_1, \ldots, f_n)$ for V^* (called the **dual** of β).

Definition. Suppose $\beta = (x_1, \ldots, x_n)$ is an ordered basis for V (a vector space over $\mathbb{F}$).

(1) Define $f_1, \ldots, f_n : V \to \mathbb{F}$ as follows: given $v \in V$, let $[v]_\beta = (a_1, \ldots, a_n)$: then $f_i(v) = a_i$ for $i = 1, \ldots, n$.
(2) The **dual of** β is $\beta^* := (f_1, \ldots, f_n)$.

Lemma 1. *In the above situation, each f_i is a linear functional on V.*

Proof. Suppose $v, w \in V$; write $[v]_\beta = (a_1, \ldots, a_n)$ and $[w]_\beta = (b_1, \ldots, b_n)$. This means

$$\begin{aligned} v &= a_1x_1 + \cdots + a_nx_n \\ w &= b_1x_1 + \cdots + b_nx_n. \end{aligned}$$

So obviously

$$v + w = (a_1 + b_1)x_1 + \cdots + (a_n + b_n)x_n.$$

So $[v + w]_\beta = (a_1 + b_1, \ldots, a_n + b_n)$. Thus

$$f_i(v + w) = a_i + b_i = f_i(v) + f_i(w),$$

proving each f_i preserves $+$. A similar proof shows they preserve scalar mult. □

So in the situation of the definition, we have $f_1, \ldots, f_n \in V^*$, and in fact it can be shown that $\{f_1, \ldots, f_n\}$ is a basis for V^*. Here are some hints as to how one might prove this.

Facts. Suppose $V, x_1, \ldots, x_n, f_1, \ldots, f_n$ are as in the definition.

(1) $f_i(x_i) = 1 \qquad \text{while} \qquad f_j(x_i) = 0 \quad \text{for } j \neq i.$
[Hint: what is $[x_i]_\beta$?]

(2) Using this, you can show that $f_1, \ldots, f_n$ are linearly independent.
[Hint: if $a_1 f_1 + \cdots + a_n f_n = \widehat{0}$, then in particular

$$(a_1 f_1 + \cdots + a_n f_n)(x_i) = 0 \quad \text{for each } i = 1, \ldots, n$$

i.e., $a_1 f_1(x_i) + \cdots + a_n f_n(x_i) = 0$ for $i = 1, \ldots, n$, which simplifies...]

(3) For all $g \in V^*$, if $a_j = g(x_j)$ for $j = 1, \ldots, n$ and $h := \sum_{j=1}^{n} a_j f_j$, then $h(x_i) = g(x_i)$ for all $i = 1, \ldots, n$. [Hint: evaluate $h(x_i)$]

(4) But if $g, h : V \to \mathbb{F}$ are linear transformations which agree on the basis $\{x_1, \ldots, x_n\}$, then $g = h$.

(5) Note that the previous item implies $\{f_1, \ldots, f_n\}$ spans V^*.

Corollary. *If V is finite-dimensional, then* $\dim(V) = \dim(V^*)$.

2.5

$\{v_1, \cdots, v_n\}$ a basis of V

Last time $V^n \leftrightarrow F^w$

$\vec{v} \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

where

linear transformation $T: V^n \to W^m$ $\leftrightarrow$ matrix $M_{m\times n}(F)$

addition, scalar mult $\Longrightarrow$ same

Matrix Representation of a linear Transformation

Recall Let $T: V \to W$ linear transformation $\beta = \{v_1, \cdots, v_n\}$ a basis for V $\gamma = \{w_1, \cdots, w_m\}$ a basis for W

Define: $[T]_\beta^\gamma = [\,[T(v_1)]_\gamma, [T(v_2)]_\gamma, \cdots, [T(v_n)]_\gamma\,]$

$[T]_\beta^\gamma \in M_{m\times n}(F)$, Note: If $T: V \to V$, $[T]_\beta^\beta = [T]_\beta$

Def'n Let V and W are vector spaces, Denote $\mathcal{L}(V, W) = \{T: V \to W \text{ linear}\}$

Define $T, U: V \to W$, $(T+U)(x) = T(x) + U(x)$

If $c \in F$, we define $cT: V \to W$, $(cT)(x) = c\,T(x)$

Theorem ① If $T, U \in \mathcal{L}(V, W)$, then, $T+U \in \mathcal{L}(V, W)$ · $cT \in \mathcal{L}(V, W)$ for all $c \in F$

② $\mathcal{L}(V, W)$ is a vector space under these operations

Notation: $\mathcal{L}(V, V) = \mathcal{L}(V)$

Theorem 2 If $T, U \in \mathcal{L}(V, W)$, $\beta = \{v_1, \cdots, v_n\}$ a basis for V , $\gamma = \{w_1, \cdots, w_m\}$ a basis for W

Then ① $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ (Matrix equalities)

② $[cT]_\beta^\gamma = c \cdot [T]_\beta^\gamma$

Proof $[T+U]_\beta^\gamma = [\,[(T+U)(v_1)]_\gamma, \cdots\cdots, [(T+U)(v_n)]_\gamma\,]$

1^{st} column $[(T+U)(v_1)]_\gamma \overset{\text{def. } T+U}{=\!=} [T(v_1) + U(v_1)]_\gamma \overset{\text{last time}}{=\!=} [T(v_1)]_\gamma + [U(v_1)]_\gamma$

So $[T+U]_\beta^\gamma = \left[([T(v_1)]_\gamma + [U(v_1)]_\gamma) \; \cdots \; \cdots ([T(v_n)]_\gamma + [U(v_n)]_\gamma) \right]$

$= [T]_\beta^\gamma + [U]_\beta^\gamma$

Theorem 3 If $V \xrightarrow{T} W \xrightarrow{U} Z$ where V, W, Z are vector spaces

$\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_m\}$, $\gamma = \{z_1, \cdots, z_p\}$ bases of V, W, Z respectively

(1) $U \cdot T = U \circ T : V \to Z$, $U \circ T(x) = U(T(x))$. UT is linear

Motivation to introduce matrix multiplication

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "UT"'] & W \arrow[r, "U"] & Z \end{tikzcd}$$

$A = [U]_\beta^\gamma \in M_{p \times m}(F)$

$B = [T]_\alpha^\beta \in M_{m \times n}(F)$

$C = [UT]_\alpha^\gamma \in M_{p \times n}(F)$

$\parallel$ define

AB

Question: What's the relation between A, B, C

Answer: $C = [UT]_\alpha^\gamma = [[UT(v_1)]_\gamma, \cdots, [UT(v_n)]_\gamma]$

Notation $C = (C_{ij})$ row column

$$= \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & \vdots & & \\ C_{p1} & C_{p2} & & C_{pn} \end{bmatrix}$$

The j^{th} column of C $[UT(v_j)]_\gamma = [U(T(v_j))]_\gamma$

$B = [[T(v_1)]_\beta, \cdots, [T(v_n)]_\beta]$, $[T(v_n)]_\beta = \begin{bmatrix} b_{11} & b_{12} & \cdots & \cdots \\ b_{21} & b_{22} & \cdots & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$

$[T(v_j)]_\beta = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$, It means $T(v_j) = b_{1j} w_1 + b_{2j} w_2 + \cdots + b_{mj} w_m$

$= \sum_{k=1}^m b_{kj} \cdot w_k$

Remark: If $B = [T]_\beta^\gamma \Leftrightarrow T(v_j) = \sum_{k=1}^m b_{kj} w_k$

So $[U T(v_j)]_\gamma = [U(\sum_{k=1}^m b_{kj} w_k)]_\gamma$

$\overset{U:\text{ linear}}{=} [\sum_{k=1}^m b_{kj} U(w_k)]_\gamma$

$= \sum_{k=1}^m b_{kj} [U(w_k)]_\gamma$ k^{th} column of A $\begin{bmatrix} a_{1k} \\ \vdots \\ a_{pk} \end{bmatrix}$

$= \sum_{k=1}^m b_{kj} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{pk} \end{bmatrix}$ then $C_{ij} = \sum_{k=1}^m a_{ik} \cdot b_{kj}$

Conclusion:

$$\begin{tikzcd} \underset{\alpha}{V} \arrow[r, "T"] \arrow[rr, bend right, "C = AB"'] & \underset{\beta}{W} \arrow[r, "U"] & \underset{\gamma}{Z} \end{tikzcd}$$

A B

$[UT]_\alpha^\gamma$

and

$$C_{ij} = \sum_{k=1}^m a_{ik} \cdot b_{kj}$$

Definition Let $A \in M_{m \times n}(\mathbb{F})$, $B \in M_{n \times p}(\mathbb{F})$

Define $(AB)_{ij} = \sum_{k=1}^{n} (A_{ik}) \cdot (B_{kj})$

Example $A \in M_{2 \times 3}(\mathbb{R})$, $B \in \mathbb{R}^3$ $(M_{3 \times 1}(\mathbb{R}))$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1a + 2b + 3c \\ 4a + 5b + 6c \end{bmatrix}$

2.7

Matrix Multiplication (Cont'd)

Recall ① Matrix Representation of a linear transformation

$T: V \to W$ linear

$\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_m\}$ bases of V and W, respectively

Define $[T]_\alpha^\beta = [[T(v_1)]_\beta, \cdots, [T(v_n)]_\beta] \in M_{m \times n}(\mathbb{F})$

② $V \xrightarrow{T} W \xrightarrow{U} Z$, T, U linear, V, W, Z vector spaces

$\alpha = \{v_1, \cdots, v_n\}$ $\beta = \{w_1, \cdots, w_m\}$ $\gamma = \{z_1, \cdots, z_p\}$ bases of V, W, Z respectively

$B = [T]_\alpha^\beta$, $A = [U]_\beta^\gamma$, $C = [UT]_\alpha^\gamma$

$V^n \xrightarrow{T} W^m \xrightarrow{U} Z^p$

$\boxed{\begin{array}{c} B = [T]_\alpha^\beta \in M_{m \times n}(\mathbb{F}) \\ UT \\ A = [U]_\beta^\gamma \in M_{p \times m}(\mathbb{F}) \end{array}}$

$C = [UT]_\alpha^\gamma \in M_{p \times n}(\mathbb{F})$

$$C_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}$$

row column

$a_{i1}, a_{i2}, \cdots a_{im}$

b_{1j}
b_{2j}
$\vdots$
b_{mj}

The entry of matrix C at row i^{th}, column j^{th}

$=$ (i^{th} row of A) Compose Multiplication (j^{th} column of B)

Denote $C = AB$ Product of the Matrix A and Matrix B

Def'n Matrix multiplication: Let $A \in M_{m \times n}(\mathbb{F})$, $B \in M_{n \times p}(\mathbb{F})$, define the product of A and B, $AB \in M_{m \times p}(\mathbb{F})$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj} \quad \forall \begin{array}{l} 1 \le i \le m \\ 1 \le j \le p \end{array}$$

Question: Find AB given A and B

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} b_{11} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{matrix}} & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & & \vdots \\ b_{n1} & & b_{np} \end{pmatrix}}_{B}$$

Ex ① $A \in M_{m \times n}(\mathbb{R})$, $x \in \mathbb{R}^n = M_{n \times 1}(\mathbb{R})$ then $Ax \in \mathbb{R}^m$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} a_{1k} \cdot x_k \\ \sum_{k=1}^{n} a_{2k} x_k \\ \vdots \\ \sum_{k=1}^{n} a_{mk} \cdot x_k \end{pmatrix} \in \mathbb{R}^m$$

Special observation

① $A \cdot 0_n = 0_m$

② Denote $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$

$Ae_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = 1^{st}$ column of A $\qquad A = [Ae_1, \cdots, Ae_n]$

$\vdots$

$Ae_k = k^{th}$ Column of A

③ $T: V \longrightarrow W$

$\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_m\}$ are bases of V and W

$A = [T]_\alpha^\beta \in M_{m \times n}(\mathbb{F})$

Take $x \in V$, $x = c_1 v_1 + \dots + c_n v_n$ where $c_1, \cdots c_n \in \mathbb{F}$ $\qquad \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [x]_\alpha$

$T(x) = T(c_1 v_1 + \dots + c_n v_n)$

$\overset{\text{linear}}{=} c_1 T(v_1) + \cdots + c_n T(v_n)$

$[T(x)]_\beta = [c_1 T(v_1) + \cdots + c_n T(v_n)]_\beta$

$= c_1 [T(v_1)]_\beta + \cdots + c_n [T(v_n)]_\beta$

Lemma $A \in M_{m \times n}(\mathbb{F})$, $x \in \mathbb{F}^n$. Denote $a_1, \cdots, a_n$ be column of A by that order : $A = [a_1 \; a_2 \; , \cdots \; a_n]$ $\quad a_k \in \mathbb{F}^m$

Then $Ax = x_1 a_1 + \cdots + x_n a_n$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

linear combination of the column of A)

Proof $\quad$ RHS $= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$

$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{pmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn} \end{pmatrix}$

Recall $[T(x)]_\beta = c_1 [T(v_1)]_\beta + \cdots + c_1 [T(v_1)]_\beta$

$\overset{\text{lemma}}{=} [[T(v_1)]_\beta, [T(v_2)]_\beta, \cdots [T(v_n)]_\beta] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\underset{\in F^m}{[T(x)]_\beta} = \underset{M_{m\times n}(F)}{[T]_\alpha^\beta} \underset{F^n}{[x]_\alpha}$

$T: V^n \to W^m$

$\begin{matrix} \text{IIS} & & \text{IIS} \\ F^n & \xrightarrow{?} & F^m \end{matrix}$

(4) Consider $T: F^n \to F^m$, defined as follows $F^n \to F^m$

$T(x) := \underset{m\times n}{A}\underset{n\times 1}{x}$ Provided $A \in M_{m\times n}(F)$

Claim ①: T is linear

Claim ②: $[T]_\alpha^\beta = A$ where α, β are standard bases of F^n, F^m

Proof ① T is linear $\Leftrightarrow T(cx+y) = cT(x) + T(y)$ where $c \in F$, $x, y \in F^n$

$\Leftrightarrow A(cx+y) = cAx + Ay$ Leave it for next time

② $[T]_\alpha^\beta \overset{\text{def}}{=} [[T(e_1)]_\beta, \cdots, [T(e_m)]_\beta]$

$\overset{\text{def of }T}{=} [[Ae_1]_\beta, \cdots [Ae_n]_\beta]$ β. standard basis of F^m, $z \in F^m$, $[z]_\beta = z$

$z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$

$= [Ae_1, \cdots, Ae_n]$

$= A$

Given $A \in M_{m\times n}(F)$ Then we can define a linear transformation $T: F^n \to F^m$, $T(x) = Ax$

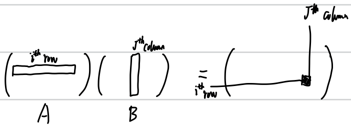
Conversely, given $T: F^n \to F^m$ linear

Does it exist a matrix A so that $T(x) = Ax$? If yes, Find A with respect to T?

2.10

Matrix Multiplications (Properties)

$\underset{m\times p}{C} = \underset{m\times n}{A} \cdot \underset{n\times p}{B}$, $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$



② Given $A \in M_{m\times n}(\mathbb{F})$, $e_1, \dots, e_n$ standard basis in $\mathbb{F}^n$

Then $\underset{m\times n}{A}\underset{n\times 1}{e_k} = \underset{m\times 1}{k^{th} \text{ Column of } A}$

$A = [Ae_1, \dots, Ae_n]$

③ Extension $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$, $B = \{V_1\ V_2 \cdots V_p\}$, $\overset{\text{row-column}}{V_k \in \mathbb{F}^n}$

$M_{m\times p}(\mathbb{F}) \ni AB = A[V_1\ V_2 \cdots V_n] \overset{\text{claim}}{=} [AV_1\ AV_2 \cdots AV_n]$

Proof : J^{th} Column of $AB = \begin{bmatrix} (AB)_{1J} \\ (AB)_{2J} \\ \vdots \\ (AB)_{mJ} \end{bmatrix} \overset{\text{Matrix Product}}{=} \begin{bmatrix} \sum_{k=1}^{n} A_{1k}B_{kJ} \\ \sum_{k=1}^{n} A_{2k}B_{kJ} \\ \vdots \\ \sum_{k=1}^{n} A_{mk}B_{kJ} \end{bmatrix}$

J^{th} Column of $[AV_1\ AV_2 \cdots AV_p]$ is

$$AV_j = A\begin{bmatrix} B_{1J} \\ B_{2J} \\ \vdots \\ B_{mJ} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ - & - & - & - \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}\begin{pmatrix} B_{1J} \\ B_{2J} \\ \vdots \\ B_{nJ} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} A_{1k}B_{kJ} \\ \sum_{k=1}^{n} A_{2k}B_{kJ} \\ \vdots \\ \sum_{k=1}^{n} A_{mk}B_{kJ} \end{pmatrix}$$

$V_j = J^{th}$ Column of B

So $AB = [AV_1\ AV_2 \cdots AV_p]$ ■

Special Case $\underset{m\times n}{A}[e_1\ e_2 \cdots \underset{n\times n}{e_n}] = [Ae_1\ Ae_2 \cdots Ae_n] = A$ where $e_1 \dots e_n \in \mathbb{F}^n$ $[e_1\ e_2 \cdots e_n]$

$$= \underset{n\times n}{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \vdots & \cdots & \cdots & 1 \end{bmatrix}} = I_n \text{ (the identity matrix of } n\times n)$$

Next Question $\underset{m\times n}{I} A = A$ (exercise)

Kroncker delta : $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$, $(I_n)_{ij} = \delta_{ij}$

Properties ① $A, B \in M_{m\times n}(\mathbb{F})$, $C \in M_{n\times p}(\mathbb{F})$

$$\underset{m\times n \;\; n\times p \to m\times p}{(A+B)\cdot C} = \underset{m\times n \;\; n\times p \to m\times p}{AC + BC}$$

② $A \in M_{m\times n}(\mathbb{F})$, $C, D \in M_{n\times p}(\mathbb{F})$

$A(C+D) = AC + AD$

③ $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$, $C \in \mathbb{F}$

$C\cdot(AB) = (CA)\cdot B = A(CB)$

Proof $((A+B)\cdot C)_{ij} \overset{\text{matrix product}}{=} \sum_{k=1}^{n} (A+B)_{ik} \cdot C_{kj}$

$\overset{\text{def } A+B}{=} \sum_{k=1}^{n} (A_{ik} + B_{ik}) C_{kj}$

$\overset{\text{distribution}}{=} \sum_{k=1}^{n} (A_{ik}C_{kj} + B_{ik}C_{kj})$

$= \sum_{k=1}^{n} A_{ik}C_{kj} + \sum_{k=1}^{n} B_{ik}C_{kj}$ So $(A+B)C = AC+BC = (AC)_{ij} + (BC)_{ij} = (AC+BC)_{ij}$

Properties ④ $A \in M_{m\times n}(\mathbb{F})$. $B \in M_{n\times p}(\mathbb{F})$

$A^t \in M_{n\times m}(\mathbb{F})$, $B^t \in M_{p\times n}(\mathbb{F})$

$$\underset{\substack{n\times p \\ p\times m}}{(AB)^t} = \underset{p\times n \;\; n\times m \to p\times m}{B^t \cdot A^t}$$

Proof exercise check entry by entry

⑤ $\overset{m\times n}{A} I_n = A$

Remember $(A^t)^t = A$

$I_m A \overset{4}{=} (A^t \cdot I_m)^t$

$I_m^t = I_m$

$= (A^t)^t$

$= A$

⑥ V vector space . $\dim V = n$, β an ordered basis for V $= \{v_1, \cdots, v_n\}$

identity mapping $I_V : V \to V$, $I_V(x) = x$, then $[I_V]_\beta = [[I_V(v_1)]_\beta \; [I_V(v_2)]_\beta \cdots\cdots [I_V(v_n)]_\beta]$

$I_m^t = I_m = \underset{m\times n}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}} = I_n$, So $[I_V]_\beta = I_n$

$\parallel$

$[[I_V(v_1)]_\beta \; [I_V(v_2)]_\beta \cdots\cdots [I_V(v_n)]_\beta]$

Question Describe all linear transformation $T: V^n \to W^m$

$$\begin{tikzcd} V^n \arrow[r, "T"] \arrow[d, "IS"] & W^m \arrow[d, "IS"] \\ F^n \arrow[r] & F^m \end{tikzcd}$$

α $\quad$ β bases

Describe all linear transformations $T: F^n \to F^m$

$T(x) = Ax$ $\quad$ $f(x) = x^2$ non linear $\quad$ $(x+y)^2 \neq x^2 + y^2$

$f(x) = 2x$ $\quad$ $2(x+y) = 2x+2y$ linear!

Theorem Let $A \in M_{m \times n}(\mathbb{F})$ Define $L_A: F^n \to F^m$, $L_A(x) = \underset{m\times n}{A}\underset{n\times 1}{x}$ Then ① L_A is linear

$m \times 1$

$x \in F^n$

② $[L_A]_\alpha^\beta = A$ where α, β are standard ordered basis of F^n and F^m

③ $L_{A+B}: F^n \to F^m$ (Assume $A, B \in M_{m\times n}(\mathbb{F})$)

$L_{A+B}(x) = (A+B)x$

$L_{A+B}(x) = (A+B)x$
$= Ax + Bx$
$= L_A(x) + L_B(x)$
$= (L_A + L_B)(x)$

$c \in \mathbb{F}$, $L_{cA} = cL_A$

④ 3rd Way to Prove $A=B$ $\to$ entry wise, Column-wise, Matrix equality Theorem

Given $A, B \in M_{m\times n}(\mathbb{F})$, then $A=B$ iff $Ax = Bx$ for all $x \in F^n$

2.12

Matrices

① Left matrix multiplication : $L_A : F^n \to F^m$

Given $A \in M_{m\times n}(\mathbb{F})$, define $L_A(X) = AX$

① L_A is linear (check this)

② Matrix Equation theorem. Given $A, B \in M_{m\times n}(\mathbb{F})$

Then $A = B$ iff $AX = BX$ for every $X \in \mathbb{F}^n$

iff $L_A = L_B$

Proof: $\Rightarrow$ Trivial

$\Leftarrow$ Given $AX = BX$ for all $X \in \mathbb{F}^n$. Prove $A = B$, let $e_1, \cdots, e_n$ the standard basis in F^n

From assumption $\underbrace{Ae_k}_{k^{th}\text{ column of } A} = \underbrace{Be_k}_{k^{th}\text{ column of } B}$, $\forall 1 \le k \le n$, So $A = B$

Fun Question : How to represent a row of A in terms of matrix product

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, 1^{st} row: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2^{nd} row: $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

More Properties $L_A : F^n \to F^m$, $L_A(X) = AX$

① $\underbrace{L_{AB}}_{\text{matrix Product}} = \underbrace{L_A L_B}_{\text{Composition of mapping}}$, $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$

$L_{AB} : {}_{AB(y)}$ $F^n \to F^m$

Proof: Let $\{e_1, \cdots, e_p\}$ be the standard basis for $\mathbb{F}^p$, to know L_{AB}, we only need $L_{AB}(e_k)$, $e_k \in \mathbb{F}^p$

$1 \le k \le p$, $L_{AB}(e_k) = (AB)e_k = (A[b_1, \cdots, b_p])e_k$

$= k^{th}$ column of $(Ab_1, \cdots\cdots, Ab_p)$

$= Ab_k$

$L_A L_B(e_k) = L_A(L_B(e_k))$

$= L_A(Be_k)$

$= L_A(b_k)$

$= Ab_k$

$= L_{AB}(e_k)$ So $L_A L_B = L_{AB}$

② $A\cdot(BC) = (AB)\cdot C$

$\Updownarrow$

Proof

$L_{A(BC)} = L_{(AB)C}$

$L_{A(BC)} \overset{①}{=} L_A L_{BC} \overset{①}{=} L_A(L_B L_C)$

$\overset{\text{association}}{=\!=} (L_A L_B) L_C$

$\underset{①}{\overset{\text{of the composite}}{=\!=}} L_{AB} L_C$

$= L_{(AB)C}$

③ If $T: F^n \to F^m$ linear, then there exists a unique matrix $C \in M_{m\times n}(\mathbb{F})$ s.t. $T(x) = Cx$ for every $x \in \mathbb{F}^n$

Proof Existence, Let $\alpha = \{e_1, \cdots, e_n\}$ be standard basis in $\mathbb{F}^n$

$\beta = \{\varepsilon_1, \cdots, \varepsilon_m\}$ be the standard basis for $\mathbb{F}^m$

$[x]_\alpha = x, \ \forall x \in F^n$

$[T]_\alpha^\beta \in M_{m\times n}(\mathbb{F})$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad [x]_\alpha = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$ for every $x \in \mathbb{F}^n$

matrix product

$T(x) = [T]_\alpha^\beta x$ Done

Uniquess Assume $T(x) = C_1 x = C_2 x \overset{?}{\Rightarrow} C_1 = C_2$

$\forall x$ Yes! From Matrix Equality Theorem

Inverseable Matries

Question ① $\overset{m\times n \ n\times p}{AB}, \overset{n\times p \ m\times n}{BA}$ well-defined? Answer: $A: m\times n$, $B: n\times m$

$\#\text{col}A = \#\text{row}B$ $\quad \#\text{col}B = \#\text{row}A$

② $\underset{m\times m}{AB} = \underset{n\times n}{BA}$? In general, $AB \neq BA$

Defn Let $A \in M_{n\times m}(\mathbb{F})$, A is call inversiable matrix, if there exists a matrix $B \in M_{m\times n}(\mathbb{F})$ s.t. $AB = BA = I_n$

Ex. $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ is inverseable since $B = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$ and $AB = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $BA = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Lemma If $AB = BA = I_n$, $AC = CA = I_n$, then $B = C$ (an inverse a matrix (if exites) is unique)

Proof $B = BI_n = B(AC) = (BA)\cdot C = I_n C = C$. $B = A^{-1}$. (the inverse of A) $\quad A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$, $A^{-1} \neq \begin{pmatrix} \frac{1}{5} & \frac{1}{7} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$

Basic Proplies of A^{-1} (if exists)

① if A and B are $n\times n$ inversiable matries, then (AB) is also inverseable, and $(AB)^{-1} = B^{-1}A^{-1}$

Proof Check $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$, $(AB)\cdot(B^{-1}A^{-1}) = A(B\cdot B^{-1})\cdot A^{-1} = AI_nA^{-1} = A\cdot A^{-1} = I_n$

Similarly, $(B^{-1}A^{-1})(AB) = I_n$

② If A is invertiable, then A^{-1} is also invertiable and $(A^{-1})^{-1} = A$

Remark $A_{3\times 2}$, $B_{2\times 3}$. $AB \in M_{3\times 3}$, $BA \in M_{2\times 2}$, $(BA)^{-1} = \underline{A^{-1}B^{-1}}$

in vertical $\quad$ invertical $\quad$ not well defined

2.14

Inversible linear transformation : $T: V \to W$ linear, and there exist a function $U: W \to V$ s.t. $UT = I_V$ and $TU = I_W$

Theorem: $T: V \to W$ linear, then T is an injective linear transformation iff T is an isomorphism

Proof : $\Rightarrow$ Given $T: V \to W$ linear invertible, So $\exists$ $U: W \to V$ (a function) $TU = I_W$, $UT = I_V$

T is one to one $\quad T(x) = T(y)$

$$\underset{x}{\underset{\parallel}{U(T(x))}} = \underset{y}{\underset{\parallel}{U(T(y))}}$$

So T is one to one.

T is onto take $z \in W$, then $T(U(z)) = z$, So T is onto

$\Leftarrow$ exercise

Lemma If $f: X \to Y$ $\quad g: Y \to Z$ function of sets

$g \circ f : X \to Z$ If $g \circ f$ is bijective then $\left\{ \begin{array}{l} f \text{ injective} \\ g \text{ surjective} \end{array} \right.$

Proof : Consider $f(x) = f(y)$

$g(f(x)) = g(f(y))$

Because $g \circ f$ is one to one, $x = y$

Invertible linear transformations $=$ isomorphisms

$\Updownarrow$

Invertible matrices

Theorem: Given $T: V \to W$ linear, if T is invertable, then such function $U: W \to V$; $UT = I_V$, $TU = I_W$, is unique, Denote the inverse of T as T^{-1}

More over T^{-1} is also linear

Proof Uniqueness Suppose $U_1 T = I_V$ and $TU_1 = I_W$

$U_2 T = I_V$ and $TU_2 = I_W$

$$U_1, U_2 : W \to V$$

$$W \xrightarrow{I_W} W \xrightarrow{U_1} V$$

$U_1 = U_1 I_W = U_1(TU_2)$

$= (U_1 T) U_2$

$= I_V U_2 = U_2$

Linear $T^{-1}: W \to V$, $TT^{-1} = I_W$, $T^{-1}T = I_V$

Take $x, y \in W$, $c \in F$, Want $T^{-1}(cx+y) = cT^{-1}(x) + T^{-1}(y)$

$T(T^{-1}(cx+y)) = I_W(cx+y) = cx+y$

$T(cT^{-1}(x) + T^{-1}(y)) \overset{T \text{ linear}}{=\!=} c \cdot T(T^{-1}(x)) + T(T^{-1}(y)) = cx+y$

So $T(T^{-1}(cx+y)) = T(cT^{-1}(x) + T^{-1}(y))$, Since T is invertable, T is a bijection, T is one to one

So $T^{-1}(cx+y) = cT^{-1}(x) + T^{-1}(y)$

Theorem ② Let V and W be finite dimensional vector spaces α and β be ordered bases of V, W respectively
$T: V \to W$ linear. then T is invertiable iff $[T]_\alpha^\beta$ is invertiable

Applications ① Given $A \in M_{n\times n}(\mathbb{F})$, $L_A: \mathbb{F}^n \to \mathbb{F}^n$, $L_A(x) = Ax$, L_A is invertiable iff A is an invertiable matrix

② Lemma a: If $A, B \in M_{n\times n}(\mathbb{F})$, and AB is invertiable. then A and B are invertiable

Proof: AB is invertiable, so L_{AB} is invertiable (bijective)

$L_A L_B$ is bijective, L_A surjective, L_B injective

$L_A: \mathbb{F}^n \to \mathbb{F}^n$, $L_B: \mathbb{F}^n \to \mathbb{F}^n$

Recall: a linear $T: V \to W$, $\dim V = \dim W$, T is injective $\Longleftrightarrow$ T is surjective
$\Longleftrightarrow$ T is bijective

$\Rightarrow L_A, L_B$ invertiable

$\Rightarrow A, B$ are invertiable matrices

③ **Invertiable Matrix Theorem Part ①**

$A \in M_{n\times n}(\mathbb{F})$, If there exists a matrix $C \in M_{n\times n}(\mathbb{F})$ that either $AC = I_n$ or $CA = I_n$, then A is invertiable

Proof Given $AC = I_n$, Prove $CA = I_n$, $I_n^{-1} = I_n$ AC is invertiable so A, C are invertiable

$$(AC)^{-1} = I_n$$
$$C^{-1}A^{-1} = I_n$$
$$C \cdot (C^{-1} \cdot A^{-1}) = C$$
$$A^{-1} = C$$
, so $CA = A^{-1}A = I_n$

2.24

Recall: Matrix of a linear Representation | Change of Coordinate matrix

① $V \xrightarrow{T} W \xrightarrow{U} Z$ V, W, Z : finite dim vector spaces

T, U : linear α, β, γ : ordered bases

Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

② $T: \underset{\alpha}{V} \to \underset{\beta}{W}$ linear, α, β : ordered bases

$\forall x \in V$, then $[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$

③ $T: \underset{\alpha}{V} \to \underset{\beta}{W}$, $\dim V, \dim W < \infty$, T linear, then T is isomorphism $\iff$ $[T]_\alpha^\beta$ is invertiable

Question: Given a finite dim' vector space V. α, β two ordered bases on V $x \in V$ $[x]_\alpha \overset{?}{=} [x]_\beta$

Relation, whether that relation depends on x?

Theorem Let α and β be two ordered bases for a finite dim vector space V

Then, ① $[x]_\beta = [I_V]_\alpha^\beta [x]_\alpha$

② $Q = [I_V]_\alpha^\beta$ is invertiable, Q is called the change of coordinate matrix from α to β

Remark $[I_V]_\alpha^\beta = \left[[I_V(v_1)]_\beta \; [I_V(v_2)]_\beta \cdots [I_V(v_n)]_\beta \right]$ where $\alpha = \{v_1, \cdots, v_n\}$

$[I_V]_\alpha^\beta = [\,[v_1]_\beta, \cdots [v_n]_\beta\,]$

Proof $[x]_\beta = [I_V(x)]_\beta = [I_V]_\alpha^\beta [x]_\alpha$ ∎

apply ② for I_V

Proof Since I_V is an isomorphism, we have the conclusions

Question: Given $Q = [I_V]_\alpha^\beta$, $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_n\}$. What's relation between v_i, w_i?

jth column of Q

$$\begin{bmatrix} Q_{1j} \\ \vdots \\ Q_{nj} \end{bmatrix} = [v_j]_\beta$$

$v_j = Q_{1j} w_1 + \cdots + Q_{nj} w_n$

$v_j = \sum_{i=1}^{n} Q_{ij} w_i$

'

Theorem $T: V \to V$ linear, $\dim V < \infty$, α, β be two ordered bases of V

Q: the change of coordinate matrix from α to β

Then, $[T]_\alpha = Q^{-1}[T]_\beta Q$

$$[T]_\alpha = Q^{-1}[T]_\beta Q$$

$$\iff Q[T]_\alpha = QQ^{-1}[T]_\beta Q$$

$$\iff Q[T]_\alpha = [T]_\beta Q$$

Composition

LHS: $Q[T]_\alpha = [I_V]_\alpha^\beta \cdot [T]_\alpha^\alpha = [I_V T]_\alpha^\beta = [T]_\alpha^\beta$

RHS $= [T]_\beta Q = [T]_\beta^\beta \cdot [I_V]_\alpha^\beta = [T I_V]_\alpha^\beta = [T]_\alpha^\beta$

So $Q[T]_\alpha = [T]_\beta Q$ ∎

Def'n: Let $A, B \in M_{n \times n}(F)$, then A and B are said to be <u>similar</u> (A is similar to B)

If there exists an invertable matrix Q s.t. $B = Q^{-1}AQ$

Recall: Two vector spaces V and W are isomorphic to each other ($V \cong W$) if there exists an isomorphism $T: V \to W$
linear, 1-1, onto

Theorem of chapter 2: Let V and W be finite dim V.S. Then $V \cong W$ iff $\dim V = \dim W$ (Proof: exercise)

Chapter. 3 : Elementary Matrix Operations

Def'n Given $A \in M_{m \times n}(\mathbb{F})$

Elementary row Operations

① Interchanging two rows $R_i \leftrightarrow R_j$ (ith row of A)

② Multiply a row by a non-zero scalar $R_i \leftarrow cR_i$ $c \neq 0$

③ Adding a row by a scalar multiplication of another row $R_i \leftarrow R_i + cR_j$

Ex. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -9 & -10 & -11 & -12 \end{pmatrix}$

$R_1 \leftrightarrow R_3$ — $\begin{pmatrix} -9 & -10 & -11 & -12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix} = E_1 A$

$R_2 \times 3$ — $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 15 & 18 & 21 & 24 \\ -9 & -10 & -11 & -12 \end{pmatrix} = E_2 A$

$R_3 + 4R_1$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & 1 & 4 \end{pmatrix} = E_3 A$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_1 \Leftrightarrow R_3$ $E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$R_2 \times 3$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_3 + 4R_1$ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

Remark ① $R_i + cR_j$ puts into R_j is not an elementary row operation

② We can define elementary column operations

3.1

Def'n Elementary Matrices

$E \in M_{m \times m}(\mathbb{F})$ is obtained from I_m using elementary column | row operations

Ex. Elementary matrices corresponding to the row operations of A in previous examples are

Theorem Let $A \in M_{m\times n}(\mathbb{F})$ and suppose B is obtained from A by performing an elementary row operation. Then $\exists$ an elementary matrix $E \in M_{m\times m}(\mathbb{F})$ s.t. $B = EA$

Explicitly: ① E is obtained from I_m using the same operation as that which was performed on A to get B

② Conversely, if E is an elementary matrix, $E \in M_{m\times m}(\mathbb{F})$ then $B = EA$ is the matrix obtained from A using the same operations to get E from I_m

Proof Verify the theorem for three types of elementary column operations

① $Col_i \leftrightarrow Col_j$ Wlog, we assume $i<j$

$$A = [a_1, \cdots, a_i, \cdots, a_j, \cdots, a_n] \xrightarrow{Col_i \leftrightarrow Col_j} B = [a_1, \cdots, a_j, \cdots, a_i, \cdots, a_n]$$

$$I_n = [e_1, e_2, \cdots, e_i, \cdots e_n] \xleftrightarrow{Col_i \leftrightarrow Col_j} E = [e_1, e_2, \cdots, e_j, \cdots, e_i, \cdots e_n]$$

$$\begin{aligned} AE &= [Ae_1, \cdots, Ae_j, \cdots, Ae_i, \cdots, Ae_n] \\ &= [a_1, \cdots a_2, \cdots, a_j, \cdots, a_i, \cdots a_n] \\ &= B \end{aligned}$$

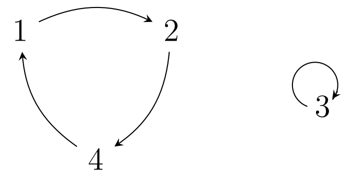
MATH 146 **Feb 24 – corrected** **Tutorial notes**

In normal language, a *permutation* of $\{1, 2, 3, 4\}$ is any ordering of its elements, for example $(2, 4, 3, 1)$. For us, however, a permutation of $\{1, 2, 3, 4\}$ is any bijection $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$, for example,

$$\sigma(1) = 2, \quad \sigma(2) = 4, \quad \sigma(3) = 3, \quad \sigma(4) = 1.$$

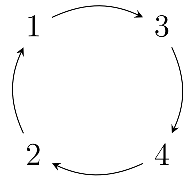
One advantage of viewing permutations as functions is that we can interpret them graphically, for example,

σ:



Another advantage is that we can compose two permutations of the same set. For example, if τ is the permutation

τ :

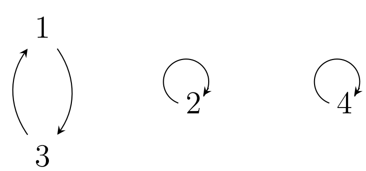


then $\sigma \circ \tau$ (usually written $\sigma\tau$) is the permutation defined by

$$\begin{array}{rcccl} \sigma\tau(1) & = & \sigma(3) & = & 3 \\ \sigma\tau(2) & = & \sigma(1) & = & 2 \\ \sigma\tau(3) & = & \sigma(4) & = & 1 \\ \sigma\tau(4) & = & \sigma(2) & = & 4 \end{array}$$

so $\sigma\tau$ displayed graphically is

$\sigma\tau$:



Cycle notation. We write $\sigma = (1\ 2\ 4)(3)$, $\tau = (1\ 3\ 4\ 2)$, and $\sigma\tau = (1\ 3)(2)(4)$, indicating the "cycles" (reading left to right). It's also OK to leave out the 1-element cycles, writing $\sigma = (1\ 2\ 4)$ and $\sigma\tau = (1\ 3)$, as long as it is understood the permutations are acting on $\{1, 2, 3, 4\}$.

Definition. S_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

Definition. Given $\sigma \in S_n$ and $A, B \in \mathsf{M}_{n\times n}(\mathbb{F})$, we write $A \xrightarrow{R:\sigma} B$ to mean that B is obtained from A by moving $\mathrm{Row}_1(A)$ to row $\sigma(1)$, moving $\mathrm{Row}_2(A)$ to row $\sigma(2)$, etc.

For example, if σ is the permutation on $\{1, 2, 3, 4\}$ defined at the start of the lecture, i.e., $\sigma = (1\ 2\ 4)$, and A is a 4×4 matrix with rows $r_1, \ldots, r_4$, then

$$A \;=\; \begin{pmatrix} \text{---}\; r_1 \;\text{---} \\ \text{---}\; r_2 \;\text{---} \\ \text{---}\; r_3 \;\text{---} \\ \text{---}\; r_4 \;\text{---} \end{pmatrix} \quad \xrightarrow{R:\sigma} \quad \begin{pmatrix} \text{---}\; r_4 \;\text{---} \\ \text{---}\; r_1 \;\text{---} \\ \text{---}\; r_3 \;\text{---} \\ \text{---}\; r_2 \;\text{---} \end{pmatrix} \;=\; B.$$

Observe that

$$B = \begin{pmatrix} \text{—} & r_{\sigma^{-1}(1)} & \text{—} \\ \text{—} & r_{\sigma^{-1}(2)} & \text{—} \\ \text{—} & r_{\sigma^{-1}(3)} & \text{—} \\ \text{—} & r_{\sigma^{-1}(4)} & \text{—} \end{pmatrix}.$$

In general, if σ is a permutation of $\{1, 2, \ldots, n\}$ then

$$\begin{pmatrix} \text{—} & r_1 & \text{—} \\ \text{—} & r_2 & \text{—} \\ & \vdots & \\ \text{—} & r_n & \text{—} \end{pmatrix} \xrightarrow{R:\sigma} \begin{pmatrix} \text{—} & r_{\sigma^{-1}(1)} & \text{—} \\ \text{—} & r_{\sigma^{-1}(2)} & \text{—} \\ & \vdots & \\ \text{—} & r_{\sigma^{-1}(n)} & \text{—} \end{pmatrix}.$$

Definition. Given a permutation σ on $\{1, 2, \ldots, n\}$, the **permutation matrix** associated to σ is the matrix P_σ obtained from I_n by $\xrightarrow{R:\sigma}$. That is, $I_n \xrightarrow{R:\sigma} P_\sigma$. Thus

$$P_\sigma = \begin{pmatrix} \text{—} & e_{\sigma^{-1}(1)} & \text{—} \\ \text{—} & e_{\sigma^{-1}(2)} & \text{—} \\ & \vdots & \\ \text{—} & e_{\sigma^{-1}(n)} & \text{—} \end{pmatrix}$$

For example, the permutation matrix associated to the permutation σ above is

$$P_\sigma = \begin{pmatrix} \text{—} & e_4 & \text{—} \\ \text{—} & e_1 & \text{—} \\ \text{—} & e_3 & \text{—} \\ \text{—} & e_2 & \text{—} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \overset{\text{Note}}{=} \begin{pmatrix} | & | & | & | \\ e_2 & e_4 & e_3 & e_1 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ e_{\sigma(1)} & e_{\sigma(2)} & e_{\sigma(3)} & e_{\sigma(4)} \\ | & | & | & | \end{pmatrix}.$$

In general:

- P_σ has one 1 in each row and in each column; all other entries are 0.
- $\mathrm{Row}_i(P_\sigma) = e_{\sigma^{-1}(i)}$.
- $\mathrm{Col}_j(P_\sigma) = e_{\sigma(j)}$.
- $P_{\sigma^{-1}} = (P_\sigma)^t$ (by the previous two bullets).
- For any j, $P_\sigma e_j = e_{\sigma(j)}$. (Because in general, $Ae_j = \mathrm{Col}_j(A)$.)

Theorem. *If $\sigma \in S_n$ and $A \in \mathsf{M}_{n\times n}(\mathbb{F})$, then*

(1) *$P_\sigma A$ is the result of applying σ to the rows of A; i.e., $A \xrightarrow{R:\sigma} P_\sigma A$.*

(2) *AP_σ is the result of applying σ^{-1} to the columns of A, i.e., $A \xrightarrow{C:\sigma^{-1}} AP_\sigma$.*

Proof of (2). Write

$$A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix}. \qquad \text{Applying } \sigma^{-1} \text{ to the columns gives} \quad \begin{pmatrix} | & | & & | \\ c_{\sigma(1)} & c_{\sigma(2)} & \cdots & c_{\sigma(n)} \\ | & | & & | \end{pmatrix}.$$

To prove this second matrix equals AP_σ, we just need to prove that $\mathrm{Col}_j(AP_\sigma) = c_{\sigma(j)}$ for all $j = 1, \ldots, n$. In fact,

$$\mathrm{Col}_j(AP_\sigma) = A \cdot \mathrm{Col}_j(P_\sigma) = A \cdot e_{\sigma(j)} = \mathrm{Col}_{\sigma(j)}(A)$$

as required. (Proof of (1) is "dual" to this, using $\mathrm{Row}_i(P_\sigma A) = \mathrm{Row}_i(P_\sigma) \cdot A$.) □

Corollary. *$P_\sigma P_\tau = P_{\sigma\tau}$ for any $\sigma, \tau \in S_n$.*

Proof. On Assignment 4. □

2.26

Elemene Matrix Operation, Rank of a Matrix

Theorem ① Let $A \in M_{m\times n}(\mathbb{F})$ and suppose B is obtained from A by performing an elementary row operation. Then $\exists$ an elemenary matrix $E \in M_{m\times n}(\mathbb{F})$ s.t. $B = EA$

Explicitly: ① E is obtained from I_m using the same operation as that which was performed on A to get B

② Conversely, if E is an elemenery matrix, $E \in M_{m\times m}(\mathbb{F})$ then $B = EA$ is the matrix obtained from A using the same operations to get E from I_m

Proof Verify the theorem for three types of elementry column operations

① $Col_i \leftrightarrow Col_j$ Wlog, we assume $i<j$

$$A = [a_1, \cdots, a_i, \cdots, a_j, \cdots, a_n] \xrightarrow{Col_i \leftrightarrow Col_j} B = [a_1, \cdots, a_j, \cdots, a_i, \cdots, a_n]$$

$$I_n = [e_1, e_2, \cdots, e_i, \cdots e_n] \xleftrightarrow{Col_i \leftrightarrow Col_j} E = [e_1, e_2, \cdots, e_j, \cdots, e_i, \cdots e_n]$$

$$\begin{aligned} AE &= [Ae_1, \cdots, Ae_j, \cdots, Ae_i, \cdots, Ae_n] \\ &= [a_1, \cdots a_2, \cdots, a_j, \cdots, a_i, \cdots a_n] \\ &= B \end{aligned}$$

Therem ② Elementry matrices are invertiable

Proof: Let E be an elemenary column matrix, $\hat{E}E = E\hat{E} = I_n$

Idea: Apply theorem ① with $B = I$, $A = E$, then $I = E\hat{E}$. done

Take: $E \xrightarrow[\text{operation}]{\text{an elemeny matrix}} I$

We will verify through the way we get E

Type 1: $I_n \xrightarrow{Col_i \leftrightarrow Col_j} E$

$E \xleftrightarrow{Col_i \leftrightarrow Col_j} I_n$

Type 2 $I_n \xrightarrow[C\,Col_i]{Col_i \leftarrow C\,Col_i,\ C\neq 0} E$

$E \xrightarrow{C^{-1}\,Col_i} I_n$

Type 3 $Col_i \leftarrow Col_i + C\,Col_j$

$I_n \longrightarrow E$

$E \xrightarrow{Col_i \leftarrow Col_i - C \cdot Col_j} I_n$

Rank of a matrix

Recall ① $T: V \to W$ linear, $\dim R(T) < \infty$

② $A_{m\times n}$, $L_A: F^n \to F^m$ linear, $L_A(x) = Ax$, We define $\text{rank } A = \text{rank } L_A$

$= \dim R(L_A)$

$= \dim L_A(F^n)$

Theorem Let $A_{m\times n}$, $P_{m\times m}$ $Q_{n\times n}$ Suppose P and Q are invertiable

① $\text{rank}(PA) = \text{rank } A$

② $\text{rank}(AQ) = \text{rank } A$

③ $\text{rank}(PAQ) = \text{rank } A$

Proof: ① $\text{rank}(PA) \overset{\text{def}}{=} \dim L_{PA}(F^n)$

$\dim L_A(F^n) = \dim L_P(L_A(F^n))$ $\quad (L_{PA} = L_P L_A)$

Since P is invertiable, L_P is an isomorphism

Result $T: V \xrightarrow{\text{isomorphism}} W$, V_0 is a finite dimentional subspace of V, $\dim V_0 = \dim T(V_0)$

② $\text{rank}(AQ) \overset{\text{def}}{=} \dim L_{AQ}(F^n)$ $\quad AQ: F^n \to F^m$

$= \dim L_A(L_Q(F^n))$ $\quad L_Q: F^n \to F^n$

$= \dim L_A(F^n)$ $\quad$ an isomorphism

$= \text{rank } A$ $\quad L_Q(F^n) = F^n$

Invertiable matrix, Theorem Part ② $A_{n\times n}$ is invertiable iff $\text{rank } A = n$

Proof $\Rightarrow$ $\text{rank } A = \text{rank}(A \cdot A^{-1})$

$= \text{rank}(I_n)$

$= \text{rank}(I_{F^n})$

$= \dim F^n$

$= n$

$\Leftarrow$ Given $\text{rank } A = n$, we need to show A is invertiable

$n = \text{rank } A = \text{rank } L_A$, $L_A: F^n \to F^n$

So L_A is onto, L_A is an isomorphism, so A is invertiable

Question find Rank A given $A_{m\times n}$

Solution ① $A \to L_A$ find dim Range (L_A) quite complicated

Solution ② work on A itself

Corollary Every elementary operation preserves the rank of the matrix

$$A \xrightarrow[\text{Rank } B = \text{Rank } A]{\text{an elementary row (column) operation}} B$$

row operation matrix $B = EA$

for some elementary matrix E

E is invertible, so $\text{rank} B = \text{rank} A$

$A \to$ sparse matrices (a lot of 0s entries)

Theorem $A \xrightarrow{\text{a finite number of elementary row/column operation}} D = \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$

$$= \left[\begin{array}{c|c} I_n & O_1 \\ \hline O_2 & O_3 \end{array}\right]$$

$\text{rank } A = \text{rank } D = n$

2.28

Today Rank of a matrix $A \longleftrightarrow$ Column of A, rows of A $\Big\}$ Create 4 special vector spaces, generated from A
4 Fundamental subspaces. ~~~ Rank nullity Theorem

Prop Let $A_{m\times n}$, then $\operatorname{rank} A = \#$ of maximal independent columns of A

Proof: $\operatorname{Rank} A = \dim R(L_A)$ $\qquad L_A: F^n \to F^m$

$= \dim L_A$ $\qquad R(L_A) = L_A(F^n)$

$= \dim \operatorname{span}\{L_A(e_1), \dots, L_A(e_n)\}$ $\qquad = \operatorname{span}\{L_A(e_1), \dots, L_A(e_n)\}$

$= \dim \operatorname{span}\{Ae_1, \dots, Ae_n\}$ $\qquad$ where $\{e_1, \dots, e_n\}$ standard basis of F^n

$= \dim \operatorname{span}\{a_1, \dots, a_n\}$

where $A = \{a_1, \dots, a_n\}$, $a_1, \dots, a_n$ are column of A

$= \#$ of maximal linearly indep. set of column of A

Ex. ① $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\operatorname{rank} A = \#$ of maximal linearly indep. set $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\} = 3$

② $D = \left[\begin{array}{c|c} I_n & O_1 \\ \hline O_2 & O_3 \end{array}\right]$ $\qquad \operatorname{rank} = n$

③ $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}$ $\qquad$ Find rank of A

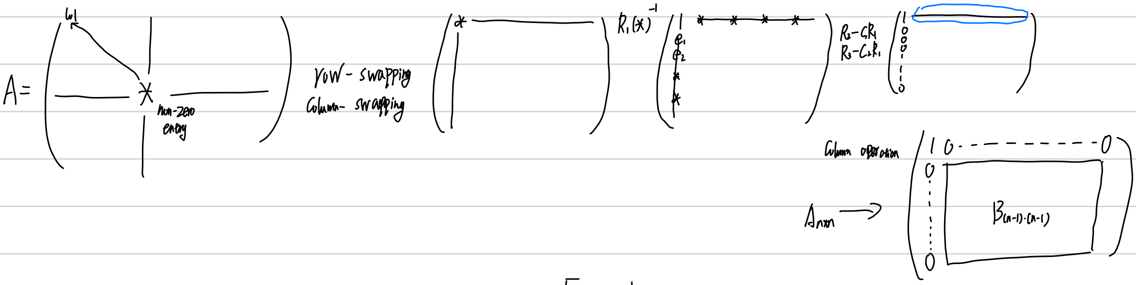
$a_{ij} = i^j$

Theorem Given $A_{m\times n} \xrightarrow[\text{of elementary row/column operations}]{\text{a finite number}} D = \left[\begin{array}{c|c} I_n & O_1 \\ \hline O_2 & O_3 \end{array}\right]$ and $\operatorname{rank} A = n$

Proof Case #1 $\quad A = 0$, $D = [0]$ $\qquad D =$

$n = 0$

Case #2 $\quad A \neq 0$ $\quad A$ has a non-zero entry



Next step: use induction hypothesis on $B_{n-1,n-1} \longrightarrow \tilde{D} = \left[\begin{array}{c|c} I_{n-1} & O_1 \\ \hline O_2 & O_3 \end{array}\right]$

$A \longrightarrow \left(\begin{array}{c|c} 1 & 0\,0 \cdots 0 \\ \hline 0 & \\ 0 & \tilde{D} \\ \vdots & \\ 0 & \end{array}\right)$ Done

Ex. $A = \begin{pmatrix} 0 & -1 & 1 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$ show how to transform $A \to \left[\begin{array}{c|c} I_n & O_1 \\ \hline O_2 & O_3 \end{array}\right]$

Solution

$$A = \begin{pmatrix} 0 & -1 & 1 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{\text{Column 2} \times (-1) \leftrightarrow \text{Column 1}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow[\text{Col 1} - 3\text{ Col}]{\text{Col 3} - \text{Col 1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{\text{Col 2} \leftrightarrow \text{Col 4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[\text{Col 4} - \text{Col 2}]{\text{Col 3} - \text{Col 2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{\text{Col 4} - 2\text{Col 3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$A \xrightarrow[\text{row operation}]{\text{elementary}} B = EA$ where $E_{m\times m}$ invertiable elementary matrix

$\xrightarrow[\text{column operation}]{\text{elementary}} C = AG$ where $G_{n\times n}$ invertiable

$A \longrightarrow D = E_p \cdot E_{p-1} \cdots, E_1 A G_1 G_2 \cdots G_q$

Denote $B = E_p, \cdots, E_1$, $C = G_1 G_2 \cdots G_q$

then B and C are invertiable, Since $E_i . G_j$ are invertiable

so $D = BAC$ where B, C are invertiable $m \times m$, $n \times n$ matrices

Theorem: Given $A_{m\times n}$, there exist invertiable matrices $B \in M_{m\times m}(\mathbb{F})$ and $C_{n\times n}$ s.t. $BAC = \left(\begin{array}{c|c} I_r & O_1 \\ \hline O_2 & O_3 \end{array}\right)$

and $r = \text{rank } A$

Corollary Let $A_{m\times n}$, Then ① $\text{Rank}(A^t) = \text{Rank } A$

Proof From previous theorem, $\exists$ invertiable matrix $B_{m\times m}$, $C_{n\times n}$ s.t. $BAC = \underbrace{\left(\begin{array}{c|c} I_r & O_1 \\ \hline O_2 & O_3 \end{array}\right)}_{D}$ where $r = \text{rank } A$

$A = B^{-1} \cdot D \cdot C^{-1}$. $A^t = (B^{-1})^t \cdot D^t \cdot (C^{-1})^t$

$D^t = \left(\begin{array}{c|c} I_r & O_2 \\ \hline O_1 & O_3 \end{array}\right)$, So $\text{rank } A^t = \text{rank } D^t = r$

② $\text{rank}(A^t) = \dim R(L_{A^t})$

$= \#$ of maximal linearly indep. $\frac{\text{Column of } A^t}{\text{row of } A}$

$=$

$\vdots$

$\text{rank}(A) = \#$ of maximal linearly. indp. rows of A

$= \#$ of maximal linearly Indp. column of A

3.2

Theorem (Rank of a matrix Product)

Given $A_{m\times n}$, $B_{n\times p}$ Then $rank(AB) \leq min(rank\ A, rank\ B)$

Application : $A_{3\times 2}$, $B_{2\times 3}$ $\Rightarrow AB_{3\times 3}$ $rank(AB) \leq min(rank A, rank B) \leq 2 < 3$

$rank A \leq min(3,2) = 2$

$rank B \leq min(2,3) = 2$

So AB is not invertiable (A square matrix $C_{n\times n}$ is invertable iff $rank\ C = n$)

Fun Questions

① $A_{m\times n}$, $B_{n\times m}$, In general $AB \neq BA$

?? Is $rank(AB)$ always equal to $rank(BA)$?

② $tr\ AB = tr(BA)$

Proof $rank(AB) \overset{?}{\leq} rank\ A)$

$Rank(AB) = dim\ R_{L_{AB}}$

$R_{L_{AB}} = \{ABx \mid x \in R^p\} \underset{\text{Sub space}}{\subseteq} \{Ay \mid y \in R^n\} = R(L_A)$

So $dim\ R(L_{AB}) \leq dim\ R(L_A)$

$rank(AB) \leq rank\ A$

$rank(AB) = rank((AB)^T) = rank(B^TA^T) \leq rank(B^T) = rank\ B$ $\square$

Fun Question $\underset{m\times n}{A}$, $\underset{n\times m}{A^T}$: $\underset{m\times m}{A \cdot A^T}$, $\underset{n\times n}{A^T \cdot A}$ Symmetric

Prove ① $Null(A)$ is a Subspace of $Null(A^T A)$

② $Rank\ A = Rank(A^T A)$ Exercise ◡̈

◡̈ Hint: Four Fundamental subspaces of a Matrix

$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 Col_1 + x_2 Col_2 \cdots + x_n Col_n$

Def'n : Given $A_{m\times n}$, define ① $Col(A) = \{Ax \mid x \in F^n\}$

$= \{$all linear combinations of column of $A\}$

$=$ column Space of A

$=$ Span $\{$Column of $A\}$

② $Row(A) = \{$all linear Combination of $A\}$

$= Col(A^T) = \{A^Ty \mid y \in F^m\}$

③ $Null(A) = \{x \in F^n \mid Ax = 0\}$ null Space of A

④ $Null(A^T) = \{y \in F^m \mid A^Ty = 0\}$ Left null Space of A

$A^Ty = 0$

$(A^Ty)^T = 0$

$y^TA = 0$

Theorem Let $A_{m\times n}$, then

① $Col(A)$ and $Null(A^T)$ are subspace of F^m

$Row(A)$ and $Null(A)$ are subspace of F^n

② $\dim Col(A) + \dim Null(A^T) = m$

$\dim Rank(A) + \dim Null(A) = n$

③ $F^m = Col\,A \oplus Null(A^T)$

$F^n = Row\,A \oplus Null(A)$

Proof ② (use Rank-Nullity-Theorem) $T: U \to V$, $\dim U < \infty$, then $\dim U = \dim R(T) + \dim N(T)$

In this case, $L_A: F^n \to F^m$, $L_A(x) = Ax$ (L_A is linear)

Apply the Rank-Nullity-Theorem we have $\dim F^n = \dim R(L_A) + \dim N(L_A)$

$\#$ of column of $A = n = rank\,A + \dim Null(A)$

Moreover: $rank$ of A = maximal number of linearly independent rows of A

$= \dim Row A$ because $Rank\,A = Span\{$all rows of $A\}$

③ First Prove $Col(A) \cap Null(A^T) = \{0\}$ (an exercise)

From ②, $\dim Col(A) + \dim Null(A^T) = m$, so $\dim Col(A) + Null(A^T) \overset{A2.Q3a}{=} \dim Col(A) + \dim Null(A^T) - \dim(Col(A) \cap Null(A^T))$

$= m - 0 = m$

So $Col(A) \oplus Null(A^T) = F^m$

Prove ① $Rank\,A = Rank(A^TA)$

② Singular Value decomposition

Summary: $Rank\,A = \dim Col(A) = \dim Row(A) = Rank(A^T)$

3.4

Inverciable Matrix Theorem Part #3. Let $A_{n\times n}$. Then, the following Statement are equivalent

1. A is invertiable
2. The column of A form a basis for F^n
3. The rows of A form a basis for F^n
4. A is a Produce of elementary matrices
5. $\text{rank} A =$ (Part 2)

Proof ① $\iff$ ②, ① $\iff$ ④ — $\text{rank}A =$ the maximal number of linearly Independence Columns of A

① A is invertiable $\iff$ $\text{rank } A = n \iff$ the columns of A form a basis for F^n
A has n columns

④ $\Rightarrow$ ① assume $A = E_1 E_2 \cdots E_p$, where $E_1, \cdots, E_p$ are elementary matrices
Since elementary matrices are invertiable, a matrix Produce of invertiable matries is invertiable
so A is invertiable, $A^{-1} = E_p^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$ (verify $A^{-1} \cdot A = I_n$)

① $\Rightarrow$ ④ Assume A is invertiable, Recall A elementary matrix operation $D = \left(\begin{array}{c|c} I_\lambda & O_1 \\ \hline O_2 & O_3 \end{array}\right)$. $\lambda = \text{rank} A$, $D = BAC$ where B and C Produces of elementary matrix

In this Case, $\lambda = n$. $D = I_n = \underline{B} A \underline{C}$

$$B^{-1} I_n C^{-1} = A$$

$$B^{-1} C^{-1} = A$$

$$B = E_1, \cdots, E_p, \quad B^{-1} = (E_1, \cdots, E_p)^{-1} = E_p^{-1} \cdots E_1^{-1}$$ ☒

Summary $A_{n\times n}$, A invertiable $\iff$ A = Produce of elementary matrices

How to find A^{-1}? elementary row operations Given A invertiable

Idea $C = (A_{n\times n} | I_n)_{n\times 2n}$

$$A^{-1}C = A^{-1}(A | I_n) = (A^{-1}A | A^{-1}I_n) = (I_n | A^{-1})$$

A^{-1} is also invertiable, $A^{-1} = E_1 E_2 \cdots E_k$. E_k : elementary matrix

$$C = (A | I_n) \xrightarrow{\text{multiply by } A^{-1} \text{ on the left}} A^{-1}C = (I_n | A^{-1})$$

$$\xrightarrow[\text{row operations}]{\text{series of elementary}} E_1 E_2 \cdots E_p \, C$$

To find A^{-1}: $(A | I_n) \xrightarrow{\text{elementary row operations}} (I_n | A^{-1})$

Question $(A | I_n) \xrightarrow[\text{row operation}]{\text{finite \# of elementary}} (I_n | B)$ is $B = A^{-1}$?

Theorem ① If A is invertiable $n \times n$, then $(A \mid I_n) \longrightarrow (I_n \mid A^{-1})$ via a certain finite set of elemenary row operations

② Conversly, if $A_{n\times n}$ and if there is a finite number of row operations $(A \mid I_n) \rightarrow (I_n \mid B)$, then A is invertiable $B = A^{-1}$

$$(A \mid I_n) \xrightarrow{\text{row operation}} (I_n \mid B)$$

$$(I_n \mid B) = \underbrace{E_q \cdots E_2 E_1}_{=G} (A \mid I_n)$$

$$(I_n \mid B) = (GA \mid GI_n)$$

$$= (GA \mid G)$$

$I_n = GA$ So A is invertiable

$B = G$ and $B = A^{-1}$

Remark $(A \mid I_n) \longrightarrow (\underset{0\,0\,0\,0}{\underline{\qquad}} \mid B)$

$k < w$

A is not invertiable

Example $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

① Find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - \frac{1}{2}R_2]{R_2 \cdot \frac{-1}{2}} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & -1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

② Write A as a elemenary matrix (exercise)

System of linear Equations

Def'n ① a system of linear equations is of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where $a_{ij}, b_j \in \mathbb{F}$ $\quad 1 \le i \le m$, $1 \le j \le n$ (m equations, n variables $x_1, \dots, x_n$)

② $$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

$A X = b$, $A \in M_{m\times n}(\mathbb{F})$, $X \in \mathbb{F}^n$ (variable), $b \in F^m$

A: the coefficient matrix of the system

b: the right hand side of the system

③ $(A|b)_{m\times(n+1)}$ = the argumented matrix of the system

④ Given $A_{m\times n}$, $b \in \mathbb{F}^m$ If $AC = b$ for some $C \in \mathbb{F}^n$, then C called a solution of the system

⑤ The set of all solutions to the system is called the solution set of the system

⑥ The system $AX = b$ is called consistent (有解) if the system $AX = b$ is called inconsistent if the solution set $= \emptyset$

⑦ The system $AX = 0$ is called homogeneous, the system $AX = b$, $b \ne 0$ is called inhomogeneous

Theorem The solution set of the homogeneous system, $AX = 0$, $A_{m\times n}$ is a subspace of $\mathbb{F}$

Solution set $= \{ \text{all } x \in \mathbb{F}^n \text{ s.t. } AX = 0 \} = \text{Null}(A) = N(L_A)$

3.6

Theorem Consider $AX=0$, $A_{m\times n}$, $X\in\mathbb{F}^n$ (m equations, n unknowns $x_1, x_2, \cdots, x_n$)

Denote: K = the solution set to $AX=0$

$= \{\text{all } X\in\mathbb{F}^n \text{ such that } AX=0\}$

Claim ① K is a subspace of $\mathbb{F}^n$

② $\dim K = n - \operatorname{rank} A$

$= \#\text{ variables} - \operatorname{rank} A$

$= \#\text{ columns of } A - \operatorname{rank} A$

Proof Note $= N(L_A) = \{X\in\mathbb{F}^n \text{ s.t. } AX=0\}$, Therefore, K is a subspace of $\mathbb{F}^n$

By the Rank-Nullity Theorem on $L_A: \mathbb{F}^n \to \mathbb{F}^m$

$n = \dim R(L_A) + \dim N(L_A)$

$= \operatorname{rank} A + \dim K$ so $\dim K = n - \operatorname{rank} A$

Theorem Consider $AX=0$, $A_{m\times n}$, $X\in\mathbb{F}^n$, $K=\{X\in\mathbb{F}^n \mid AX=0\}$

Remarks: ① $0\in K$ (the trivial solution)

② $K=\{0\} \iff \dim K = 0 \iff \operatorname{rank} A = n$

In words, the linear system $AX=0$ has a unique solution iff $\operatorname{rank} A = n \iff$ the columns of A are linearly Indep

A has full-column rank

In words more unknowns than # variables then $AX=0$ has more than one solution

③ If $m<n$, then $AX=0$ always has a non-zero solution. $\operatorname{rank} A \le \min(m,n)$

Theorem Consider $AX=b$, $A_{m\times n}$, $b_{m\times 1}$.

If c is a solution $AX=b$. Then the solution set of $AX=b$, $K=\{z\in\mathbb{F}^n, Az=b\}$, $K=\{c\}+K_H$ where $H=\{z\in\mathbb{F}^n \cdot Az=0\}$

Proof $K=\{X\in\mathbb{F}^n : AX=b\}$. $K_H=\{y\in\mathbb{F}^n : Ay=0\}$, have $Ac=b$

① $K\subset\{c\}+K_H$ Take $w\in K$ so $\left.\begin{matrix}Aw=b\\ Ac=b\end{matrix}\right\} \Rightarrow A(w-c)=0$, $w-c\in K_H$, So $w\in\{c\}+K_H$

left

Check it

② $K\supset\{c\}+K_H$

Invertiable Matrix Theorem Part 4

Given $A_{n\times n}$, then A is invertiable iff the equation $Ax=b$ has a unique solution

Proof $\Rightarrow$ A invertiable, Solve $Ax=b$

$A^{-1}Ax = A^{-1}b$

$I_n x = A^{-1}b$

$x = A^{-1}b$

$\Leftarrow$ The equation $Ax=b$ has a unique solution C

$\underset{\text{Solution set of } Ax=b}{K} = \{c\} + \underset{\text{Solution set of } Ax=0}{K_H}$

It means $K_H = \{0\}$ From the remark: $\text{rank}A = n$

So A is invertiable

Question $A_{m\times n}$, $Ax=b$, when $Ax=b$ has a unique solution?

- existence — $\text{rank}A = \text{rank}(A|b)$
- unique — $\text{rank}A = \#\text{columns}$

Theorem Given $A_{m\times n}$, $b\in\mathbb{F}^m$, the system $Ax=b$ is consistent iff $\text{rank}(A) = \text{rank}(A|b)$.

Proof $b=Ax$ has a solution

$\Leftrightarrow b\in R(L_A) = \text{Span}\{L_A(e_1),\cdots,L_A(e_n)\}$

$\Leftrightarrow b\in \text{Span}\{\text{Col}_1(A),\cdots,\text{Col}_n(A)\}$

$\Leftrightarrow \text{Span}\{\text{Col}_1(A),\cdots,\text{Col}_n(A)\} = \text{Span}\{\text{Col}_1(A),\cdots,\text{Col}_n A, b\}$

$\Leftrightarrow \dim(\quad\downarrow\quad) = \dim(\quad\downarrow\quad)$

$\Leftrightarrow \text{Rank}A = \text{Rank}\{A|b\}$

Question How to find solutions of $Ax=b$? Answer: use elementary row operations

Def'n Two linear systems $A_1x=b_1$, $A_2x=b_2$, A_1 $m\times n$, A_2 $p\times n$, $b_1\in\mathbb{F}^m$, $b_2\in\mathbb{F}^p$, are said to be equivalent if they have same solution set.

Theorem Consider $Ax=b$, $A_{m\times n}$, $b\in\mathbb{F}^m$ and C an $m\times m$ invertiable, then the system $Ax=b$ is equivalent to $(CA)x = Cb$

Proof exercise

Collary If $(A|b) \xrightarrow[\text{Operations}]{\text{elementary row}} (\hat{A}|\hat{b})$. Then, $Ax=b$ is equivalent to $\hat{A}x=\hat{b}$

Proof $C(A|b) = (CA|Cb)$

$\hat{A} \quad CAx = Cb$

$C=$ the Product of elementary matrices that transform $(A|b) \to (\hat{A}|\hat{b})$

Goal: $(A|b) \xrightarrow[\text{row. op}]{\text{elementary}} (\hat{A}|\hat{b})$ simplest form $=$ RREF

Def'n a matrix is called in Reduced row elementary form

if it satisfies the following Properties:

① all non-zero rows are above zero rows $00\cdots0$

② The first entry of each non-zero row is 1. called the leading one

③ all other entries of the column that contains the leading one must be 0

④ The leading one of a non-zero row is on the right to the leading of the Previous rows

3.4

Question: How to transform a matrix $A_{m\times n}$ to an RREF

Method: Gaussian Elimination / another method: Gauss-Jordan elimination

Step ① In the left most column, use elementary row operations to create 1 in the first row

$$A=\begin{pmatrix}1&0&1&2&4\\2&1&4&1&6\\-1&-4&-9&5&4\end{pmatrix}\xrightarrow[R_3+R_1]{R_2-2R_1}\begin{pmatrix}1&0&1&2&4\\0&1&2&-3&-2\\0&-4&-8&7&8\end{pmatrix}$$

in the first row

Step ② use type 3, row operations to make all other entries of that column to be 0

Step ③ Consider the submatrix consisting of the columns to the left of the column that we modified. below the first row

use elementary row operation to create 1 of the matrix

Step ④ use elementary row operation to make all entries of that columns between the leading one

$$\left(\begin{array}{cccc|c}1&0&1&2&4\\0&1&2&-3&-2\\0&-4&-8&7&8\end{array}\right)\xrightarrow{R_3+4R_2}\left(\begin{array}{cccc|c}1&0&1&2&4\\0&1&2&-3&-2\\0&0&0&5&0\end{array}\right)\xrightarrow{R_3\times\frac{1}{5}}\left(\begin{array}{cccc|c}1&0&1&2&4\\0&1&2&-3&-2\\0&0&0&1&0\end{array}\right)$$

to zeros

Step ⑥ Work up ward. beginning with the last non-zero row, we now operations to create zeros above the leading one

to zeros

$$\left(\begin{array}{cccc|c}1&0&1&2&4\\0&1&2&-3&-2\\0&0&0&1&0\end{array}\right)\xrightarrow[R_2+3R_3]{R_1-2R_3}\left(\begin{array}{cccc|c}1&0&1&0&4\\0&1&2&0&-2\\0&0&0&1&0\end{array}\right)$$ RREF

Step ⑦ Repeat step⑥ until we get RREF

Theorem: Gaussian elimination will transform a matrix A to its RREF

Theorem $A\xrightarrow{\text{sec of row operations}}R_1$ in RREF

$A\xrightarrow[\text{row operation}]{\text{another sec of}}R_2$ in RREF

is $R_1=R_2$? Yes!

What's RREF for? ① Solve $Ax=b$

② Study some properties of RREF

Recall $Ax=b$ $\quad C_{\text{invertible}}: CAX = Cb$

Step ① Write the argumented matrix $(A|b)$

Step ② Use elementary row operation to find the RREF of $(A|b) \longrightarrow (A'|B')_{RREF}$

Write the system of equation again : $A'x = b'$

Step③ Assign Parametric Values $t_1, t_2, \cdots,$ to the variables that $\underset{\text{does not}}{\wedge}$ correspond to the leading ones : $\underset{\text{free Variables}}{x_3 = t_3}$

For example solve $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ -1 & -4 & -9 & 5 \end{pmatrix} X = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix}$

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 4 \\ 2 & 1 & 4 & 1 & 6 \\ -1 & -4 & -9 & -5 & 4 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} 1^{st} & 2^{nd} & & 4^{th} \text{ colums} \end{array} \end{array}, \quad X \in R^4, \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{aligned} x_1 \quad + x_3 \quad &= 4 \\ x_2 + 2x_3 \quad &= -2 \\ x_4 &= 0 \end{aligned} \quad \overset{\text{Step(3)}}{\Longrightarrow} \quad \begin{aligned} x_1 &= 4 - x_3 = 4-t \\ x_2 &= -2-2t \\ x_3 &= t \\ x_4 &= 0 \end{aligned}$$

all solution $Ax=b$ is $\begin{pmatrix} 4-t \\ -2-2t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

Def'n Let R be the RREF of A to the system $Ax=b$

If the j^{th} column of R does not contain the leading one than x_j is called a free variable

$\#$ free variables $= n - \#$ of leading ones of R

$= n - \#$ of non-zero row of R

observe that the non-zero rows of R are linearly indep.

$\#$ free variables $= n - \dim \text{Row}(R)$

$= n - \operatorname{rank} R$

$= n - \operatorname{rank} A \qquad A \xrightarrow[\text{operation}]{\text{elementary row}} R$

For example, $\operatorname{rank} A = 3 = \#$ of non-zero rows of R. Column 3 of R does not contain the leading one

x_3 free variable

Recall step2/3: assign parametric variables for free variables $x_3 = t \in R$

Theorem If $Ax=b$ is consistent and if the general solution is of the form $X = X_0 + t_1u_1 + \cdots + t_ku_k$ where $t_1, \cdots, t_k \in \mathbb{F}$, $X_0, u_1, \cdots, u_k \in \mathbb{F}^n$

Then ① X_0 is solution of $Ax=b$ $(t_1 = t_2 = \cdots t_k = 0)$ Note: $t_1, \cdots, t_k$ are all free variable of $Ax=b$, $k = n - \text{rank}A$

② $k = n - \text{rank}A$, $\text{rank}A = r$

③ $\{u_1, \cdots, u_{k-r}\}$ is a basis for $AX=0$

③ $K = \{X_i : AX=b\}$, where $K_H = \{X : AX=0\}$

$= \{X_0\} + K_H$

$K_H = \{-X_0\} + K$

$= \{t_1u_1 + \cdots + t_{n-r}u_{n-r} \mid t_1, \cdots t_{n-r} \in \mathbb{F}\}$

$= \text{span}\{u_1, \cdots, u_{n-r}\}$

Since, $\dim K_H = n - \text{rank}A = n - r$, so $\{u_1, \cdots, u_{n-r}\}$ is a basis for K_H

Given $A_{m\times n}$, its RREF R ① $\text{Rank}A \times \text{Rank}R$?

② $\text{Col}(A) \times \text{Col}(R)$ → assignment ?

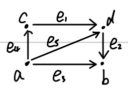
③ $\text{Row}(A) \times \text{Row}(R)$

3.9 Tutorial (Optional material)

Free Vector Spaces and Presentation of VSs

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\dfrac{x = -y + z}{W}$

$\mathbb{F}^3 / W$



$f_4: a \longmapsto c$

E = edge set

we want an as-free-as-possible V.S. V spanned by E and satisfying: signed cycles add to 0

e.g. $e_5 + e_2 - e_3 = 0$

$$(*) \begin{cases} e_1 + e_2 - e_3 + e_4 = 0 & k_1 \\ e_1 - e_5 + e_4 = 0 & k_2 \\ e_2 - e_3 + e_5 = 0 & k_3 \end{cases}$$

Goal: V over $\mathbb{F}$ spanned by E satisfying $(*)$

step① take V_0 = free V.S. generated by E over $\mathbb{F}$

step② $W := \text{span}\{k_1, k_2, k_3\}$, $W = \text{span}\{$signed cycles of graph$\}$

step③ $V := V_0 / W$

step④ Let $e_i' = e_i + W$

free vector spaces over $\mathbb{F}$, generate by B

Theorem: Fix $\mathbb{F}$, field. For everyone set B, $\exists$ a V.S. V over $\mathbb{F}$ with B as its basis

Prof: aside: 2 ways to think about free construct. ① Formal sums: $\sum a_i b_i \in V$, $b_i \in B$

e.g. $B = \{x, y, z\}$

$(\alpha_1 x + \alpha_3 z) + (\beta_1 x + \beta_2 y) = (\alpha_1 + \beta_1)x + \beta_2 y + \alpha_3 z$

② $\underbrace{\text{function from } B \text{ to } \mathbb{F}}_{\mathbb{F}^B}$ For each $b \in B$, fix $e_b \in \mathbb{F}^B$

$e_b : B \to \mathbb{F}$

$x \longmapsto \begin{cases} 1 & \text{if } x = b \\ 0, & \text{else} \end{cases}$

Let $V = \text{span}\{e_b : b \in B\}$

If $B' = \{e_b : b \in B\}$

then B' is a basis for V

$B \cong B'$ as sets

3.11

Determinants of $A_{n\times n}$

Def'n: $n=1$, $\det(a_{11}) = a_{11}$ $\qquad$ $\det: M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

$n=2$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$n>2$, $\det(A) = \sum_{J=1}^{n} (-1)^{J+1} \cdot a_{J1} \cdot \det(\tilde{A}_{J1})$, Cofactor - expansion along the first column

Where $\tilde{A}_{J1}$ is $(n-1)\times(n-1)$ matrix that is obtained from A by deleting row J, column 1

The scalar $(-1)^{J+1} \det(\tilde{A}_{ji})$ is called cofactor of entry of A in row j, column i

Ex: Denote $\det(A)$, $|A|$ determinant of A

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= (1\cdot 5\cdot 9 - 1\cdot 8\cdot 6) - (4\cdot 2\cdot 9 - 4\cdot 3\cdot 8) + (7\cdot 2\cdot 6 - 7\cdot 5\cdot 3)$$

$$= 0$$

Goal: $A \xrightarrow[\text{operations}]{\text{elementary row}}$ Simpler form Compute determinant easier

type 1, 2, 3 → how det is changed?

$A \xrightarrow{\text{type 1}} B$ $\qquad \det B = -\det A$

$\xrightarrow{cR_i \to R_i} B$ $\qquad \det B = c \det A$

$\xrightarrow{R_i + cR_j \to R_i} B$ $\qquad \det B = \det A$

Exercise

$$A = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2^2 & 3^2 & 4^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{n-1} & 3^{n-1} & 4^{n-1} & & n^{n-1} \end{vmatrix}$$

Ex. 2 $\det(O_{n\times n}) = 0$

$$\det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} = +1 \cdot \det I_{n-1} = \cdots = \det I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\det(I_n)$

Theorem $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ A is invertiable **iff** $\det A \neq 0$

If A is inverriable, then $A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$

$\Rightarrow$ If A is invertiable, we need to show $\det A \neq 0$, $\det A = a_{11}a_{22} - a_{12}a_{21}$

$\Downarrow$

$\text{rank}(A) = 2$: $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \neq \vec{0}$, $a_{11} \neq 0$ or $a_{21} \neq 0$

Case #1 $a_{11} \neq 0$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{R_2 - \frac{a_{21}}{a_{11}} R_1} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \frac{\det A}{a_{11}} \end{bmatrix}$ Rank=2, So $\det A \neq 0$

Case #2 $a_{21} \neq 0$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{R_1 - \frac{a_{11}}{a_{21}} R_2} \begin{bmatrix} 0 & -\frac{\det A}{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}$ Rank=2, So $\det A \neq 0$

$\Leftarrow$ Given $\det A \neq 0$, We need to prove A is invertiable

$\det A = a_{11}a_{22} - a_{12}a_{21}$

Let $B = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$, We will verify $AB = I_2$, $AB = \frac{1}{\det A} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

$= \frac{1}{\det A} \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= I_2$

Ex. $\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22} \cdots a_{nn}$

Proof: If $A_{n \times n}$ upper triangular matrix then $\det A = \prod_{k=1}^{n} a_{kk}$

$n=1$, Suppose the statement holds for any $(n-1) \times (n-1)$ upper triangle matrix

For : n $\det(A) = a_{11} \det \tilde{A}_{11} + 0 \det \tilde{A}_{21} + \cdots + 0 \det \tilde{A}_{n} = a_{11} \cdot (a_{22} \cdots a_{nn}) = a_{11}a_{22} \cdots a_{nn}$

Note $\tilde{A}_{11}$ $(n-1) \times (n-1)$ upper triangular matrix

Theorem $A_{n\times n}$ $\det(A)$ is a linear function of each row where remaining row are fixed

$$\det\begin{pmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - b_k + d c_k - \\ - a_n - \end{pmatrix} = \det\begin{pmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - b_k - \\ - a_n - \end{pmatrix} + d \cdot \det\begin{pmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - c_k - \\ \vdots \\ - a_n - \end{pmatrix}$$

$F(b_k + d \cdot c_k) = F(b_k) + \alpha F(c_k)$ $\quad 1 \le k \le n$

Proof by induction on n

$\underline{n=1}$ $\det(b_1 + \alpha c_1) = \det b_1 + \alpha \det c_1$

Assume the statement is true for any $(n-1)\times(n-1)$ matrix

Now, let $A_{n\times n}$ $1 \le k \le n$, we need to show $\det A = \det B + \alpha \det C$

$$\det(A) = \begin{pmatrix} - a_1 - \\ - b_k + \alpha c_k - \\ - a_n - \end{pmatrix} = \sum_{j=1}^{n} (-1)^{j+1} a_{j1} \det \tilde{A}_{j1}$$

$a_{k1} = b_{k1} + \alpha c_{k1}$

$$= \sum_{j\neq k} (-1)^{j+1} a_{j1} \det \tilde{A}_{j1} + (-1)^{k+1} a_{k1} \det \tilde{A}_{k1}$$

$\tilde{A}_{k1} = \tilde{B}_{k1} = \tilde{C}_{k1}$

$$= \sim\sim + (-1)^{k+1} (b_{k+1} + \alpha c_{k+1}) \det \tilde{A}_{k1}$$

$$\det A = \sim + (-1)^{k+1} b_{k+1} \det \tilde{B}_{k} + (-1)^{k+1} \alpha c_{k+1} \det \tilde{C}_{k1}$$

3.13

Theorem $A_{n\times n}$ $\det(A)$ is a linear function of each row where remaining row are fixed

$$\det\begin{pmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — b_k + \alpha c_k — \\ — a_n — \end{pmatrix} = \det\begin{pmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — b_k — \\ — a_n — \end{pmatrix} + \alpha \cdot \det\begin{pmatrix} — a_1 — \\ — a_2 — \\ — c_k — \\ \vdots \\ — a_n — \end{pmatrix}$$

$F(b_k + \alpha \cdot c_k) = F(b_k) + \alpha F(c_k)$ $\quad 1 \le k \le n$

Proof $\det(A) \overset{\text{by def}}{=} \sum_{j=1}^{n} (-1)^{j+1} a_{j1} \det \tilde{A}_{j1}$ from A by deleting row j column 1

Show $\det A = \det B + \alpha \det C$ Compare a_{j1} with b_{j1}, c_{j1}

$\tilde{A}_{j1}$ with $\tilde{B}_{j1}, \tilde{C}_{j1}$, $1 \le k \le n$, k fixed

$1 \le j \le k-1$ $\quad a_{j1} = b_{j1} = c_{j1}$

$j = k$ $\quad \tilde{A}_{j1}$ $\tilde{B}_{j1}$ $\tilde{C}_{j1}$

, $k+1 \le j \le n$ $\quad (n\times 1)\times(n-1)$

all row are same except for one row

By induction hypothesis $\det \tilde{A}_{j1} = \det \tilde{B}_{j1} + \alpha \cdot \det \tilde{C}_{j1}$

Similarly $k+1 \le j \le n$ $\quad a_{j1} = b_{j1} = c_{j1}$, $\det \tilde{A}_{j1} = \det \tilde{B}_{j1} + \alpha \det \tilde{C}_{j1}$

$j = k$, $a_{k1} = b_{k1} + \alpha c_{k1} \Rightarrow \tilde{A}_{k1} = \tilde{B}_{k1} = \tilde{C}_{k1}$, so $\det A = \det B + \alpha \det C$

Notation Denote now $a_1, \cdots, a_n$ are rows of a matrix, rewrite the theorem.

$\det(a_1, \cdots, a_{k-1}, b_k + \alpha c_k, a_{k+1}, \cdots, a_n) = \det(a_1, \cdots, a_{k-1}, b_k, a_{k+1}, \cdots, a_n) + \alpha \det(a_1, \cdots, a_{k-1}, c_k, a_{k+1}, \cdots, a_n)$

det: a function of row vectors multilinear function.

Note: det is not a linear function of an entry while the other entries are fixed

$\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 3 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$

Corollary $A_{n\times n}$ rows $a_1, \cdots, a_n$ one of the rows of A is 0, $\det(a_1, \cdots, 0, \cdots, a_n) = 0$

Proof $\det(a_1, \cdots, 0+0, \cdots, a_n) = \det(a_1, \cdots, 0, \cdots, a_n) + \det(a_1, \cdots, a_n)$

$\det(a_1, \cdots, 0, \cdots, a_n) = 2\det(a_1, \cdots, 0, \cdots, a_n)$ $\quad$ So $\det(a_1, \cdots, 0, \cdots, a_n) = 0$

Type ② $A_{n\times n} \xrightarrow{R_i \leftrightarrow cR_i} B$ Then $\det B = c\cdot\det A$

Proof $B = \begin{bmatrix} — a_1 — \\ — ca_i — \\ — a_n — \end{bmatrix}$

$\det B = \det(a_1, \cdots, ca_i, \cdots, a_n) = \det(a_1, \cdots, 0, a_n) + c\det(a_1, \cdots a_i, \cdots, a_n)$

$= 0 + c\cdot\det A$

$= \det A$ □

Question $\det(cA) = c^n \det A$

Type ③ $A_{n\times n} \xrightarrow{R_i \leftrightarrow R_j} B$ claim $\det B = -\det A$

Step 1 $R_i \longleftrightarrow R_{i+1}$

Lemma If a matrix $A_{n\times n}$ has two identical adjacent rows then $\det A = 0$

$A = \begin{bmatrix} — a_1 — \\ \vdots \\ — a_i — \\ — a_{i+1} = a_i — \\ \vdots \\ — a_n — \end{bmatrix}$

Proof by induction $\det A = \sum_{J=1}^{n} (-1)^{J+1} A_{J1} \det \hat{A}_{J1}$

Case 1 $J \in \{1, I+1\}$

Case 2 $j = I$ and $J = i+1$

Case 1 : $J \notin \{i, I+1\}$ $\hat{A}_{J1}$ has two equal rows by induction hypothesis $\det \hat{A}_{J1} = 0$

$\det A = \underbrace{(-1)^{j+1} A_{i1} \det \hat{A}_{j1} + (-1)^{j+2} A_{j+1}}_{\text{opposite sign}} , \det \hat{A}_{i+1,1} = 0$

$A_{i1} = A_{i+1\,1}$

$\hat{A}_{i1} = \hat{A}_{i+1,1}$

Lemma, $A \xrightarrow{R_i \leftrightarrow R_{i+1}} B$, then $\det B = \det A$

Proof: Denote $a_1, \cdots, a_n$ rows of A

$\det B = \det(a_1, \cdots, a_{i+1}, a_i, \cdots a_n)$

$\det A = \det(a_1, \cdots, a_i, a_{i+1} \cdots a_n)$

Then, $0 \overset{\text{lemma}}{=\!=} \det(a_1, \cdots, \overset{\text{row } i}{a_i + a_{i+1}}, \cdots, a_i + a_{i+1}, \cdots a_n)$

$\overset{\text{linearity}}{=\!=} \det(a_1, \cdots, a_i, a_i, \cdots, a_n) = 0$

$+ \det(a_1, \cdots, a_i, a_{i+1}, \cdots a_n) = \det B$

$+ \det(a_1, \cdots, a_{i+1}, a_i, \cdots, a_n) = \det A$

$+ \det(a_1, \cdots, a_{i+1}, a_{i+1}, \cdots a_n) = 0$

$\Longrightarrow 0 = \det B + \det A \Longrightarrow \det B = -\det A$

Lemma If A has two identical row, $\det A = 0$

Proof $A = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{bmatrix}$ $\quad a_i = a_j \quad$ Ex. $A_{5\times5}$ $\begin{bmatrix} a_1 \\ a_2 \\ a_1 \\ a_4 \\ a_5 \end{bmatrix} \xrightarrow{a_4 \leftrightarrow a_5} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_5 \\ a_4 \end{bmatrix} \xrightarrow{a_5 \leftrightarrow a_3} \begin{bmatrix} a_1 \\ a_2 \\ a_5 \\ a_3 \\ a_4 \end{bmatrix}$

By switching adjacent rows we can move a_j next to a_i $\quad \det A = -\det A + \det A = 0$

Mar 23, Lecture 28: Determinants (cont'd)

Theorem 4.5 (Determinant after a type 3 elementary row operation)**.** *Let* $A \in M_{n\times n}(\mathbb{F})$ *and* $A \xrightarrow{R_i \leftarrow R_i + cR_j} B$. *Then* $\det(B) = \det(A)$.

Proof. Suppose $a_1, \ldots, a_n$ are rows of A. We first prove for the case $i < j$. We have

$$\begin{aligned}\det(B) &= \det(a_1, \ldots, a_i + ca_j, \ldots, a_j, \ldots, a_n)\\ &= \det(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) + c\det(a_1, \ldots, a_j, \ldots, a_j, \ldots, a_n)\\ &= \det(A) + 0\\ &= \det(A).\end{aligned}$$

Similarly, we have the same result for the case $j < i$. □

In summary, let A be a square matrix and B be a matrix obtained from A by an elementary row operation. Then

- For $A \xrightarrow{R_i \leftrightarrow R_j} B$, we have $\det(B) = -\det(A)$.
- For $A \xrightarrow{R_i \leftarrow cR_i} B$, we have $\det(B) = c\det(A)$.
- For $A \xrightarrow{R_i \leftarrow R_i + cR_j}$, we have $\det(B) = \det(A)$.

Summary of some properties of determinants that we have studied so far.
Let A be an $n \times n$ matrix. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{j1} \cdot \det(\tilde{A}_{j1}),$$

where $\tilde{A}_{ij}$ the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j. The determinant of a square matrix satisfies the following properties:

1. As a function of each row, the determinant is a linear function of each row, when the remaining rows are held fixed.
2. If two adjacent rows are equal, then $\det(A) = 0$.
3. $\det(I_n) = 1$.

Note that, one can prove that a function $F : \mathbb{F}^n \times \cdots \mathbb{F}^n \to \mathbb{F}$ that satisfies those three properties is unique. (Section 4.5 in Friedberg et al.'s book).

Now, we will discuss more properties of determinants. Let $A, B \in M_{n\times n}(\mathbb{F})$. We will prove the following properties:

det(BA) $\qquad$ tr(AB) = tr(BA)

|| $\qquad$ but tr(AB) ≠ tr(A) tr(B)

1. $\det(AB) = \det(A)\det(B)$.

2. $\det(A) = \det(A^T)$.

3. $\det(A)$ can be computed using the expansion along any column or row of A.

Corollary 4.5.1. Let E be an elementary matrix obtaining from I_n by an elementary row operation. Then

det(EA) = det E det A

$A \longrightarrow EA = B$ elementary row operation

$I_n \longrightarrow E$

1. For a type 1 elementary row operation, $\det(E) = -1$.

2. For a type 2 elementary row operation with scalar $c \neq 0$, $\det(E) = c$.

3. For a type 3 elementary row operation, $\det(E) = 1$.

4. $\det(E^T) = \det(E)$.

In all cases, $\det(E) \neq 0$.

Proof. Since $I_n \xrightarrow{\text{an elementary row operation}} E$ and $\det(I_n) = 1$, we have the conclusion. Note that E^T is an elementary matrix of the same type as E, so

$$\det(E^T) = \det(E).$$

Check $A \xrightarrow{\text{row operation}} EA = B$ $\square$

Theorem 4.6. *Let E be an $n \times n$ elementary matrix and $A \in M_{n\times n}(\mathbb{F})$. Then* $\det(EA) = \det(E)\det(A)$.

Proof. We verify the equality for all types of elementary row operations.

- Type 1: $A \xrightarrow{R_i \leftrightarrow R_j} EA$. Then $I_n \xrightarrow{R_i \leftrightarrow R_j} E$. By Corollary 4.5.1, $\det(E) = -1$. By Theorem 4.4, $\det(EA) = -\det(A) = \det(E)\det(A)$.
- Type 2: $A \xrightarrow{R_i \leftarrow cR_i} EA$. Then $I_n \xrightarrow{R_i \leftarrow cR_i} E$. By Corollary 4.5.1, $\det(E) = c$. By Theorem 4.5, $\det(EA) = c\det(A) = \det(E)\det(A)$.
- Type 3: $A \xrightarrow{R_i \leftarrow cR_i} EA$. Then $I_n \xrightarrow{R_i \leftarrow R_i + cR_j} E$. By Corollary 4.5.1, $\det(E) = 1$. By Theorem 4.5, $\det(EA) = \det(A) = \det(E)\det(A)$.

$\square$

Corollary 4.6.1. Let $A \in M_{n\times n}(\mathbb{F})$ and $E_1, \ldots, E_k$ be elementary matrices. Then

1. $\det(E_1 \ldots E_k A) = \det(E_1) \ldots \det(E_k)\det(A)$. $\qquad$ det(EA) = det E det A

k times

2. $\det(E_1 \ldots E_k) = \det(E_1) \ldots \det(E_k)$. $\qquad$ $A = I_n$, det $I_n = 1$

Proof Sketch. Use Theorem 4.6 for Part 1. Part 2 is from Part 1 for $A = I_n$. $\square$

Theorem 4.7 (Invertible Matrix Theorem (part 5))**.** *Let* $A \in M_{n\times n}(\mathbb{F})$*. Then* A *is invertible if and only if* $\det(A) \neq 0$*.*

Proof. ($\Rightarrow$) Suppose A is invertible, then $A = E_1 \dots E_q$, where $E_1, \dots, E_q$ are elementary matrices. Using Theorem 4.6 and Corollary 4.5.1, we have

$$\det(A) = \det(E_1)\dots\det(E_q) \neq 0.$$

($\Leftarrow$) Given $\det(A) \neq 0$. Suppose A is not invertible.
Let R be the RREF of A. Since A is not invertible, $\operatorname{rank}(A) < n$. Since $\operatorname{rank}(A) =$ the number of nonzero rows of R, we conclude that R has at least one zero rows. Therefore, $\det(R) = 0$.
On the other hand, since R is the RREF of A, there exist elementary matrices $E_1, \dots, E_p$ such that

$$A = E_1 \dots E_p R.$$

So $\det(A) = \det(E_1)\dots\det(E_p)\det(R) = 0$, a contradiction. Therefore, the assumption is wrong and A is invertible. □

Corollary 4.7.1. Let $A \in M_{n\times n}(\mathbb{F})$. If $\operatorname{rank}(A) < n$, then $\det(A) = 0$.

Proof. If $\operatorname{rank}(A) < n$, A is not invertible. So $\det(A) = 0$. □

Theorem 4.8. *Let* $A, B \in M_{n\times n}(\mathbb{F})$*. Then* $\det(AB) = \det(A)\det(B)$*.*

Proof. Case 1: A is invertible. Then $A = E_1 \dots E_q$. Using Corollary 4.6.1, we have

$$\det(AB) = \det(E_1 \dots E_q B) = \det(E_1)\dots\det(E_q)\det(B) = \det(E_1 \dots E_q)\det(B) = \det(A)\det(B).$$

Case 2: A is not invertible. Then AB is not invertible. By the Invertible Matrix Theorem, $\det(A) = 0$ and $\det(AB) = 0$. So $\det(AB) = \det(A)\det(B)$. □

Theorem 4.9. *Let* $A \in M_{n\times n}(\mathbb{F})$*. Then* $\det(A) = \det(A^T)$*.*

Proof. Case 1: A is not invertible. Then $\operatorname{rank}(A) = \operatorname{rank}(A^T) < n$. Then by Corollary 4.7.1., we have $\det(A) = \det(A^T) = 0$.
Case 2: A is invertible. Then there exist elementary matrices $E_1, \dots, E_k$ such that $A = E_1 \dots E_k$. Then

$$\begin{aligned}
\det(A^T) &= \det(E_k^T \dots E_1^T) \\
&= \det(E_k^T)\dots\det(E_1^T) \\
&= \det(E_k)\dots\det(E_1) \\
&= \det(E_1)\dots\det(E_k) \\
&= \det(E_1 \dots E_k) \\
&= \det(A).
\end{aligned}$$

□

April 1st

Definition 46. *A polynomial* $f(t) \in \mathbb{F}[t]$ *splits over* $\mathbb{F}$ *if there are scalars* $c, a_1, \dots, a_n$ *(not necessarily distinct) in* $\mathbb{F}$ *such that*

$$f(t) = c(t - a_1)(t - a_2)\cdots(t - a_n).$$

Example 38. Consider $f(t) = t^3 + t^2 + t + 1 = (t^2 + 1)(t + 1)$.

- $f(t)$ does not split over $\mathbb{R}$, $\mathbb{Q}$.
- $f(t)$ splits over $\mathbb{C}$: $f(t) = (t + i)(t - i)(t + 1)$.
- $f(t)$ splits over $\mathbb{F}_2$: $f(t) = (t + 1)(t + 1)(t + 1)$.

Theorem 5.9. *The characteristic polynomial of any diagonalizable linear operator splits.*

Proof. Let T be a diagonalizable linear operator on an n-dimensional vector space V. Then there exists an ordered basis β of V so that $[T]_\beta$ is a diagonal matrix D. Then the characteristic polynomial of T is

$$p_D(t) = (\lambda_1 - t)\dots(\lambda_n - t) = (-1)^n(t - \lambda_1)\dots(t - \lambda_n),$$

which splits over $\mathbb{F}$. □

Definition 47. *Let* λ *be an eigenvalue of a linear operator* T *on an* n*-dimensional vector space or a matrix* $A \in M_{n\times n}(\mathbb{F})$ *with characteristic polynomial* $p(t)$.

- *The (algebraic) multiplicity of* λ *is the largest positive integer* k *for which* $(t - \lambda)^k$ *is a factor of* $p(t)$.
- *The geometric multiplicity of* λ *is* $\dim(E_\lambda)$, *the dimension of the eigenspace*

$$E_\lambda = \{v \in \mathbb{F}^n : T(v) = \lambda v\}.$$

- *Note that* $E_\lambda = Null(T - \lambda I)$ *and* E_λ *has at least one nonzero eigenvector. Therefore,*

$$1 \le \dim(E_\lambda) \le n.$$

Theorem 5.10. *Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T having algebraic multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.*

Proof. Suppose $\dim(E_\lambda) = k$. Choose an ordered basis $\{v_1, \ldots, v_k\}$ for E_λ and extend it to an ordered basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Let $A = [T]_\beta$. Since $T(v_j) = \lambda v_j$ for $1 \leq j \leq k$, we have

$$A = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix}.$$

From Assignment 5, we have

$$\begin{aligned} p_A(t) = \det(A - tI_n) &= \det \begin{pmatrix} (\lambda - t)I_k & B \\ 0 & C - tI_{n-k} \end{pmatrix} \\ &= \det((\lambda - t)I_k) \det(C - tI_{n-k}) = (\lambda - t)^k \det(C - tI_{n-k}). \end{aligned}$$

So $(\lambda - t)^k$ is a factor of $p_A(t)$, therefore, $k \leq m$, $\square$

Theorem 5.11. *Let T be a linear operator on a vector space V and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $j = 1, 2, \ldots, k$, let S_j be a finite linearly independent subset of the eigenspace E_{λ_j}. Then*

$$S = S_1 \cup S_2 \cup \cdots S_k$$

is a linearly independent subset of V.

Proof Sketch.

- Suppose $S_i = \{v_{i,1}, \ldots, v_{i,n_i}\} \subset E_{\lambda_i}$, for $1 \leq i \leq k$. Then
$$S = \{v_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq n_i\}.$$
- Consider
$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{i,j} = 0.$$
Let $w_i = \sum\limits_{j=1}^{n_i} a_{ij} v_{i,j} \in E_{\lambda_i}$. Then
$$w_1 + \cdots + w_k = 0.$$
- Claim: $w_1 = \cdots = w_k = 0$.
Suppose otherwise. By renumbering if necessary, suppose that, for $1 \leq m \leq k$, $w_i \neq 0$ for all $1 \leq i \leq m$ and $w_i = 0$ for $m < i \leq k$. The above equation becomes $w_1 + \cdots + w_m = 0$, which implies $\{w_1, \ldots, w_m\}$ is linearly dependent. On the other hand, since $w_i \in E_{\lambda_i}$ and $\lambda_i's$ are distinct, $\{w_1, \ldots, w_m\}$ is linearly independent, a contradiction. Therefore $w_1 = \cdots = w_k = 0$.
- For each i, $0 = w_i = \sum\limits_{j=1}^{n_i} a_{ij} v_{i,j}$. Since $\{v_{i,j} \mid 1 \leq j \leq n_i\}$ is linearly independent, we have $a_{ij} = 0$, for all $1 \leq j \leq n_i$.
- In conclusion, $a_{i,j} = 0$ for all $1 \leq i \leq k, 1 \leq j \leq n_i$. Therefore, S is linearly independent.

□

Theorem 5.12. *Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \ldots, \lambda_k$ be all distinct eigenvalues of T. Then*

1. *T is diagonalizable iff the algebraic multiplicity of λ_j is equal to $\dim(E_{\lambda_j})$ for all $1 \leq j \leq k$.*

2. *If T is diagonalizable and β_j is an ordered basis for E_{λ_j}, for each j, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*

Proof. Let $p(t)$ be the characteristic polynomial of T. By the assumption,

$$p(t) = c(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\ldots(t-\lambda_k)^{m_k}.$$

So $\sum\limits_{i=1}^{k} m_i = \deg(p) = n$.

1.($\Leftarrow$) Suppose $m_i = \dim(E_{\lambda_i})$ for all $1 \leq i \leq k$.

- Let S_i be a basis for E_{λ_i}, then S_i has a_i elements which are eigenvectors of T corresponding to eigenvalue λ_i.

- By Theorem 5.11, the set $S = \bigcup\limits_{i=1}^{k} S_i$ is linearly independent and S has $\sum\limits_{i=1}^{k} m_i = n$ elements. So S is a basis for V of eigenvectors of T.

1. ($\Rightarrow$) Suppose T is diagonalizable and β is an ordered basis for V of eigenvectors of T.

- Let $S_i = \beta \cap E_{\lambda_i}$. Then S_i is a linearly independent subset of E_{λ_i}. Therefore,

$$n_i = |S_i| \leq \dim(E_{\lambda_i}) = d_i \leq m_i.$$

- Since β is a set of eigenvector, we have

$$\beta = \beta \cap \bigcup_{i=1}^{k} E_{\lambda_i} = \bigcup_{i=1}^{k} (\beta \cap E_{\lambda_i}).$$

  Also, $(\beta \cap E_{\lambda_i}) \cap (\beta \cap E_{\lambda_j}) = \emptyset$ for $i \neq j$. Thus,

$$n = \sum_{i=1}^{k} n_i.$$

- We have

$$n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n.$$

- So $n_i = d_i = m_i$.

$\square$

Remark 18. The above proof provides a procedure to check whethere a square matrix is diagonalizable or not. If yes, it provides the factorization of A as PDP^{-1} where D is a diagonal matrix.

- Find all eigenvalues of A. Suppose $\lambda_1, \ldots, \lambda_k$ are all distinct eigenvalues of A and a_j is the algebraic multiplicity of λ_j, for $1 \leq j \leq k$.
- Find a basis for each eigenspace E_{λ_j}, for $1 \leq j \leq k$.
- If there exists $1 \leq j \leq k$ such that $\dim E_{\lambda_j} \neq a_j$, the matrix A is not diagonalizable.
- If $a_j = \dim E_{\lambda_j}$ for all $1 \leq j \leq k$, the matrix A is diagonalizable. Let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$, where β_j is an ordered basis for E_{λ_j}, for $1 \leq j \leq k$. Let P be a square matrix whose columns are vectors from β and let D be a diagonal matrix whose diagonal entries are eigenvalues of A corresponding to the column of P. Then

$$A = PDP^{-1}.$$

Example 39. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is diagonalizable and

$$A = [v_1 \; v_2] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} [v_1 \; v_2]^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \right)^{-1}.$$

Example 40. Check whether $A = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 1 \end{bmatrix}$ is diagonalizable. If yes, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Also, find A^{100}.

Theorem 4.10. *The determinant of A can be evaluated by cofactor expansion along any column. That is, for any $1 \leq j \leq n$, we have*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Proof. □

$\det(Col_1, Col_2, \cdots, Col_j, \cdots, Col_n)$

$= (-1)^{j-1} \det(Col_j, Col_1, Col_2, \cdots, Col_n)$

$\det(Col_1, Col_2, Col_3) \longrightarrow \det(Col_1, Col_3, Col_2) \longrightarrow \underline{\det(Col_3, Col_1, Col_2)}$

Example 32. Compute the determinant of a 4×4 matrix using elementary row operations.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\det(A) = 2 \det \begin{pmatrix} 3 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$= 2 \cdot \left(3 \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \det \begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix}\right)$$

$$= 2 \cdot (3 \cdot 5 + 4)$$

$$= 2 \cdot 19$$

$$= 38$$

det(BA)

||

$tr(AB) = tr(BA)$

but $tr(AB) \neq tr(A)\,tr(B)$

1. $\det(AB) = \det(A)\det(B)$.
2. $\det(A) = \det(A^T)$.
3. $\det(A)$ can be computed using the expansion along any column or row of A.

elementary

Main Techniques: ① $\det(EA) = \det E \det A$

MATH 146 March 23–27: Permutations and Determinants Tutorial

- Recall: a *permutation* of $\{1, 2, \ldots, n\}$ is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
- $S_n :=$ the set of all permutations of $\{1, 2, \ldots, n\}$.
- Recall: given $\sigma \in S_n$,
  - $\xrightarrow{R:\sigma}$ denotes the operation on $M_{n\times k}(\mathbb{F})$ (any $k, \mathbb{F}$) which moves row i to row $\sigma(i)$ $\forall i$.
  - P_σ is the $n \times n$ matrix such that $I_n \xrightarrow{R:\sigma} P_\sigma$.
- Recall: $P_\sigma P_\tau = P_{\sigma\tau}$ $\forall \sigma, \tau \in S_n$, and $(P_\sigma)^{-1} = P_{\sigma^{-1}} = (P_\sigma)^t$ $\forall \sigma \in S_n$.
- One can show that if $A \xrightarrow{R:\sigma} B$ then $B = P_\sigma A$.

Lemma 1. *$\forall \sigma \in S_n$, $\det(P_\sigma) \in \{1, -1\}$.*

Proof sketch. The operation $\xrightarrow{R:\sigma}$ can be simulated by a sequence of row-pair swaps (elementary row operations of type 1). Thus $I_n \xrightarrow{R:\sigma} P_\sigma$ implies

$$I_n \xrightarrow{R_{i_1} \leftrightarrows R_{j_1}} A_1 \xrightarrow{R_{i_2} \leftrightarrows R_{j_2}} A_2 \cdots \xrightarrow{R_{i_k} \leftrightarrows R_{j_k}} A_k = P_\sigma.$$

As each row-pair swap flips the sign of the determinant, $\det(P_\sigma) = (-1)^k$. □

Definition. Given $\sigma \in S_n$:

(1) The **sign** (or **signum**) of σ, denoted $\mathrm{sgn}(\sigma)$, is given by $\mathrm{sgn}(\sigma) := \det(P_\sigma)$.
(2) The **parity** of σ is 'even' or 'odd' according to whether $\mathrm{sgn}(\sigma)$ is 1 or -1.

There are a variety of equivalent ways to efficiently find the sign (or parity) of σ, including:

(1) Calculate $\det(P_\sigma)$.
(2) Factor σ into its cycles, say of lengths $\ell_1, \ell_2, \ldots, \ell_t$. As a cycle of length ℓ can be simulated by $\ell - 1$ row-pair swaps, $\mathrm{sgn}(\sigma) = (-1)^{(\ell_1-1)+\cdots+(\ell_t-1)}$.
(3) Calculate the **inversion number** $N(\sigma)$ of σ. This is defined as the number of pairs (i, j) with $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. It can be proved that $\mathrm{sgn}(\sigma) = (-1)^{N(\sigma)}$.

Example. Consider the permutation $\sigma \in S_8$ given by

x	1	2	3	4	5	6	7	8
$\sigma(x)$	4	8	6	3	5	1	2	7

(1) Using cycle notation,

$$\sigma = (1\ 4\ 3\ 6)(2\ 8\ 7)(5).$$

σ has cycles of length 4, 3 and 1, so $\mathrm{sgn}(\sigma) = (-1)^{3+2+0} = -1$.

(2) Listing all the inversions for σ,

$(1,4)$, $(1,6)$, $(1,7)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,7)$, $(2,8)$ $(3,4)$, $(3,5)$, $(3,6)$
$(3,7)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$.

$N(\sigma) = 17$ so $\mathrm{sgn}(\sigma) = (-1)^{17} = -1$.

We can now explain the *complete expansion* of $\det(-)$. Suppose A is an $n \times n$ matrix whose (i, j) entry is a_{ij}. The next theorem states that $\det(A)$ can be expressed as an alternating sum of products of entries of A, where each product contains one entry from each row and each column of A, and the sign of the product is given by the signum function.

Theorem (Complete expansion of det)**.** $\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$.

Proof. Write the first row of A as $\sum_{j=1}^{n} a_{1j}e_j$. By linearity of det in the first row,

$$\det(A) = a_{11}\det\begin{pmatrix} \text{---} & e_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_n & \text{---} \end{pmatrix} + a_{12}\det\begin{pmatrix} \text{---} & e_2 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_n & \text{---} \end{pmatrix} + \cdots + a_{1n}\det\begin{pmatrix} \text{---} & e_n & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_n & \text{---} \end{pmatrix}.$$

Repeating in the second row, we get an expression for $\det(A)$ involving n^2 terms. Eventually, we get an expression for $\det(A)$ involving n^n terms, looking like

$$\det(A) = \sum_{i_1,i_2,\ldots,i_n=1}^{n} a_{1i_1}a_{2i_2}\cdots a_{ni_n}\det\begin{pmatrix} \text{---} & e_{i_1} & \text{---} \\ \text{---} & e_{i_2} & \text{---} \\ & \vdots & \\ \text{---} & e_{i_n} & \text{---} \end{pmatrix}$$

Many terms equal 0; if $(i_1, i_2, \ldots, i_n)$ is not a permutation of $\{1, \ldots, n\}$, then the matrix with rows $e_{i_1}, \ldots, e_{i_n}$ has two equal rows, so its determinant is 0. Thus this simplifies to

$$\begin{aligned} \det(A) &= \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\det\begin{pmatrix} \text{---} & e_{\sigma(1)} & \text{---} \\ \text{---} & e_{\sigma(2)} & \text{---} \\ & \vdots & \\ \text{---} & e_{\sigma(n)} & \text{---} \end{pmatrix} \\ &= \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\det(P_{\sigma^{-1}}) \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma^{-1})a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} \end{aligned}$$

which simplifies to

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

since $\operatorname{sgn}(\sigma^{-1}) = \det(P_{\sigma^{-1}}) = \det((P_\sigma)^t) = \det(P_\sigma) = \operatorname{sgn}(\sigma)$. $\square$

Mar 25, Lecture 29: Summary–Important Facts about Determinants; Eigenvalues and Eigenvectors

Theorem 4.10. *The determinant of A can be evaluated by cofactor expansion along any column. That is, for any $1 \leq j \leq n$, we have*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Proof. Let $a_1, \ldots, a_n$ be columns of A, $A = [a_1 \ldots a_j \ldots a_n]$. Denote

$$B = [a_j \ a_1 \ a_2 \ldots a_{j-1} \ a_{j+1} \ldots a_n].$$

Observe that $\tilde{A}_{ij} = \tilde{B}_{i1}$ and $A_{ij} = B_{i1}$. Also, since A can be obtained from B by $(j-1)$ successive interchange of adjacent columns, $\det(A) = (-1)^{j-1} \det(B)$. We have

$$\det(A) = (-1)^{j-1} \sum_{i=1}^{n} (-1)^{i+1} B_{i1} \det(\tilde{B}_{i1}) = \sum_{i=1}^{n} (-1)^{j+i} A_{ij} \det(\tilde{A}_{ij}).$$

□

Summary–Important Facts About Determinants. Let $A \in M_{n \times n}(\mathbb{F})$. Then

- $n = 1 : \det(A) = A_{11}$.
- $n = 2 : \det(A) = A_{11}A_{22} - A_{12}A_{21}$.
- $n \geq 2$: Cofactor expansion along column j:

  $$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

  Cofactor expansion along row i:

  $$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

- If B is obtained from A by interchanging two rows or two columns of A, $\det(B) = -\det(A)$.
- If B is obtained from A by multiplying a row or a column of A by a scalar c, $\det(B) = c\det(A)$.
- If B is obtained from A by adding a multiple of row (column) i to row (column) j, for $i \neq j$, then $\det(B) = \det(A)$.

- If A has two equal rows (columns), then $\det(A) = 0$.
- If A has a zero row (column), then $\det(A) = 0$.
- $\det(A) = \det(A^T)$.
- The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, $\det(I_n) = 1$.
- A is invertible if and only if $\det(A) \neq 0$. If A is invertible, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
- If $B \in M_{n\times n}(\mathbb{F})$, then $\det(AB) = \det(A)\det(B)$.

Example 32. Compute the determinant of a 4×4 matrix using elementary row operations.

$R_1 \leftarrow R_1 + R_2 + R_3 + R_4$

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

3 3 3 3 ones

$$\det(A_1) = \begin{vmatrix} 3 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \begin{matrix} \\ \leftarrow R_2 - R_1 \\ \leftarrow R_3 - R_1 \\ \leftarrow R_4 - R_1 \end{matrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= 3 \cdot (-1)^3$$

$$= -3$$

5 Diagonalization

5.1 Eigenvalues and Eigenvectors

Motivation: Reference from Introduction to Linear Algebra by Gilbert Strang. Consider $A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$. Suppose we would like to know A^{2020}. Here are some first powers of A:

$$A^2 = \begin{bmatrix} 0.84 & 0.48 \\ 0.16 & 0.52 \end{bmatrix}, A^3 = \begin{bmatrix} 0.804 & 0.588 \\ 0.196 & 0.412 \end{bmatrix}, \dots, A^{2020} = \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} + \text{very small vector.}$$

A^{2020} was found by using the eigenvalues of A, not by multiplying 2020 matrices.

Given $A \in M_{n\times n}(\mathbb{F})$. For almost all vector $v \in \mathbb{F}^n$, Av is not in the same direction as v, that is $Av \neq \lambda v$ for any $\lambda \in \mathbb{F}$. Certain exceptional vectors (excluding the zero vector) v are in the same direction as Av, which we call eigenvectors. The scalar $\lambda \in \mathbb{F}$ such that $Av = \lambda v$ for some $v \in \mathbb{F}^n - \{0\}$ is called an eigenvalue of A.

Definition 41. *Let $A \in M_{n\times n}(\mathbb{F})$.*

- *A nonzero vector $v \in \mathbb{F}^n$ is called an eigenvector of A if there exists a scalar $\lambda \in \mathbb{F}$ such that $Av = \lambda v$. Such λ is called the eigenvalue corresponding to the eigenvector v.*
- *If $\lambda \in \mathbb{F}$ is an eigenvalue of A, the set*

$$E_\lambda = \{v \in \mathbb{F}^n : Av = \lambda v\} \underset{\neq \emptyset}{=} \{\text{eigenvectors of } A \text{ corresponding to } \lambda\} \cup \{0\}$$

If $AV = \lambda V$

$(A - \lambda I_V)V = 0$

is called the eigenspace of A corresponding to λ. Note that

$$E_\lambda = Null(A - \lambda I_n).$$

$n \geq \dim E_\lambda \geq 0$

Theorem 5.1. *Let $A \in M_{n\times n}(\mathbb{F})$. Then a scalar $\overset{EF}{\lambda}$ is an eigenvalue of A if and only if $\det(\underbrace{A - \lambda I_n}) = 0$.* ← application

Proof. λ is an eigenvalue of $A \Leftrightarrow \exists v \in \mathbb{F}^n, v \neq 0$ such that $Av = \lambda v$.
$\Leftrightarrow \exists v \in \mathbb{F}^n, v \neq 0$ such that $(A - \lambda I_n)v = 0$.
$\Leftrightarrow (A - \lambda I_n)$ is not invertible.
$\Leftrightarrow \det(A - \lambda I_n) = 0$. □

If $A - \lambda I_n$ is invertiable
$(A - \lambda I_n) V = 0$
$(A - \lambda I_n)^{-1} (A - \lambda I_n) V = 0$
$V = 0$

Definition 42. *Let $A \in M_{n\times n}(\mathbb{F})$. The $n-$th degree polynomial $\det(A - tI_n)$ of variable t is called the* characteristic polynomial *of A, denoted by $p_A(t)$.*

$$p_A(t) = \det(A - tI_n) = \begin{vmatrix} a_{11} - t & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - t & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - t \end{vmatrix}.$$

Example 33. Find all eigenvalues of $A = \begin{bmatrix} -1 & 6 & 3 \\ 1 & 0 & 1 \\ -3 & 6 & 5 \end{bmatrix}$.

Proof. We have

$$\det(A - tI) = \begin{vmatrix} -1-t & 6 & 3 \\ 1 & -t & 1 \\ -3 & 6 & 5-t \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -t & -1 \\ 6 & 5-t \end{vmatrix} + (-1)6\begin{vmatrix} 1 & -1 \\ -3 & 5-t \end{vmatrix} + 3\begin{vmatrix} 1 & -t \\ -3 & 6 \end{vmatrix}$$
$$= -t^3 + 4t^2 - 4t = -t(t-2)^2.$$

So the eigenvalues of A is 0 and 2. □

Theorem 5.2 (Properties of Charateristic Polynomials)**.** *Let $A \in M_{n\times n}(\mathbb{F})$. Then*

1. *$p_A(t)$ is a polynomial in $\mathbb{F}[t]$ of degree n and its leading coefficient is $(-1)^n$.*
2. *The coefficient of t^{n-1} in $p_A(t)$ is $(-1)^{n-1} tr(A)$.*
3. *The constant coefficient is $\det(A)$.*
4. *A has at most n eigenvalues.*
5. *If $B \in M_{n\times n}(\mathbb{F})$ is similar to A, then $p_B(t) = p_A(t)$. (Definition: Let $A, B \in M_{n\times n}(\mathbb{F})$. The matrix B is said to be* similar *to A if there exists an invertible matrix P such that $B = P^{-1}AP$.)*

Fundamental Theorem of Algebra

a Polynomial of degree n has at most n solutions

$\det(A - \lambda I_n) = (-1)^n t^n + (-1)^{n-1} tr(A)\, t^{n-1} + \dots\dots + \det A$

Proof Sketch. (1-2). The characteristic polynomial can be computed by cofactor expansion along the first column

$$p_A(t) = (a_{11} - t)(a_{22} - t) \dots (a_{nn} - t) + \text{terms of degree ¡ n-1.}$$

The coefficients of t^n and t^{n-1} in $p_A(t)$ come entirely from $(a_{11} - t)(a_{22} - t) \dots (a_{nn} - t)$, which implies (1) and (2).
(3). $p_A(t) = \det(A - tI) = (-1)^n t^n + (-1)^{n-1} tr(A) + \dots + c_0$. Let $t = 0$, we have $c_0 = p_A(0) = \det(A - 0\ I) = \det(A)$.
(4). Since a polynomial of degree n has at most n roots, A has at most n eigenvalues.
(5). $\det(B - tI) = \det(P^{-1}(A - tI)P) = \det(P^{-1}) \det(A - tI) \det(P) = p_A(t) \det(P^{-1}) \det(P) = p_A(t) \det(P^{-1}P) = p_A(t)$. □

Mar 27, Lecture 30: Eigenvalues and Eigenvectors(cont'd)

Last time: compute the determinant of the following matrix

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Proof. Denote A_n the $n \times n$ matrix of such form.

$$\det(A_4) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \overset{col_1 - col_2}{=} \begin{vmatrix} -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{vmatrix} = -\det(A_3) + \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}.$$

$$B_3 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix} \underset{R_3+R_1}{\overset{R_2+R_1}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

So

$$\det(A_4) = 1 - \det(A_3), \quad \det(A_3) = 1 - \det(A_2), \quad \det(A_2) = 1.$$

So $\det(A_4) = \det(A_2) = 1$ and $\det(A_3) = 0$. $\square$

Recall: Let $A \in M_{n\times n}(\mathbb{F})$.

- An eigenpair of A is $(\lambda, v) \in \mathbb{F} \times (\mathbb{F}^n - \{0\})$ such that $Av = \lambda v$.
- A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $p_A(t) = \det(A - tI_n) = 0$. The corresponding eigevectors are solutions of $(A - \lambda I)v = 0$.
- The characteristic polynomial of A is

  $$p_A(t) = \det(A - tI_n) = (-1)^n t^n + (-1)^{n-1}\mathrm{tr}(A)\, t^{n-1} + \cdots + \det(A).$$

- A has at most n eigenvalues.

Example 34. Find the eigenvalues and a basis for the eigenspaces of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Proof. We have

$$\det(A - tI) = \begin{vmatrix} 1-t & 2 \\ 3 & 2-t \end{vmatrix} = (1-t)(2-t) - 6 = t^2 - 3t - 4 = (t+1)(t-4).$$

So the eigenvalues of A are -1 and 4.
For $\lambda_1 = -1$, find the corresponding eigenvectors:

$$A - \lambda_1 I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

$(A - \lambda_2 I) V = 0$

For $\lambda_2 = 4$, find the corresponding eigenvectors:

$$A - \lambda_2 I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}.$$

So a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$.

□

Definition 43. *Let $A, B \in M_{n\times n}(\mathbb{F})$. Then B is similar to A if there exists an invertible matrix P such that $B = P^{-1}AP$.*

Theorem 5.3. *Let $A, B \in M_{n\times n}(\mathbb{F})$ and B be simlar to A. Then*

1. *A and B have the same charateristic polynomial, that is, $p_A(t) = p_B(t)$.*
2. *$\det(A) = \det(B)$.*
3. *$\mathrm{tr}(A) = \mathrm{tr}(B)$.*
4. *$\mathrm{rank}(A) = \mathrm{rank}(B)$.*

Proof. 1. Since B is similar to A, there exists an invertible matrix P such that $B = P^{-1}AP$. Then

$$p_B(t) = \det(B - tI_n) = \det(P^{-1}AP - tP^{-1}I_nP) = \det(P^{-1}(A - tI_n)P)$$
$$= \det(P^{-1})p_A(t)\det(P) = p_A(t)\det(P^{-1})\det(P) = p_A(t).$$

3.Hint: Use the following property: $\mathrm{tr}(CD) = \mathrm{tr}(DC)$ for any $C \in M_{m\times n}(\mathbb{F})$ and $D \in M_{n\times m}(\mathbb{F})$.
4.Hint: You may first want to prove that for any invertible matrix $Q \in M_{n\times n}(\mathbb{F})$, we have $\mathrm{rank}(A) = \mathrm{rank}(AQ) = \mathrm{rank}(QA)$. □

Next, we define eigenvalues and eigenvectors for linear operators.

Definition 44.
- *Let $T : V \to V$ be a linear mapping (called a linear operator) on a vector space V. A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of the linear operator T if there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$. Such vector v is called an eigenvector of T corresponding to the eigenvalue λ.*
- *Let $T : V \to V$ be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial of T to be the characteristic polynomial of $A = [T]_\beta$.*

Example 35.
1. Consider $T : \mathbb{R}[x] \to \mathbb{R}[x], T(p(x)) = p'(x)$. If $(\lambda, p(x))$ is an eigenpair of T then

$$T(p(x)) = \lambda p(x) \Rightarrow p'(x) = \lambda p(x) \Rightarrow p(x) = ce^{\lambda x}, \text{ where } c \in \mathbb{R}.$$

2. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the plane through an angle of 90 degree counterclockwise. Then T has no eigenvectors and no eigenvalues.

Theorem 5.4. *Let $T : V \to V$ be a linear operator on a vector space V. Then*

1. *A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $(T - \lambda I)$ is not invertible.*
2. *Let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.*

Proof. Exercise. □

Lemma 14. *Let V be an n-dimensional vector space with ordered basis β. Then the characteristic polynomial of the linear operator T does not depend on the chosen basis. That is, if α is another ordered basis for V, the characteristic polynomial of T is also the characteristic polynomial of $[T]_\alpha$.*

Proof Sketch. Recall: If $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of an n-dimensional vector space V, then

$$[T]_\beta = \Big[[T(v_1)]_\beta \quad [T(v_2)]_\beta \quad \cdots \quad [T(v_n)]_\beta \Big] \in M_{n\times n}(\mathbb{F}).$$

If α is another ordered basis of V, then

$$[T]_\beta = Q^{-1}[T]_\alpha Q,$$

where Q is the change of coordinate matrix that changes β-coordinates into α-coordinates,

$$Q = [I_V]_\beta^\alpha = \begin{bmatrix} [v_1]_\alpha & [v_2]_\alpha & \cdots & [v_n]_\alpha \end{bmatrix}.$$

That means $[T]_\beta$ is similar to $[T]_\alpha$. From the previous theorem, $[T]_\beta$ and $[T]_\alpha$ have the same characteristic polynomial. Therefore, the characteristic polynomial of the linear operator T does not depend on the chosen basis. □

Definition 45. • *A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.*

- *Version 1: A square matrix A is called diagonalizable if L_A is diagonalizable.*

Theorem 5.5. *Let $T : V \to V$ be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if there is an ordered basis β for V consisting of eigenvectors of T.*

Proof. ($\Rightarrow$) Suppose T is diagonalizable. By definition, there exists an ordered basis $\beta = \{v_1, \dots, v_n\}$ such that $[T]_\beta$ is diagonal. That is

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [T]_\beta = \begin{bmatrix} [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_n)]_\beta \end{bmatrix}.$$

So $[T(v_k)]_\beta = \lambda_k e_k$, for $1 \le k \le n$, where $\{e_1, \dots, e_n\}$ is the standard ordered basis of $\mathbb{F}^n$. Therefore,

$$T(v_k) = \lambda_k v_k, \quad 1 \le k \le n.$$

Since v_k is an element in a basis of V, $v_k \neq 0$. Therefore, v_k is an eigenvector of T for $1 \le k \le n$.

($\Leftarrow$) Conversely, suppose V has an ordered basis $\beta = \{v_1, \dots, v_n\}$ of eigenvectors of T. Then, there are scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that

$$T(v_k) = \lambda_k v_k, \quad \forall 1 \le k \le n.$$

Hence, $[T(v_k)]_\beta = \lambda_k e_k, \ \forall 1 \le k \le n$, and

$$[T]_\beta = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

□

Below is the version of Theorem 5.4 for the linear operator $L_A : \mathbb{F}^n \to \mathbb{F}^n$, where $A \in M_{n\times n}(\mathbb{F})$. Note that eigenvectors of the linear operator L_A are eigenvectors of the matrix A.

Theorem 5.6. *Let $A \in M_{n\times n}(\mathbb{F})$. Then A is diagonalizable if and only if there is an ordered basis β for $\mathbb{F}^n$ of eigenvectors of A.*

Example 36. Recall to the previous theorem. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$. Two corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since v_1 is not a multiple scalar of v_2, the set $\{v_1, v_2\}$ is linearly independent. So $\{v_1, v_2\}$ is a basis of $\mathbb{F}^n$. Hence, A is diagonalizable.

Theorem 5.7 (Diagonalizable Matrix – Definition Version 2)**.** *Let $A \in M_{n\times n}(\mathbb{F})$. Then A is diagonalizable if and only if there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.*

Proof. Note that $A = [L_A]_\alpha$, where $\alpha = \{e_1, \ldots, e_n\}$ is the standard ordered basis of $\mathbb{F}^n$. Then
A is diagonalizable $\Leftrightarrow \exists$ an ordered basis $\beta = \{v_1, \ldots, v_n\}$ for $\mathbb{F}^n$ such that $[L_A]_\beta = D$, a diagonal matrix.
$\Leftrightarrow D = P^{-1}AP$, where $P = [I_{L_A}]_\beta^\alpha = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$
$\Leftrightarrow PDP^{-1} = A$, where $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$. $\square$

Remark 16. The above proof provides a procedure to check whethere a square matrix is diagonalizable or not. If yes, it provides the factorization of A as PDP^{-1} where D is a diagonal matrix.

- Find all eigenvalues of A.
- Find a basis for each eigenspace of A.
- If $\sum\limits_{\lambda:\text{ eigenvalue of } A} E_\lambda = n$, A is diagonalizable. Let P be a square matrix whose columns are eigenbases of A and let D be a diagonal matrix whose diagonal entries are eigenvalues of A corresponding to the column of P. Then
$$A = PDP^{-1}.$$
- If $\sum\limits_{\lambda:\text{ eigenvalue of } A} E_\lambda \neq n$, A is not diagonalizable.

Example 37. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is diagonalizable and

$$A = [v_1 \ v_2] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} [v_1 \ v_2]^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \right)^{-1}.$$

Theorem 5.8. *Let T be a linear operator on a vector space V and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \ldots, v_k$ are eigenvectors of T such that $T(v_i) = \lambda_i v_i$, $(1 \leq i \leq k)$, then $\{v_1, \ldots, v_k\}$ is linearly independent.*

Proof. □

Mar 30, Lecture 31: Eigenvalues and Eigenvectors (cont'd)

Recall: Let T be a linear operator on a finite-dimensional vector space V. Then T is diagonalizable $\Leftrightarrow$ There is an ordered basis for V such that $[T]_\beta$ is diagonal $\Leftrightarrow$ There is an ordered basis β for V of eigenvectors of T.

Recall a part of the proof. ($\Leftarrow$) Conversely, suppose V has an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of eigenvectors of T. Then, there are scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that

$$T(v_k) = \lambda_k v_k, \quad \forall 1 \le k \le n.$$

Hence, $[T(v_k)]_\beta = \lambda_k e_k$, $\forall 1 \le k \le n$, and

$$[T]_\beta = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

□

Lemma 15. *If A diagonalizable, and $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis for $\mathbb{F}^n$ of eigenvectors of A, then*

1. *$D = [L_A]_\beta$ is diagonal and D_{kk} is the eigenvalue corresponding to v_k for $1 \le k \le n$.*
2. *$P^{-1}AP = D$, where P be the $n \times n$ matrix whose k-th column is the k-th vector of β, $P = [v_1 \cdots v_n]$.*

Proof. 1. Proved last time.
2. Using the following lemma (prove this –Hint: Use the change of coordinates matrix theorem).

Lemma 16. *Let $A \in M_{n\times n}(\mathbb{F})$ and let γ be an ordered basis for $\mathbb{F}^n$. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose k-th column is the k-th vector of γ.*

□

Theorem 5.7 (Diagonalizable Matrix – Definition Version 2)**.** *Let $A \in M_{n\times n}(\mathbb{F})$. Then A is diagonalizable if and only if there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.* $\longrightarrow$ $A = PDP^{-1}$

$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$

$A^m = PD^mP^{-1}$

Proof. ($\Rightarrow$) Proved in Lemma 15.
($\Leftarrow$) Suppose there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Since P is invertible, the columns of P, $v_1, \ldots, v_n$, form a basis for $\mathbb{F}^n$. Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for $\mathbb{F}^n$. By Lemma 16, $[L_A]_\beta = P^{-1}AP = D$. So A is diagonalizable. □

Remark 16. If there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, then the columns of P are eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the columns of P. It answers the question that the factorization $A = PDP^{-1}$, where P is invertible and D is diagonal, if exists, is not unique. Even if we sort the diagonal entries of D in a given order (such as decreasing or increasing order), D is then unique, but P is still not unique.

Question: how to find a basis for $\mathbb{F}^n$ from the eigenvectors of A (if such basis exists)?

Theorem 5.8. *Let $T : V \to V$ be a linear operator and $\dim V = n$. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \ldots, v_k$ are eigenvectors of T corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, \ldots, v_k\}$ is linearly independent.*

Proof. Prove by induction on k.
For $k = 1$, since v_1 is an eigenvector of T, $v_1 \neq 0$, so $\{v_1\}$ is linearly independent.
Assume the theorem holds for $k-1$ distinct eigenvalues, where $k-1 \geq 1$, and we have k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose $v_1, \ldots, v_k$ are eigenvectors of T corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$.
Consider

$$c_1v_1 + \cdots + c_{k_1}v_{k-1} + c_kv_k = 0, \quad \text{where} \quad c_1, \ldots, c_k \in \mathbb{F}. \tag{2}$$

Applying $T - \lambda_k I$ to both sides of Equation (2), we have

$$c_1(\lambda_1 - \lambda_k)v_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0.$$

By the induction hypothesis, $\{v_1, \ldots, v_{k-1}\}$ is linearly independent. Therefore,

$$c_1(\lambda_1 - \lambda_k) = c_2(\lambda_2 - \lambda_k) = c_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$

Since $\lambda_j \neq \lambda_k$ for $1 \leq j \leq k-1$, we have $(\lambda_j - \lambda_k) \neq 0$ for $1 \leq j \leq k-1$. Hence

$$c_1 = \ldots = c_{k-1} = 0.$$

Then Equation (2) reduces to $c_k v_k = 0$, which leads to $c_k = 0$ since $v_k \neq 0$. Consequently, $c_1 = \cdots = c_k = 0$, and it follows that $\{v_1, \ldots, v_k\}$ is linearly independent. □

Corollary 5.8.1. Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Example 37. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ has two distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$. So A is diagonalizable.

Remark 17. The converse of Corollary 5.8.1 is not true. That is, there exists a linear operator T on an n-dimensional vector space such that T is diagonalizable but T does not have n distinct eigenvalues. For example, the identity matrix is diagonalizable, even though I_n has only one eigenvalue $\lambda = 1$:

$$A = I_n = P^{-1} I_n P, \quad \text{for any invertible matrix } P.$$

Definition 46. *A polynomial $f(t) \in \mathbb{F}[t]$ splits over $\mathbb{F}$ if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in $\mathbb{F}$ such that*

$$f(t) = c(t - a_1)(t - a_2) \cdots (t - a_n).$$

Example 38. Consider $f(t) = t^3 + t^2 + t + 1 = (t^2 + 1)(t + 1)$.

- $f(t)$ does not split over $\mathbb{R}$, $\mathbb{Q}$.
- $f(t)$ splits over $\mathbb{C}$: $f(t) = (t + i)(t - i)(t + 1)$.
- $f(t)$ splits over $\mathbb{F}_2$: $f(t) = (t + 1)(t + 1)(t + 1)$.

Theorem 5.9. *The characteristic polynomial of any diagonalizable linear operator splits.*

Proof. Let T be a diagonalizable linear operator on an n-dimensional vector space V. Then there exists an ordered basis β of V so that $[T]_\beta$ is a diagonal matrix D. Then the characteristic polynomial of T is

$$p_D(t) = (\lambda_1 - t) \ldots (\lambda_n - t) = (-1)^n (t - \lambda_1) \ldots (t - \lambda_n),$$

which splits over $\mathbb{F}$. □

Definition 47. *Let λ be an eigenvalue of a linear operator T on an n-dimensional vector space or a matrix $A \in M_{n\times n}(\mathbb{F})$ with characteristic polynomial $p(t)$.*

- *The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor fo $p(t)$.*
- *The geometric multiplicity of λ is $\dim(E_\lambda)$, the dimension of the eigenspace*

$$E_\lambda = \{v \in \mathbb{F}^n : T(v) = \lambda v\}.$$

- *Note that $E_\lambda = Null(T - \lambda I)$ and E_λ has at least one nonzero eigenvector. Therefore,*

$$1 \leq \dim(E_\lambda) \leq n.$$

Theorem 5.10. *Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T having algebraic multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.*

Proof. Suppose $\dim(E_\lambda) = k$. Choose an ordered basis $\{v_1, \dots, v_k\}$ for E_λ and extend it to an ordered basis $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V. Let $A = [T]_\beta$. Since $T(v_j) = \lambda v_j$ for $1 \leq j \leq k$, we have

$$A = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix}.$$

From Assignment 5, we have

$$\begin{aligned} p_A(t) = \det(A - tI_n) = \det \begin{pmatrix} (\lambda - t)I_k & B \\ 0 & C - tI_{n-k} \end{pmatrix} \\ = \det((\lambda - t)I_k)\det(C - tI_{n-k}) = (\lambda - t)^k \det(C - tI_{n-k}). \end{aligned}$$

So $(\lambda - t)^k$ is a factor of $p_A(t)$, therefore, $k \leq m$, $\square$

Theorem 5.11. *Let T be a linear operator on a vector space V and let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T. For each $j = 1, 2, \dots, k$, let S_j be a finite linearly independent subset of the eigenspace E_{λ_j}. Then*

$$S = S_1 \cup S_2 \cup \cdots S_k$$

is a linearly independent subset of V.

Proof. □

Theorem 5.12. *Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \dots, \lambda_k$ be all distinct eigenvalues of T. Then*

1. *T is diagonalizable iff the algebraic multiplicity of λ_j is equal to $\dim(E_{\lambda_j})$ for all $1 \leq j \leq k$.*
2. *If T is diagonalizable and β_j is an ordered basis for E_{λ_j}, for each j, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*

Proof. $\square$

Remark 18. The above proof provides a procedure to check whethere a square matrix is diagonalizable or not. If yes, it provides the factorization of A as PDP^{-1} where D is a diagonal matrix.

- Find all eigenvalues of A. Suppose $\lambda_1, \ldots, \lambda_k$ are all distinct eigenvalues of A and a_j is the algebraic multiplicity of λ_j, for $1 \leq j \leq k$.
- Find a basis for each eigenspace E_{λ_j}, for $1 \leq j \leq k$.
- If there exists $1 \leq j \leq k$ such that $\dim E_{\lambda_j} \neq a_j$, the matrix A is not diagonalizable.
- If $a_j = \dim E_{\lambda_j}$ for all $1 \leq j \leq k$, the matrix A is diagonalizable. Let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$, where β_j is an ordered basis for E_{λ_j}, for $1 \leq j \leq k$. Let P be a square matrix whose columns are vectors from β and let D be a diagonal matrix whose diagonal entries are eigenvalues of A corresponding to the column of P. Then

$$A = PDP^{-1}.$$

Example 39. The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is diagonalizable and

$$A = [v_1 \ v_2] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} [v_1 \ v_2]^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \right)^{-1}.$$

Example 40. Check whether $A = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 1 \end{bmatrix}$ is diagonalizable. If yes, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Also, find A^{100}.