Class Notes

Math 147
Calculus I
Advanced Level
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Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1} x + C$

9.4

Real Numbers

$N = \{1, 2, 3, 4 \cdots\}$

$Z = \{0, 1, -1, 2, -2 \cdots\}$

$Q = \{\frac{p}{q},\ p, q \in Z,\ q \neq 0\}$ – rational numbers

– field

Well-ordering principle

最小

A set has the well-ordering principle if every non-empty subset has a lease member ("least" means, a number $x \in$ set such there $x \leq a$ for every set)

Ex. Z does not have WoP

Pf: Consider the subset Z it has not least element (无最小 element)

Ex. N has WoP

Proof: let S be a non-empty subset of N. let $S_1 \in S$ [use contradiction)

If $1 \in S$, then 1 is the least element.

If $1 \notin S$, but $2 \in S$, then 2 is the least element

Repeat this process terminals with a least element in at most S steps

Ex.

$Q^+ = \{x \in Q,\ x \geq 0\}$

Does not have WoP because the subset $S = \{x \in Q, x > 0\}$

because if $\frac{p}{q}$ is a least element of S, then $\frac{p}{2q} \in S$, and $\frac{p}{2q} < \frac{p}{q}$. So $\frac{p}{q}$ was not a least element after all.

Ex. $\sqrt{2} \notin \mathbb{Q}$

Pf Suppose $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, p, q are coprime

$$2q^2 = p^2$$

$\Rightarrow p^2$ is even $\Rightarrow p$ is even

Say $p = 2k$ for some integer

$$2q^2 = (2k)^2 = 4k^2$$

$$q^2 = 2k^2$$

$\Rightarrow q^2$ is even $\Rightarrow q$ is even

p, q are assumed to be coprime, but both are even.

So that's a contradiction

9.6

Principle of Mathmatical Induction

Theorem (abslute true) let $P(n)$ be a statement about $n \in N$

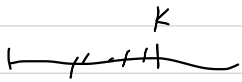
Suppose ① $P(1)$ is true

② $P(k+1)$ is true whenever $P(k)$ is true, $k \in N$

Then $P(n)$ is true for all $n \in N$

Proof (give a proof by contradiction)

Suppose there are some $n \in N$ with $P(n)$ not true



let $S = \{ n \in N : P(n) \text{ is not true} \}$

Then S is not empty. So by WOP. S has a least element

Hence, there is some $n_1 \in S$ with $n_1 \leq k$ for all $k \in S$

Notice $n \neq 1$ Since $P(1)$ is true by assumption.

Then $n_1 - 1 \in N$ so $n_1 - 1 \notin S$

Thus $P(n_1 - 1)$ is true

By (2) $P(n_1 - 1 + 1) = P(n_1)$ is true — This is a contradiction

Therefore $P(n)$ is true for every $n \in N$

ex. Prove $r_1 + \cdots + r^n = \frac{r - r^{n+1}}{1-r}$ for $r \neq 1$, $n \in N$

Pf: let $P(n)$ be the statment $r_1 + \cdots + r^n = \frac{r - r^{n+1}}{1-r}$

$P(1)$ is the statment $r = \frac{r - r^2}{1-r}$

This is true $\frac{r - r^2}{1-r} = \frac{r(1-r)}{1-r} = r$

now assume $P(k)$ holds. show $P(k+1)$ holds

LHS of $P(k+1) = r + \cdots + r^{k+1} = \frac{r - r^{k+1}}{1-r} + r^{k+1}$ Since $P(k)$ is true

$= \frac{r - r^{k+1} + r^{k+1}(1-r)}{1-r}$

$= \frac{r - r^{k+2}}{1-r}$ = RHS of $P(k+1)$, therefore $P(k+1)$ is true

Ex. Prove $2^n > n^2$ for $n \geq 5$

let $P(r)$ be the statement $2^{r+4} > (r+4)^2 \Longrightarrow 16 \cdot 2^r > r^2 + 8r + 16$

When $n=1$, $2^{4+1} = 32 > (1+4)^2 = 25$, so $P(1)$ is true

Assume for $k \in N$. $P(k)$ is true, now show $P(k+1)$ is true

$P(k+1) = 2^{k+1+4} = 2^{k+5} = 32 \cdot 2^k$, $(k+1+4)^2 = (k+5)^2 = k^2 + 10k + 25$

$$32 \cdot 2^k = 2 \cdot 16 \cdot 2^k > 2 \cdot (k^2 + 8k + 16) > 2k^2 + 16k + 32 > k^2 + 10 + 25$$

So $P(k+1)$ is true if $P(k)$ is true

let $n = r+4$

Then (variation) let $P(n)$ be a statement above $n \geq 5$

Suppose (1) $P(5)$ is true and (2) $P(k+1)$ is true if $P(k)$ is true

Then $P(n)$ is true for all $n \geq 5$

Principle of Strong Induction

let $P(n)$ is a statement about $n \in N$

Suppose (1) $P(1)$ is true

(2) $P(k+1)$ is true whenever $P(1), P(2), \dots P(k)$ is true

Then $P(n)$ is true for all $n \in N$

ex. Suppose f is defined on N

by when $f(1)=1$, $f(2)=2$ and $f(n+2) = \frac{1}{2}(f(n+1)+f(n))$

Prove Range $f \subseteq Q$ and is $f(n) \leq 2$

Pf First Prove Rang $f \subseteq Q$

$P(n)$ $f(n) \in Q$

$P(1)$ is true Since $f(1)=1 \in Q$, $f(2)=2 \in Q$

Suppose $P(1), P(2), \dots, P(k)$ are true and check $P(k+1)$ is true

so Suppose $f(1), f(2), \dots f(k) \in Q$

$f(k+1) = \frac{1}{2}(f(k)+f(k-1)) \in Q$

Check $P(k+1)$ is true, By induction $P(n)$ is true for all $n \in N$ if $f(n) \in Q$ for all $n \in N$

Start to Prove $f(n) \leq 2$

let the $F(n)$ be the statement $f(n) \leq 2$, for $f(n) \in Q$

$n=1$, $f(1)=1$, $F(1)$ is true

$n=2$, $f(2)=2$, $F(2)$ is true

Assume $F(1), F(2), \dots \dots, F(r)$ are true for $r \in N$, and check $F(r+1)$ is true

$f(r+1) = \frac{1}{2}(f(r)+f(r-1)) \leq \frac{1}{2}(2+2) = 2$, $F(r+1)$ is true if $F(r), F(r-1)$ are true

Check $F(k+1)$ is true, By strong induction $F(n)$ is true for all $n \in N$ if $f(n) \leq 2$ for all $n \in N$

9.9

Absolute Value

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

$$|a|^2 = a^2$$

$$\sqrt{a^2} = |a|$$

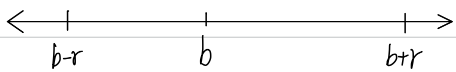
$|x| \leq r \iff -r \leq x \leq r$

$|a-b| \leq r \iff -r \leq a-b \leq r \iff b-r \leq x \leq b+r$

$|a-b| = |-(a-b)| = |b-a| \leq r$

Find x such that $|x-b| < r$

$b-r < x < b+r$



Same as finding all x such that distance from x to b is $< r$

Theorem $\triangle$ inequality

$|a+b| \leq |a| + |b|$

$|a| = |(a-b)+b| \leq |a-b| + |b|$

Pf: $-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

$-|a| - |b| \leq a+b \leq |a| + |b|$

$|a| - |b| \leq |a-b|$

Similarly $|a-b| = |b-a| \geq |b| - |a|$

$\therefore \underbrace{-(|a|+|b|)}_{\text{Consider as } -r} \leq a+b \leq \underbrace{|a|+|b|}_{\text{Consider as } r}$

$|a+b| \leq |a| + |b|$

Ex. Find the value c such that $|f(x)| \leq c$ where $f(x) = \frac{x^3-4x-1}{2x-1}$ for $2 \leq x \leq 3$

Assume $|f(x)| = \frac{|x^3-4x-1|}{|2x-1|} \leq \frac{|x^3|+|-4x|+|-1|}{|2x-1|} \leq \frac{27+12+1}{3} = \frac{40}{3}$ (分子放, 分母缩) ($x=3$, $x=2$)

Real numbers

Properties
① field ($\pm, \times, \div$)
② "order" ($a<b$)
③ contains $\mathbb{N}$ (and contain all $\mathbb{Q}$)

④ Biggie!

Definition: a non-empty subset S of an order set X
we call bounded above if $\exists A \in X$ (存在 $A \in X$)
such that $A \geq a$ for every $a \in S$
Similarly: ($A \leq a$) - - - - - - (below)

Any such number A call an upper bound (lower bound) for S
If S has both an upper and lower bounded. then we say S is bounded

Ex. $X = \mathbb{R}$
$S = \mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$
$0, -1$, any negative - and all $x \geq 0$ for all $x \in S$. So S is bounded below
No number >0 will be a lower bounded for S
S is not bounded above

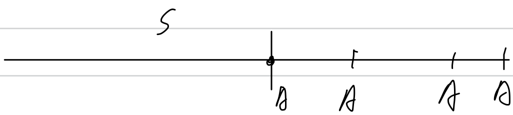
Ex. $S = [1, \pi) = \{x : 1 \leq x < \pi\}$
1 is a lower bound
π is an upper bound
> both equal

$1 \in S$
$\pi \notin S$

9.11

Recall

Defination: Say $S \subseteq X$ (order set) is bounded above, if that is some $A \in X$ so that $x \leq A$ for every $x \in S$. Any A with this property is called an upper bound for S



eg. $S = (1, \pi] = \{x : 1 < x \leq \pi\}$

Upper B: π, 1

Lower B: 1, 0

Defin: $A \in X$ is a least upper bound (greatest lower bound) - glb for the subset S of
Lub

1. A is an upper bound (lower bound)
2. if $B \in X$ any other upper bound (lower bound) for S then $B \geq A$ $(B \leq A)$

Suppose A_1 and A_2 are both lub for S. Then A_1 and A_2 are both upper bound for S

By (2) $A_2 \geq A_1$ (taking $A = A_1$, $B = A_2$) but also by (2) (taking $A = A_2, B = A_1$) $A_1 \geq A_2$

Therefore $A_1 = A_2$

LUB is also called supremum or Sup

GLB - - - - - - - infimum or inf

Ex. $S = \{\frac{n}{n+1}, n \in N\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$

$S \in (0,1)$ - bounded

$GLB = \frac{1}{2}$, $LuB = 1$

Characterization of LUB (GLB)

Theorem A is the LUB/GLB for S

if and only if

(1) A is an upper bound for S

(2) for every $Z < A$ ($>$), there exist $X \in S$, such that $X > Z$ ($<$)

Proof. First assume A is the lub(s)

(1') A is an upper bound for S

(2') let $Z < A$ Since $A = lub(s)$, Z cannot be an upper bound for S

So something S_2 call it X, is bigger than Z. i.e. $X > Z$

Secondly assume (1')+(2') for A

(1) ✓

(2) let B be any other upper bound for S

assume $B < A$, by (2') there exists $X \in S$ with $X > B$. That means B cannot be an upper bound for S. that's a contradiction. Therefore $B > A$. thus. $A = lub(s)$

Note (2') is same as saying for every $\varepsilon > 0$, there is some $X \in S$ with $X > A - \varepsilon$

(think of $Z = A - \varepsilon$ or $\varepsilon = A - Z$)

Critical Property defining R

Completeness Axiom of R (completeness Property)

Every non-empty Sub set if is bounded above it has at least upper bound

$\underline{\underline{R}}$ order field, Contaiinning N, and has the Completeness axism

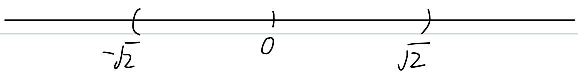
Pretend Q is entire numbers in Universe

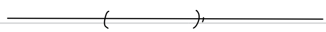
let $S = \{x \in Q,\ x^2 < 2\}$

Bded set in our numberline

No lub in number line

"no holds"



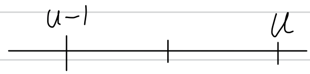


Archimedian Property

Given any real number x, there is some $N \in \mathbb{N}$ with $N > x$

Proof : Suppose this is false. Then there exist $x \in R$
Such that $x \geq N$ for every $N \in \mathbb{N}$

That means $\mathbb{N}$ is a bounded set



let u be a lub for $\mathbb{N}$

By our charterization theorem for LUB. (taking $z = u-1 < u$)
there must some $n \in \mathbb{N}$, Such that $n > u-1$, but $n+1 > u$
There is a contradiction since $n+1 \in \mathbb{N}$ and u was supposed to be an UB for $\mathbb{N}$

9.13

Completeness Axiom

Archimedean Property — Given any $r \in \mathbb{R}$

There is some $n \in \mathbb{N}$ with $n > r$

Corollary : Cor $GLB\{\frac{1}{n}, n \in \mathbb{N}\} = 0$

推论

Proof: 0 is certainly a lower bound (Want to find $n \in \mathbb{N}$ with $\frac{1}{n} < R$)

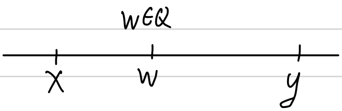
let $R > 0$. Then $\frac{1}{R} \in \mathbb{R}$

By Archimedean Property get $n \in \mathbb{N}$ with $n > \frac{1}{R}$

Then $\frac{1}{n} < R$. Hence R is not a lower bound for $\{\frac{1}{n} : n \in \mathbb{N}\}$

Therefore $0 = GLB\{\frac{1}{n} : n \in \mathbb{N}\}$

Theorem (Density of $\mathbb{Q}$) if $x, y \in \mathbb{R}$ with $x<y$, Then $w \in \mathbb{Q}$ Such that $x \leq w < y$



Proof: Do case $x \geq 0$, As $y-x>0$

By our property of Corollary, there is some $N_0 \in \mathbb{N}$

Such that $\frac{1}{N_0} < y-x \iff N_0 y > 1+N_0 x$

let $S = \{n \in \mathbb{N},\ n \geq N_0 x\}$

By the Archimedean Property, S is not empty

Also $S \subseteq \mathbb{N}$, by WOP of $\mathbb{N}$, the set has a least element

Call it N_1

If $N_1 = 1$, then $N_1 \leq N_0 x + 1$

if $N_1 \neq 1$, therefore $N_1 - 1 \notin S$

Therefore $N_1 - 1 \leq N_0 \cdot x \iff N_1 \leq N_0 x + 1$

So $N_0 x \leq N_1 \leq N_0 x + 1$ (Since $N_1 \in S$)

by N_0 $x \leq \frac{N_1}{N_0} < y$, take $w = \frac{N_1}{N_0}$

exercise: Prove there is some $w \in \mathbb{Q}$ with $x < w < y$

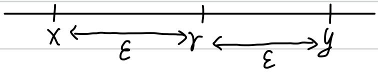
When $x<0$, as $y-x>0$ By our property of Corollary, there is some $N_0 \in \mathbb{N}$

Such that $\frac{1}{N_0} < y-x \iff N_0 x + 1 < N_0 y$

Face Irrational number are also dense

if $x<y$, then there is some $t \notin Q$, $x<t<y$

Proof (using 4b in Assi 1)



let $\frac{x+y}{2} \in R$. let $\varepsilon = y - r$

By Corollary to Archimedean Property, get $n \in N$ with $\frac{1}{n} < \varepsilon$

By 4b in Assi 1, get $t \notin Q$

with $|r-t| < \frac{1}{n} < \varepsilon$

$r-\varepsilon \leq r-\frac{1}{n} < t < r+\frac{1}{n} \leq r+\varepsilon$

证明方法参考Assignment 1的4b

Remark $\sqrt{2}$ is irrational

What we mean by $\sqrt{2}$ is a possible solution to $x^2=2$

Pf: let $S = \{x \in R, x^2<2\}$ bounded above, non-empty

So S has a lub - call it $A \in R$ (of course $A \geq 0$)

Enough to show $A^2=2$. because then $A=\sqrt{2}$

Suppose $A^2 \neq 2$

Show that this implies some $A-\frac{1}{n}$ is still an UB for S and that's contradiction

Case 1 $A^2<2$, say $A^2=2-\delta$ for some $\delta>0$, Consider $(A+\frac{1}{N})^2 = A^2 + \frac{2A}{N} + \frac{1}{N^2}$

$= A^2 + \frac{1}{N}(2A+\frac{1}{N}) \leq A^2 + \frac{1}{N}(2A+1)$

by taking $N > \frac{2A+1}{\delta} \in R$

(Archimedean Property)

Then $(A+\frac{1}{N})^2 \leq A + (\frac{2A+1}{N}) < A^2 + \delta = 2$

$\Rightarrow A+\frac{1}{N} \in S$, that contradices the fact that $A = lub(S)$

Case 2 $A^2>2$, Say $A^2-P=2$ for some $P>0$

Consider $(A-\frac{1}{N})^2 = A^2 - \frac{2A}{N} + \frac{1}{N^2}$

$= A^2 + \frac{1}{N}(\frac{1}{N}-2A) > A^2 - \frac{2A}{N}$

by taking $N > \frac{2A}{P}$ (Archimedean Property)

$(A-\frac{1}{N})^2 > A^2 - \frac{2A}{N} > A^2 - P = 2$

$\Rightarrow A-\frac{1}{N} \notin S$, and $A > A-\frac{1}{n}$

that contradices the fact that $A = lub(S)$

Conclusion. Since $A^2<2$ or $A^2>2$ are impossible, then $A^2=2$

9.16

Sequence

a Sequence a that $x_1, x_2, x_3 \cdots\cdots$

also written as $(x_n)_{n=1}^{\infty}$ or x_n

Examples ①: $1, 1, 1, 1, \cdots\cdots$

② $x_n = \frac{1}{n}$: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\cdots$

③ $x_n = (-1)^{n+1}$: $1, -1, 1, -1, \cdots\cdots$

④ $x_1 = 1$, $x_2 = \sqrt{2}$, $x_{n+1} = x_n + x_{n-1}$, $n \geq 2$

$x_3 = 1 + \sqrt{2}$

$x_4 = 1 + 2\sqrt{2}$ $\longrightarrow$ Recursively defined

Convergence

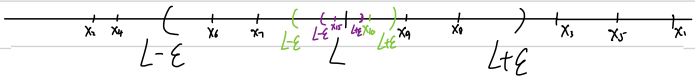
Def'n We say the sequence $(x_n)_{n=1}^{\infty}$ Converges to the real number L if for every $\varepsilon > 0$, there is some $N_0 \in \mathbb{N}$ with the property that $|x_n - L| < \varepsilon$ for all $n \geq N_0$

In this case we say that L is the limit of the sequence and write with

$\lim_{n \to \infty} x_n = L$ or $(x_n)_{n=1}^{\infty} \xrightarrow[n\to\infty]{} L$ or $(x_n) \to L$ or $x_n \to L$

If there is no such L, then we say the sequence (x_n) diverges

$$|x_n - L| < \varepsilon \iff L - \varepsilon < x_n < L + \varepsilon$$



Ex. ① $x_n = 2$ for every n

$(x_n) \to 2$ and we can take $N=1$ for every choice of ε

② $x_n = \frac{1}{n}$. Guess $L=0$

Rough work : $\left|\frac{1}{n} - 0\right| < \varepsilon$ for $n \geq N$

$\frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

Assume $\lim_{n\to\infty} \frac{1}{n} = 0$

可缩写

Pf let $\varepsilon > 0$. ~~By the Archimedean Principle~~, there is an integer $N > \frac{1}{\varepsilon}$

If $n \geq N$, then

$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, hence $|x_n - 0| < \varepsilon$ for all $n \geq N$. That proves $\lim_{n\to\infty} x_n = 0$

③ $x_n = \frac{n}{n+1}$, Guess $L=1$

Rough work : $\left|\frac{n}{n+1} - 1\right| < \varepsilon$

$\frac{1}{n+1} = \left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| < \varepsilon$. $N > \frac{1}{\varepsilon} - 1 \Longrightarrow N+1 > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{N+1} < \varepsilon$

$\frac{1}{n+1} \leq \frac{1}{n} < \varepsilon$

Pf let $\varepsilon > 0$, By the Ach. P. choose integer $N > \frac{1}{\varepsilon}$. Then if $n \geq N$

$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} \leq \frac{1}{n} < \varepsilon$

Hence $\lim_{n\to\infty} \frac{n}{n+1} = 1$

④ $x_n = \frac{(-1)^n}{2n^2-5}$, Guess $L=0$

Rough work $\left|\frac{(-1)^n}{2n^2-5} - 0\right| = \frac{1}{2n^2-5} < \varepsilon$ if $n \geq 2$

Notice $2n^2 - 5 \geq n^2$ if $n^2 \geq 5$ if $n \geq 3$

$\frac{1}{2n^2-5} \leq \frac{1}{n^2}$ if $n \geq 3$, $\frac{1}{n^2} < \varepsilon$ if $n \geq \sqrt{\frac{1}{\varepsilon}}$

Assume : let $\varepsilon > 0$. By A.P. take $N > \max\left(3, \sqrt{\frac{1}{\varepsilon}}\right)$. Then if $n \geq N$

We have $\left|\frac{(-1)^n}{2n^2-5} - 0\right| = \left|\frac{1}{2n^2-5}\right| \leq \frac{1}{n^2} < \varepsilon$

$\therefore \lim_{n\to\infty} \frac{(-1)^n}{2n^2-5} = 0$

Alternative

$\frac{1}{2n^2-5} \leq \frac{1}{n^2} \leq \frac{1}{n} < \varepsilon$

$n = \frac{1}{\varepsilon}$ works

Wrong guess

① $x_n = \frac{1}{n}$ Guess $L=1$

Suppose to be able to satisfy

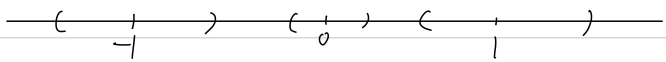
$\left|\frac{1}{n}-1\right| < \frac{1}{2}$ "eventually" (ie. these assume N such that this holds for all $n \geq N$)

$\frac{1}{2} < \frac{1}{n} < \frac{3}{2}$

This cannot be true for any $n>1$

② $x_n = (-1)^n$, $-1, 1, -1, 1, -1, 1$

This sequence divergence



Pf Take any $L \in \mathbb{R}$

Take $\varepsilon = \frac{1}{2}$ and contradict the interval $(L-\varepsilon, L+\varepsilon) \Rightarrow (L-\frac{1}{2}, L+\frac{1}{2})$

There is an interval of length 1 If there is some N so $x_n \in (L-\frac{1}{2}, L+\frac{1}{2})$ for all $n \geq N$. then both $\pm 1 \in (L-\frac{1}{2}, L+\frac{1}{2})$

but $|1-(-1)| = 2$. Some is impossible . Hence no L can be the limit.

9.18

Convergence of Sequences

Def'n Say $(x_n)_{n=1}^{\infty}$ Converge to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there is an integer N so that $|x_n - L| < \varepsilon$ for all $n \geq N$

$$L - \varepsilon < x_n < L + \varepsilon$$

Ex. Show $\lim_{n \to \infty} r^n = 0$ if $|r| < 1$

Rough work: $\left(\frac{1}{|r|}\right)^n = (1+\Delta)^n$ for some $\Delta > 0$ $\quad (1+\Delta)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \Delta^{n-k} = 1 + n\Delta + \underbrace{\cdots + \Delta^n}_{>0}$

$$\Rightarrow |r|^n < \frac{1}{n\Delta} < \frac{1}{\Delta} \cdot \frac{1}{n}$$

We want to prove $|r^n - 0| < \varepsilon \Rightarrow |r|^n < \varepsilon$. Want to find N is true for all $n \geq N$

$$|r|^n \leq \frac{1}{\Delta} \cdot \frac{1}{n} < \varepsilon$$

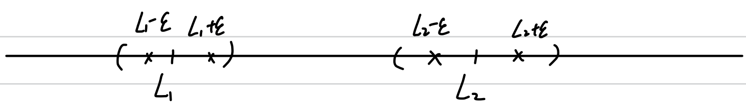
Proof: let $\varepsilon > 0$, and suppose $\frac{1}{|r|} = 1 + \Delta$ for all $\Delta > 0$, take $N > \frac{1}{\Delta} \cdot \frac{1}{\varepsilon}$

Then

$$|r|^n = |r^n - 0| < \frac{1}{\Delta} \cdot \frac{1}{n} < \varepsilon$$

Therefore, $\lim_{n \to \infty} r^n = 0$

Fact: Limit are unique $\qquad \varepsilon \leq \frac{|L_2 - L_1|}{2}$



Proof Suppse $(x_n) \to L_1$ and L_2 $\quad (L_1 \neq L_2)$

Take $\varepsilon = \frac{|L_2 - L_1|}{3} > 0$

Since $(x_n) \to L_1$, we know there is some N_1 so $|x_n - L_1| < \varepsilon$ for all $n \geq N_1$

Similarly, since $(x_n) \to L_2$ we know there is some N_2 so $|x_n - L_2| < \varepsilon$ if $n \geq N_2$

let $n = \max\{N_1, N_2\}$ Then both $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$

Then $|L_1 - L_2| = |L_1 - x_n + x_n - L_2| \leq |L_1 - x_n| + |x_n - L_2| < \varepsilon + \varepsilon = 2\varepsilon = \frac{2}{3}|L_1 - L_2|$

Not true - Contradiction, Hence $L_1 = L_2$

Squeeze Theorem

Suppose $x_n \leq y_n \leq z_n$ for all n

assume $(x_n) \to L$ and $(z_n) \to L$. Then $(y_n) \to L$

Proof let $\varepsilon > 0$. get N_1, N_2 so $|x_n - L| < \varepsilon$ for all $n \geq N_1$ and $|z_n - L| < \varepsilon$ for all $n \geq N_2$

Take $N = \max\{N_1, N_2\}$; let $n \geq N$

$L - \varepsilon < x_n \leq y_n \leq z_n \leq L + \varepsilon$

$\Rightarrow |y_n - L| < \varepsilon$ for all $n \geq N$

Ex. $x_n = -1$, $y_n = (-1)^n$, $z_n = 1$

$x_n \leq y_n \leq z_n$ for all n

$(x_n) \to -1$, $(z_n) \to 1$: But (y_n) does not converge

Ex: Prove $\lim\limits_{n\to\infty} \frac{n^2}{5^n} = 0$ Strategy: Prove: $n^2 \leq 4^n$ (give an induction prove)

Then $\underbrace{0}_{\downarrow 0} \leq \underbrace{\frac{n^2}{5^n}}_{\to 0 \text{ by squeeze}} \leq \frac{4^n}{5^n} = \underbrace{\left(\frac{4}{5}\right)^n}_{\to 0}$

Bounded Sequence

Say (x_n) is bounded if there is some C with $|x_n| \leq C$ for all n

Fact: Every convergent sequence is bounded (But the converse is not true - eg. $(-1)^n$)

Pf: Take N so $|x_n - L| < 1$ for all $n \geq N$ where $L = \lim x_n$

Then $|x_n| = |(x_n - L) + L| \leq |x_n - L| + |L| \leq 1 + |L|$ for all $n \geq N$

Take $C = \max\{|x_1|, |x_2|, |x_3|, \cdots, |x_n|\}$

Then $|x_n| \leq C$ for all n

Limit Laws

Suppose $(x_n) \to K$, $(y_n) \to L$

$(x_n + y_n) \to K+L$

$(x_n - y_n) \to K-L$

Produce $(x_n \cdot y_n) \to KL$

$\left(\frac{x_n}{y_n}\right) \to \frac{K}{L}$, $y_n \neq 0$, $L \neq 0$

Proof: Produce: Think $|x_n y_n - KL| < \varepsilon$, Pick N so $|x_n - K| < \frac{\varepsilon}{2(|L|+1)}$ for all $n \geq 0$

know $|x_n - K| < \varepsilon$, $|y_n - L| < \varepsilon$

$|y_n - L| < \frac{\varepsilon}{2C}$, $C \neq 0$, C is a bound for the convergent sequence (x_n)

let $\varepsilon > 0$

$$
\begin{aligned}
|x_n y_n - KL| &= |x_n y_n - x_n L + x_n L - KL| \\
&\leq |x_n y_n - x_n L| + |x_n L - KL| \\
&= |x_n| \cdot |y_n - L| + |L| \cdot |x_n - K| \\
&\leq C \cdot |y_n - L| + \frac{\varepsilon}{2(|L|+1)} \cdot |L| \\
&< C \cdot \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } n \geq N
\end{aligned}
$$

9.20

Monotonic Sequence

Def'n Say (x_n) is increasing if $x_{n+1} \geq x_n$ for every n $(x_1 \leq x_2 \leq x_3 \leq \cdots)$
(x_n) increasing if $x_{n+1} \leq x_n$ for every n
If (x_n) is either increasing / decreasing it is called monotonic

Ex. $(-1)^n$ is not monotonic and diverges
$\left(\frac{(-1)^n}{n}\right)$ is not monotonic but converges

Theorem (Monotonic Converge Theorm : MCT)

if x_n is a monotonic sequence that is bounded, then x_n converge
ex. $x_n = n$ - monotonic, unbounded, therefore diverge

Proof Assume x_n is an increasing sequence that is bounded above

let $A = \{x_n : n = 1, 2, 3, \cdots\}$ - No empty
and bounded above. By the Completeness Axiom of $\mathbb{R}$, it has a LUB $= L$

Claim : $x_n \to L$
let $\varepsilon > 0$ Since L is an upper bound for A, $x_n \leq L$ for all n
Hence $x_n < L + \varepsilon$ (for all n). Since $L = LUB(A)$ and $L - \varepsilon < L$,
we know that is some $x_N \in A$ with $x_N > L - \varepsilon$
(increasing) Since $x_n \geq x_N > L - \varepsilon$ if $n \geq N$. Hence for all $n \geq N$, $L - \varepsilon < x_n < L + \varepsilon$
Therefore $x_n \to L$

Ex. $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ Does x_n Converge

$x_{n+1} = x_n + \frac{1}{(n+1)^2} \geq x_n$ So x_n is increasing

$x_n = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{\leq \frac{2}{2^2} = \frac{1}{2}} + \underbrace{\frac{1}{4^2} + \cdots + \frac{1}{7^2}}_{\leq \frac{4}{4^2} = \frac{1}{4}} + \frac{1}{8^2} \cdots + \frac{1}{n^2} \leq 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2^k} \leq 2$

So x_n is bounded too, therefore by MCT x_n Converge

Ex. Let $a_1 = 1$, $a_{n+1} = \frac{2a_n+5}{6}$, $a_2 = \frac{7}{6}$, $a_3 = \frac{1}{6}(\frac{14}{6}+5) = \frac{11}{9}$

$a_1 \leq a_2 \leq a_3 \cdots$ Hope it is increasing. Guess $a_n \leq 2$

Proceed by induction to prove thee (if true)

(1) Prove $a_n \leq a_{n+1}$ for all n Because $a_1 \leq a_2$ is true

Assume $a_n \leq a_{n+1}$, and check $a_{n+1} \leq a_{n+2}$

Well $a_{n+1} = \frac{2a_n+5}{6} \leq \frac{2a_{n+1}+5}{6} = a_{n+2}$

(2) Prove $a_n \leq 2$ for all n, Base Case $a_1 \leq 2$ is true

Assume $a_n \leq 2$, Prove $a_{n+1} \leq 2$

Well $a_{n+1} = \frac{2a_n+5}{6} \leq \frac{2 \cdot 2+5}{6} < 2$, we got it!

Since a_n is bounded and increasing, by MCT it Converge, say $a_n \to L$

Think about the sequence $b_n = \frac{2a_n+5}{6} = a_{n+1}$

By Limit $a_{n+1} = b_n \to \frac{2L+5}{6}$ $\therefore$ $L = \frac{2L+5}{6} \Rightarrow L = \frac{5}{4}$

Nested Interval Property

Theorem: Suppose we have collection of nested interval
$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$
with the property that $(b_n - a_n) \longrightarrow 0$
Then there is a unique real number $r \in [a_n, b_n]$ for every n

Comments: ① This fail to be true if we don't have $(b_n - a_n) \longrightarrow 0$
② This fails for if interval $[a_n, \infty)$ e.g. $a_n = 0$

e.g. $[a_n, b_n] = [0, 1]$ for all n $\to$ fail uniqueness

$\bigcap_{n=1}^{\infty} [n, \infty) =$ empty ,
If $r \in [n, +\infty)$ for all n, then $r \geq n$ for every $n \in \mathbb{N}$. That's false

e.g. (a_n, b_n) nested $b_n - a_n \to 0$ and uniqueness fail

9.23

Recall : Nested interval Property

$x_0 \in \bigcap_{n=1}^{\infty} [a_n, b_n] \longrightarrow$ means x_0 belong to each $[a_n, b_n]$

What happened replace $[a_n, b_n]$ with (a_n, b_n)

Say $x_0 \in (a_n, b_n) \Rightarrow 0 < x_0 < \frac{1}{n}$ for all n

false - Archimedian Property

Proof Think about Sequence $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$

$a_1 \le a_2 \le \cdots \le a_n \le b_n \le b_{n-1} \le \cdots \le b_1$

Hence a_n is an increasing sequence bounded above by b_1 (acctually any b_k)

By MCT, a_n converges to $a = LUB\{a_j : j=1,2,3\cdots\}$

Similarly, b_n converges to $b = GLB\{b_j : j=1,2,3\cdots\}$

We have $a \ge a_n$ ($a = LUB(a_n)$) and $b \le b_n$ for all n (since $b = GLB(b_n)$)

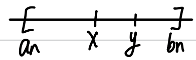
Furthermore, $a_i \le b_j$ for all i, j.

Therefore (by homework) $a \le b$, $a_n \le a \le b \le b_n$ for all n

So a (and b) belong to $[a_n, b_n]$ for every n

Uniqueness : Suppose $x, y \in [a_n, b_n]$ for all n, $x \ne y$

$\Rightarrow \underbrace{|x-y|}_{= \varepsilon} \le b_n - a_n$ for every n



but $(b_n - a_n) \longrightarrow 0$ so this is impossible - Contradiction.

Subsequences

Pick $n_1 < n_2 < n_3 < \cdots\cdots$

Given (x_n), Consider the sequence $(y_k)_{k=1}^{\infty}$ where $y_k = x_k$

The sequence (y_k) is a subsequence of (x_n)

e.g. $x_1, x_2, \underset{n_1}{\overset{||}{x_3}}, x_4, x_5, x_6, \underset{n_2}{\overset{||}{x_7}}, \underset{n_3}{\overset{||}{x_8}}, x_9, \underset{n_4}{x_{10}}$

Subsequence (y_k) where $y_1 = x_3$, $y_2 = x_7$, $y_3 = x_8$, $y_4 = x_{12}$

Notice if $x_n \to L$ then $x_k \to L$ for all subsequence

if x_n does not converge, what thing can happen with subsequence?

e.g. $x_n = (-1)^n$ then $x_{2n} \to 1$ and $x_{2n+1} \to -1$

if $x_n = n$ - every subsequence diverge too

Prop Any sequence has a monotonic subsequence

Proof We call the term x_k a Peak Point of sequence of $x_k \geq x_{k+1}, x_{k+2}, x_{k+3}$

let x_n decreasing $\Rightarrow$ every Point is a Peak Point

Case ① The sequence has infinitely many Peak Points

let x_{n_k} be the k th peak Point then x_{n_k} is decreasing since $x_{n_k} \geq x_{n_{k+1}}$

Case ② There are only finitely many Peak Points (Could be none)
有限的

Take x_{n_1} to be the first term in sequence after the least Peak Point

Then x_{n_1} is not a Peak Point. So there must be some $n_2 > n_1$ with $x_{n_1} < x_{n_2}$

But also x_{n_2} is not a Peak Point so there is some $n_3 > n_2$ with $x_{n_2} < x_{n_3}$

Repeat, Then x_{n_k} is an increasing subsequence

Bolzano - Weierstrass Theorm

Every bounded sequence has a convergent subsequence

Proof By Prop, there is a monotone subsequence that subsequence is bounded since the original sequence is bounded by MCT, the subsequence converges.

Def'n A sequence x_n is called Cauchy if for every $\varepsilon > 0$, there is some N so that

$|x_n - x_m| < \varepsilon$ if $n, m \geq N$

9.25

Def'n A sequency x_n is called <u>Cauchy</u> if for every $\varepsilon > 0$, there is some N so that
$|x_n - x_m| < \varepsilon$ if $n, m \geqslant N$

Fact: ① Any convergence sequence is Cauchy
② Any Cauchy sequence is bounded

Pf ① Given $\varepsilon > 0$, pick N so
$|x_n - L| < \frac{\varepsilon}{2}$ for all $n \geqslant N$
(where $L = \lim x_n$) Then if $n, m \geqslant N$
$|x_n - x_m| \leq |x_n - L| + |L - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

② Pick N so if $n, m \geqslant N$
$|x_n - x_m| < 1$
This means $|x_n - x_m| < 1$ for $n \geqslant N$ $\Rightarrow |x_n| \leq 1 + |x_m|$ for all $n \geqslant N$
$C = \max\{|x_1|, |x_2|, \cdots |x_m|\}$. Then $|x_n| \leq C$ for all n

Theorm Every Cauchy sequence converge

Proof Since the sequence x_n is Cauchy, it is bounded. By the Bolzano-W theorm it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ with limit L

Goal: check $x_n \to L$

let $\varepsilon > 0$, since x_n is Cauchy, there is some N such that
$|x_n - x_m| < \frac{\varepsilon}{2}$ ② if $n, m \geqslant N$. Furthermore, since $x_{n_k} \to L$, there is some index $n_k > N$
so $|x_{n_k} - L| < \frac{\varepsilon}{2}$ ①, let $n \geqslant N$ Then $|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \underbrace{\frac{\varepsilon}{2}}_{\text{by ②}} + \underbrace{\frac{\varepsilon}{2}}_{\text{by ①}} = \varepsilon$

Example. Suppose $|x_n - x_{n+1}| < \frac{1}{2^n}$ for all n, claim x_n is Cauchy

Pf Rough work, assume $n > m$

look at $|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m|$

$$\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$

$$\leq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} \leq \sum_{i=m}^{\infty} \frac{1}{2^i} = \frac{1}{2^m} \sum_{0}^{\infty} \frac{1}{2^i} = \frac{2}{2^m} = \frac{1}{2^{m-1}}$$

Given $\varepsilon > 0$, have to find N so $n > m \geq N$. given $|x_n - x_m| < \varepsilon$

know $|x_n - x_m| \leq \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$

for big enough N

Proof, Given $\varepsilon > 0$, Pick $N > 0$, $\frac{1}{2^{N-1}} < \varepsilon$, Then if $n \geq m \geq N$

$|x_n - x_m| \leq \cdots \leq \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$ and there x_n is Cauchy

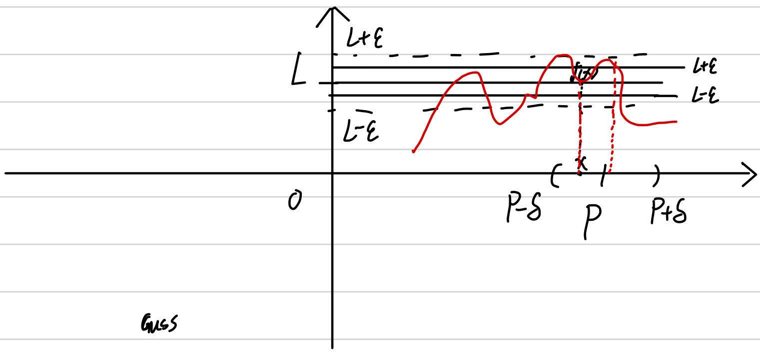
Fonctions - Limits (new chapter)

$f: A \subseteq \mathbb{R} \to \mathbb{R}$

$\uparrow$ domain

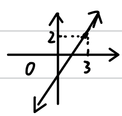
Def'n Say f has limit $L \in \mathbb{R}$ at point P if for every $\varepsilon > 0$, there is some $\delta > 0$ so that wherever $0 < |x-p| < \delta$, $x \neq p$, then $|f(x) - L| < \varepsilon$

$0 < |x-p| < \delta$, $x \neq p$, $p-\delta < x < p+\delta$, $L-\varepsilon < f(x) < L+\varepsilon$



$\lim_{x \to p} f(x) = L$

ex. 1. $\lim_{x \to 3} 2x - 4 = 2$ (Guess)



Rough work $|(2x-4) - 2| < \varepsilon$ when $|x-3| < \delta$

$|2x-6| = 2|x-3| < \varepsilon$ $\quad \delta = \frac{\varepsilon}{2}$

Pf let $\varepsilon > 0$ and take $\delta = \frac{\varepsilon}{2}$

if $0 < |x-3| < \delta$, then

$|f(x) - 2| = |2x-4-2| = 2|x-3| < 2 \cdot \delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$

Then $\lim_{x \to 3} 2x-4 = 2$

ex. $\lim_{x\to 2} x^2 = 4$ Guess

$\to (x-2)(x+2)$

Rough work $|x^2-4| < \varepsilon$

if $|x-2| < \varepsilon$ want $|x-2| \cdot |x+2| < \varepsilon$

$2-\delta < x < 2+\delta$

First require $\delta < 1 \Rightarrow 1 < x < 3$

$\Rightarrow |x+2| \leq 5 \to |x-2| \cdot |x+2| < 5|x-2| < \varepsilon$

Pf Take $\delta = \min(1, \frac{\varepsilon}{5})$, if $|x-2| < \delta$ then $1 < x < 3$

So $|x^2-4| = |x-2| \cdot |x+2| \leq 5|x-2| < 5 \cdot \delta \leq 5 \cdot \frac{\varepsilon}{5} = \varepsilon$

9.27

Limit of function at a Point

$\lim_{x \to a} f(x) = L$

Means: For every $\varepsilon > 0$ there is some $\delta > 0$ Such that if $0 < |x-a| < \delta$. then $|f(x) - L| < \varepsilon$

ex. (1) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$

Rough work

Force $\delta \leq 1 \Rightarrow x \in (2,4) \Rightarrow 3x \geq 6$

$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| < \frac{\delta}{6} < \varepsilon$

$x \in (3-\delta, 3+\delta)$ $\qquad \delta \leq 6\varepsilon$

Prove: let $\varepsilon > 0$, take $\delta = \min(1, 6\varepsilon)$

If $0 < |x-3| < \delta$, then in Particular

$x \in (2,4)$ So $|3x| = 3x \geq 6$

Hence $\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| < \frac{\delta}{6} \leq \varepsilon$

Ex. $\lim_{x \to 1} x^3 - 1 = 0$

Rough work

Force $\delta \leq 1, \frac{\varepsilon}{7} \to |x-1| < \delta \to x \leq 2$

$|x^3 - 1 - 0| = |x^3 - 1| = |x-1| \cdot |x^2 + x + 1|$

Prove let $\varepsilon > 0$ and take $\delta = \min(1, \frac{\varepsilon}{7})$

Then $0 < |x-1| < \delta$. then $0 < x < 2$

So $|x^3 - 1 - 0| = |x-1| \cdot |x^2 + x + 1| \leq 7 \cdot |x-1| < 7\delta \leq \varepsilon$

Ex. $\lim_{x \to 2} \frac{2x^2-8}{x-2}$

Rough work $\frac{2x^2-8}{x-2} = \frac{2(x^2-4)}{x-2} = 2(x+2)$, Guess $L = 8$

$$\left| \frac{2x^2-8}{x-2} - 8 \right| = |2 \cdot (x+2) - 8| = |2x-4| = 2|x-2| \qquad \delta = \frac{\varepsilon}{2}$$

Prove let $\varepsilon > 0$ and take $\delta = \frac{\varepsilon}{2}$

If $0 < |x-2| < \delta$

Then $\left| \frac{2x^2-8}{x-2} - 8 \right| = |2(x+2) - 8| = 2|x-2| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$

Ex. Let $f(x) = \begin{cases} 1 , & \text{if } x \in Q \\ 0 , & \text{if } x \notin Q \end{cases}$

Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? 极限不存在!

It is not exist!

Pf Take $\varepsilon = \frac{1}{3}$ and any $\delta > 0$. Since there are both rational and irrational Points x Satisfying $0 < |x - 0| < \delta$, we must have $|1 - L| < \varepsilon$ and $|0 - L| < \varepsilon$ (if δ is to "work" in the def')

$\Rightarrow 1 \in (L - \frac{1}{3}, L + \frac{1}{3})$ and $0 \in (L - \frac{1}{3}, L + \frac{1}{3})$

But this is impossiable

Common: Same prove shows $\lim_{x \to a} f(x)$ does not exist at any point a

Ex.

$$y = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ \text{undefined} & \text{at } x = 0 \end{cases}$$

. Find $\lim_{x \to 0} y = ?$ $\longrightarrow$ Does not exist

One sided limit

$\lim_{x \to a^+} f(x) = L$ to mean for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon$

left hand limit : check $0 < a - x < \delta$ or $-\delta < x - a < 0$ $(x < a)$, written $\lim_{x \to a^-} f(x) = L$

Ex.

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 3 \\ \frac{1}{x} & \text{if } x > 3 \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to 3^+} f(x) &= \frac{1}{3} \\ \lim_{x \to 3^-} f(x) &= 7 \end{aligned}\right] \text{— no limit at } 3$$

9.30

Limits

$\lim_{x \to a} f(x) = L$ means for every $\varepsilon > 0$, there is some $\delta > 0$ such that if $0 < |x-a| < \delta$

Then $|f(x) - L| < \varepsilon$

Notation: $f(x) \to L$ as $x \to a$

Limit Laws: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$

(1) $(f \pm g)(x) \longrightarrow L \pm K$ as $x \to a$

(2) $(f \cdot g)(x) \longrightarrow L \cdot K$ as $x \longrightarrow a$

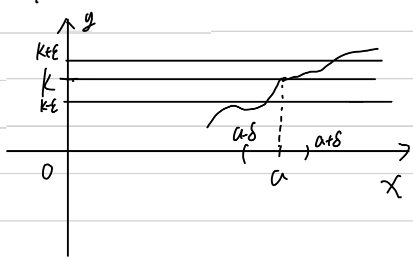
(3) $\frac{f}{g}(x) \longrightarrow \frac{L}{K}$ as $x \longrightarrow a$, provided $K \neq 0$

Pf (1) look at $|f(x) + g(x) - (L+K)| \leq |f(x) - L| + |g(x) - L|$

So given $\varepsilon > 0$, choose $\delta > 0$, then $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - L| < \frac{\varepsilon}{2}$

Then $0 < |x-a| < \delta \Rightarrow |f(x) + g(x) - (L+K)| < \varepsilon$

Suppose $g(x) \to K \neq 0$ as $x \to 0$



if we pick $\varepsilon = \frac{|K|}{2} > 0$, get δ from defn of limit

then $|g(x) - K| < \frac{|K|}{2}$ for $0 < |x-a| < \delta$

$\Rightarrow g(x) \neq 0$ for such x

Squeeze Theorm

if $f(x) \leq g(x) \leq h(x)$ for all x "near" a, if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then also $\lim_{x \to a} g(x) = L$

Continuous Function

Assume $f: A \subseteq \mathbb{R} \to \mathbb{R}$

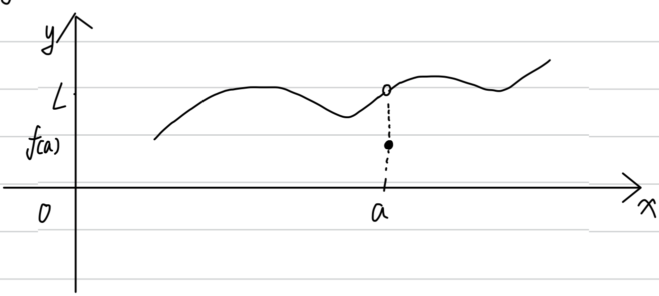
Def'n Say f is continuous at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$ (where we understand the limit to be the one sided limit if a is a endpoint of A)

That means for every $\varepsilon > 0$, there is some $\delta > 0$ so that $|x-a| < \delta$ implies

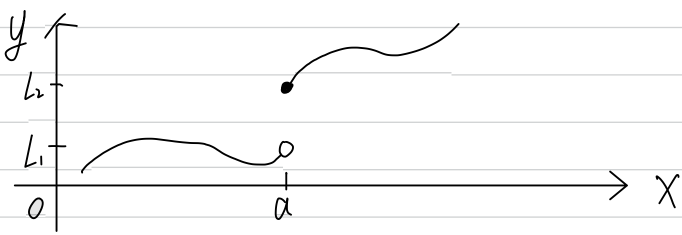
$$|f(x) - f(a)| < \varepsilon$$

(with obvious modification for one-sided limit)

Say f is continuous if f is continuous at every point at $a \in A$



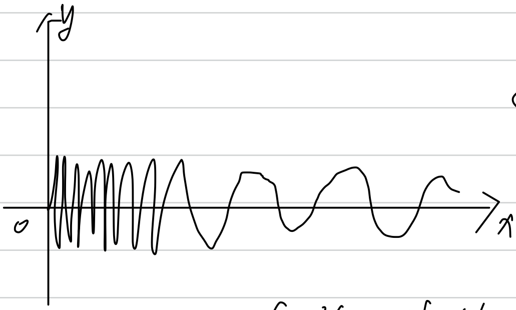
not continuous because $\lim_{x \to a} f$ exists, but is not $f(a)$



$\lim_{x \to a^-} f(x) = L_1 \neq L_2 = \lim_{x \to a^+} f(x)$

$\therefore$ $\lim_{x \to a} f(x)$ do not exist

not continuous at a



$$y = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ not Continue at $a=0$

$\lim_{x\to 0} f(x)$ does not exist

(not even RHS and LHS limits exist)

Ex. $f(x) \begin{cases} \frac{2x^2-8}{x-2}, & \text{if } x \neq 2 \\ 3, & \text{if } x = 2 \end{cases}$

$\lim_{x\to a} f(x) = 8 \neq 3 = f(2)$, so $f(x)$ is not continuous at $x=2$

Theorem: $f(x): A \to \mathbb{R}$, is Continous at $a \in A$ if and only if whenever x_n is a Sequence from A, (meaning every $x_n \in A$) with $x_n \to a$, then $f(x_n) \to f(a)$

Proof ① ($\Rightarrow$) Assume $f(x)$ is Continuous at a, let $(x_n) \to a$ with $x_n \in A$ for all n
We need to Prove $(f(x))_{n=1}^{\infty} \to f(a)$

We have to Prove for every $\varepsilon > 0$, there is some N so $n \geq N$ implies $|f(x_n) - f(a)| < \varepsilon$

Since $f(x)$ is Continuous at a, we know there is some $\delta > 0$ so $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Because $x_n \to a$, we also know there is some N, so $|x_n - a| < \delta$ for $n \geq N$

Hence if $n \geq N$, $|x_n - a| < \delta$. and that implies $|f(x_n) - f(a)| < \varepsilon$. Does it!

② ($\Leftarrow$) Suppose $f(x)$ is not Continuous at a. That means there is an $\varepsilon > 0$. that failed the def'n of Continuity. That mean no $\delta > 0$ works for there $\varepsilon > 0$, in Particular $\delta = \frac{1}{n}$ fails to work for this ε for every $n \in X$

For each n, Choose $x_n \in A$ with $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \varepsilon$

Hence the sequence $x_n \to a$, but $f(x_n) \not\to f(a)$, Since $|f(x_n) - f(a)| \geq \varepsilon$ for every n

This Contradicts the assumption

Hence $f(x)$ is Continuous at a

Ex. $f(x) \begin{cases} 1, & \text{if } x \in Q \\ 0, & \text{if } x \notin Q \end{cases}$

— not continuous anywhere

Ex Homework

$f(x) \begin{cases} 0 & \text{if } x \notin Q \\ 1 & \text{at } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \ p \in Z, \ q \in N \ (p, q) \text{ coprime} \end{cases}$

$f(x)$ is continuous at every $a \notin Q$ and discontinuous at $a \in Q$

Super bounus: Show there is no function continuous on the rational and not continuous on irrational

10.2

Continuous Function

$f: A \longrightarrow R$ (A: interval)

Say f is continuous at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$

Say f is continuous if it continue at every $a \in A$

$\Updownarrow$

For every $\varepsilon > 0$, there exise $\delta > 0$

So if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

Charterization Theom f is continuous at a if and only if where $x_n \to a$, $x_n \in A$, then $f(x_n) \longrightarrow f(a)$

Facts if f, g $A \to R$ is continuous at $a \in A$

them so are
1. $f \pm g$
2. $c \cdot f$ (c is constant)
3. $f \cdot g$
4. $\frac{f}{g}$ if $g(a) \neq 0 \Rightarrow g(x) \neq 0$ "near" a

Pf $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a)$

Cor (推论): Polynomials are continuous

Rational function are continuous (except at zero of denominator)

Pf let $f(x) = x$ is continuous and $g(x) = $ constant is continuous

Rational function $\frac{P(x)}{Q(x)}$, P, Q are polynomial

Ex. $f(x) = \begin{cases} 3x+1 & \text{if } x>0 \\ 1-x^2 & \text{if } x\le 0 \end{cases}$

$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x+1 = 1 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1-x^2 = 1 \end{array}\right] \Rightarrow \lim_{x\to 0} f(x) = 1$

$f(0) = 1-x^2\big|_{x=0} = 1$

Is f Continuous?

$\lim_{x\to 0} f(x) = f(0)$ and $f(x)$ is Continuous at all $x \neq 0$ being Polynomials there

Composition of Functions $\quad g\circ f(x) = g(f(x))$

$f: A \longrightarrow \mathbb{R}$
$g: B \longrightarrow \mathbb{R}$, $B \supseteq$ Range of f

Theorem If f is Continuous at $a \in A$ and g is Continuous at $f(a)$, then $g\circ f$ is Continuous at a

Proof Show if $x_n \to a$ ($x_n \in A$), then $g\circ f(x_n) \longrightarrow g\circ f(a)$

let $x_n \to a$, $x_n \in A$, f is Continuous at a Thus $\underset{\underset{y_n}{\parallel}}{f(x_n)} \longrightarrow f(a)$

$(y_n) \to f(a)$ and g is Continuous at $f(a)$
$g\circ f(x_n) = g(y_n) \longrightarrow g(f(a)) = g\circ f(a)$

Ex. If f is Continuous, then $|f|$ is Continuous

Pf $g(x) = |x| = \left.\begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x<0 \end{cases}\right]$ Continuous Function

Therefore $g\circ f = |f|$ is Continuous

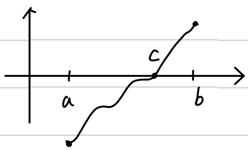
Ex. $f(x) = \sqrt{P(x)}$, $P(x)$ - polynomial with $P(x) \geq 0$

Recall if $x_n \to a$, then $\sqrt{x_n} \to \sqrt{a}$, $\therefore \sqrt{P(x)}$ is Continuous

Intermediate Value Theorem (IVT)

Suppose $f: [a,b] \to R$, that is continuous. Assume $f(a) < 0$, $f(b) > 0$

Then there exist $c \in [a,b]$ with $f(c) = 0$



Proof Let $A = \{x \in [a,b] : f(x) < 0\}$ A is not empty as $a \in A$

Also A is bounded. By Completeness axiom, A has a LUB, Call it L

$a \leq L \leq b$ (Since b is an upper bound for A). Hence f is continuous at L

Notice that : $L - \frac{1}{n} < L$ for all $n \in N$

Since $L = LUB(A)$, there must exist $x_n \in [L - \frac{1}{n}, L] \cap A$

$x_n \to L$. Since f is continuous at L, $\underset{\overset{\wedge}{0} \text{ for } x_n \in A}{f(x_n)} \to f(L) \Rightarrow f(L) \leq 0$

$\Rightarrow L \neq b$. $L < b$

Hence there exist N so $a \leq L + \frac{1}{N} < b$ and $L + \frac{1}{n} < b$ for all $n \geq N$

Hence $(L + \frac{1}{n})_{n=N}^{\infty}$ is a sequence in $[a,b]$

$(L + \frac{1}{n}) \to L \Rightarrow f(L + \frac{1}{n}) \to f(L)$

But $L + \frac{1}{n} > L = LUB(A)$

$\therefore L + \frac{1}{n} \notin A$ and that means $f(L + \frac{1}{n}) \geq 0 \Rightarrow f(L) \geq 0$

Together, this implies $f(L) = 0$, so L is the "c" of the theorem

Cor if $f: [a,b] \to$ is continuous, and z satisfies $f(a) < z < f(b)$
Then there is some $c \in [a,b]$ with $z = f(c)$

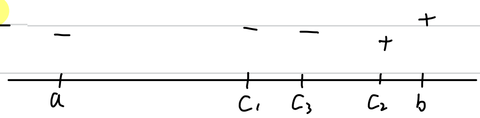
Proof let $g(x) = f(x) - z$ - continuous
$g(a) = f(a) - z < 0$, $g(b) = f(b) - z > 0$
By I.V.T. there is $c \in [a,b]$ with $g(c) = 0 \Rightarrow f(c) - z = 0 \Rightarrow f(c) = z$

Cor: any polynomial of odd degree has a real root

Pf $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, n is odd, $a_n \neq 0$
WLOG $a_n = 1$, $P(x) \to \infty$ if $x \to \infty$
不失一般性
$P(x) \to -\infty$ if $x \to -\infty$

So $P(x)$ takes on both positive and negative values and since P is continuous, by I.V.T. it has a root

Bissection Method



10.4

Extreme Value Theorm

Say f is <u>bounded above</u> if there is some $C \in R$ so $f(x) \le C$ for every $x \in$ Domain f

Equvalently, Range $f = \{f(x): x \in D(f)\}$ is bounded above.

Ex. (1) $f: R \to R$, $f(x) = x$, neither bounded above / below

(2) $f: (0,1) \to R$ $f(x) = x$, bounded but no max/min

(3) $f (0,1] \to R$, $f(x) = \frac{1}{x}$. bounded below + has a min, not bounded above

(4) $f [1,\infty) \to R$, $f(x) = \frac{1}{x}$, bounded above + below, has a max but no min

Theorm (E.V.T.)

If $f: [a,b] \to R$ is continuus, then f is bounded and there are $c, d \in [a,b]$

So that $\underset{min}{f(c)} \le f(x) \le \underset{max}{f(d)}$ for every $[a,b]$

Proof Step ① Show f is bounded above

假设不成立

Suppose not, Then for every $n \in N$, there is some $x_n \in [a,b]$ so that $f(x_n) > n$

Given us a sequence x_n is $[a,b]$, Bounded sequence, by Bolzano-W theorm, it has a convergent subsequence

Say x_{n_k} with limit L, Hence f is continuous at L. Thus $f(x_{n_k}) \longrightarrow f(L)$

On the other hand, $f(x_{n_k})$ is not bounded, so it can't be converging —— Contradiction

That prove f is bounded above and we can immediately show f is bounded below

Step ② Look at Range $f = \{f(x): x \in [a,b]\}$, by step ①, this set is bounded. by completeness axiom of R

Range f has LUB and GLB, Call LUB $= Z$, know $f(x) \le Z$ for all $x \in [a,b]$

Since $Z =$ LUB (Range f), there must be an element of Range f, say W_n with $Z - \frac{1}{n} < W_n \le Z$

Hence $W_n = f(x_n)$ for some $x_n \in [a,b]$, This give us a sequence x_n in $[a,b)$. Appealing again to Bolzano-W Theorm

There is a subsequence x_{n_k} converging to $k \in [a,b]$. f is continuous at k, so $f(x_{n_k}) \to f(k)$

Notice Squeeze theorm, implies $\underset{f(x_n)}{\underset{\shortparallel}{W_n}} \to Z$ $\Longrightarrow$ $f(x_{n_k}) \longrightarrow Z$ $\therefore$ $Z = f(k)$.

Take $d = k$ in the statement of theorm so $f(x) \le Z = f(k) = f(d)$ for all $x \in [a,b]$

Inverse Function

Def'n Say f is one to one funcion if $a \neq b \Rightarrow f(a) \neq f(b)$

Ex. of not one to one funcion

① $y = x^2$

② $y = \sin x$

Ex. of one to one funcion

① $y = x^3$

One to one funcions are inverseable funcion

$y = f(x)$, Define $f^{-1}(y) = x$ (unique x with $f(x) = y$)

More customary to write ($f^{-1}(x) = y$ where $f(y) = x$)

e.g. $f(x) = y = x^3 + 1$, Solve $f(y) = x \Rightarrow y^3 + 1 = x \Rightarrow y^3 = x - 1 \Rightarrow y = \sqrt[3]{x-1}$

f : Domain $f \longrightarrow$ Range f

f^{-1} : Range $f \longrightarrow$ Domain f

Domain f^{-1} = Range f

Range f^{-1} = Domain f

$f^{-1} \circ f(x) = f^{-1}(y) = x$, where $y = f(x)$

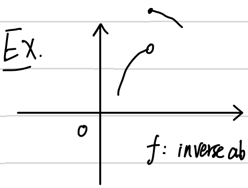
$f \circ f^{-1}(x) = f(y) = x$ where $f(y) = x$

10.7

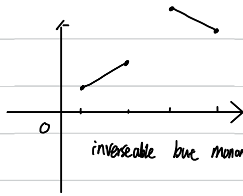
Increasing / Decreasing Function

f is (strictly) increasing if whenever $x_2 > x_1$ then $f(x_2) \geq f(x_1)$ ($>$)

Ex.



f: inverseable, not monotonic



inverseable but monotonic

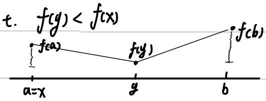
Theorem If $f:[a,b] \to \mathbb{R}$ is Continuous and one to one then it is either strictly increasing / decreasing

Proof f is one to one so $f(a) \neq f(b)$

Assume $f(a) < f(b)$, we can see that f is strictly increasing (leave $f(a) < f(b)$ as exercise)

Assume this is false then $\exists\, y > x$ s.t. $f(y) < f(x)$

Case 1 $x = a$ Then $y \neq b$ as $f(b) > f(a)$

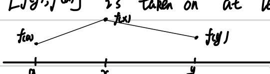


By I.V.T. f takes on all the values between $[f(y), f(a)]$ over the interval $[a, y]$

Similarly, f takes on all values in $[f(y), f(b)]$ over the inteval $[y, b]$

That means each value in $[f(y), f(a)]$ is taken on at least twice, that contradicts the assumption that f is one to one

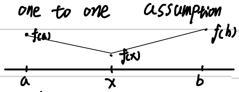
Case 2 $x \neq a$ (i) $f(x) > f(a)$



Another application of IVT shows the value between $[\max(f(a), f(y)), f(x)]$ are taken at least twice.

It contradicts one to one assumption

(ii) $f(x) < f(a)$



again we contradict one to one assumption by using IVT

Consequence If $f:[a,b] \to \mathbb{R}$ is Continuous and one to one, then Range $f = [c,d]$

Why? Say f is stricly increasing. Claim Range $f = [f(a), f(b)]$ if $a < x < b$ then $f(a) < f(x) < f(b)$

by strictly increasing Property, and we get full inteval by IVT.

Theorem If $f : [a,b] \to R$ is Continuous and one to one

Then Range $f = [c,d]$ and $f^{-1} : [c,d] \to [a,b]$ is Continuous.

Proof We have already Range $f = [c,d]$, we want to prove f^{-1} is continuous at $x_0 \in [c,d]$

Take $x_n \to x_0$, we need to show $f^{-1}(x_n) \to f^{-1}(x_0)$, let $y_n = f^{-1}(x_n)$ $f(y_n) = x_n$ and $y_0 = f^{-1}(x_0)$ $f(y_0) = x_0$, $y_n, y_0 \in [a,b]$

y_n is a bounded sequence to by Bolzano-W theorem it has a convergent subsequence, say $y_{n_k} \to z \in [a,b]$

f is continuous at z. Thus
$$\left.\begin{matrix} f(y_{n_k}) \to f(z) \\ \| \\ x_{n_k} \to x_0 \end{matrix}\right\} \Rightarrow f(z) = x_0 = f(y_0)$$

Since f is one to one, $z = y_0$

Hence $f(y_{n_k}) \to z = y_0$

Suppose y_n does not converge to y_0

Then there exists some $\varepsilon > 0$ s.t. for every N there is some $n > N$, s.t.: $|y_n - y_0| > \varepsilon$

Start with $N = 1$ and pick $n > N$ s.t. $|y_{n_1} - y_0| > \varepsilon$

Take $N_1 = n_1 + 1$ and pick $n_2 > N_1$ s.t. $|y_{n_2} - y_0| \geq \varepsilon$

Produce Subsequence y_{n_k} s.t. $|y_{n_k} - y_0| \geq \varepsilon$

Look at this sequence $y_{n_k} = a_k$

Apply previous reasoning this sequence has further subsequence $a_{k_j} = y_{n_{k_j}}$ (a subsequence of original seq (y_n))

which by BW Theorem converge and from our previous reasoning this subsequence converge to y_0

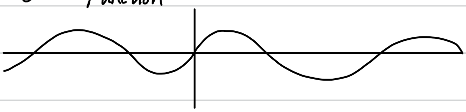
That is a contradiction since $|y_{n_k} - y_0| \geq \varepsilon$ for every term in that subsequence

This contradiction prove the theorem.

10.9

Inverse Trig Functions

Sin Function



Take Sin funcaon and in to domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Hence Sin is one to one. ∴ inverseable

Called $y = \arcsin(x)$: $[-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Updownarrow$

$\sin y = x$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

e.g. $\arcsin(-1) = y$ where $\sin y = -1$ & $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

∴ $y = -\frac{\pi}{2}$

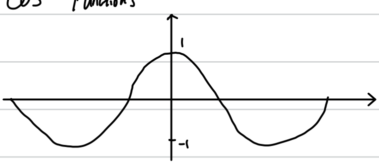
$\sin(\arcsin x) = x$, $\sin y = x$ & $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin(\underset{y}{\sin x}) = \theta$, if $\sin\theta = \overset{\sin x}{y}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin(\sin 2\pi) = \arcsin 0$

$2\pi \neq 0$

Cos Functions

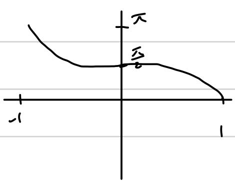


Take Cos reserve it to $[0,\pi]$. there it's inverseable and the arc cos

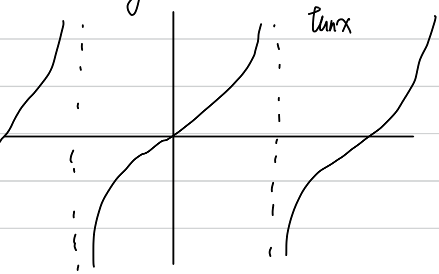
Called arc cos

$\arccos [-1,1] \to [0,\pi]$

$\arccos x = y$ where $\cos y = x$ & $y \in [0,\pi]$

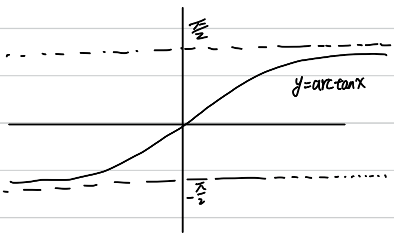


Arc Tangent



Restrict tan to $(-\frac{\pi}{2}, \frac{\pi}{2})$

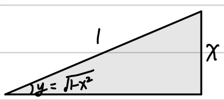
arctan: $(-\infty, \infty) \to (-\frac{\pi}{2}, \frac{\pi}{2})$



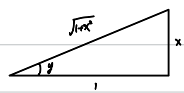
$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$ means for $\varepsilon > 0$, there exist N such that $|\arctan x - \frac{\pi}{2}| < \varepsilon$ where $x \geq N$

$-(-\frac{\pi}{2})$ where $x \leq -N$

$\underbrace{\cos(\arcsin x)}_{\arcsin x \,=\, y} = \sqrt{1-x^2}$, $\sin y = x$ any $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

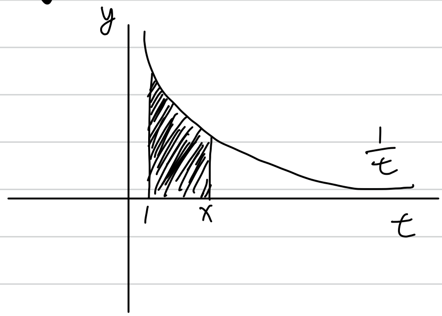


$\sec(\arctan x) = \sqrt{1+x^2}$



End of Midterm Material

Logarithm Function



Domain : $(0, +\infty)$ For $x>0$. let A_x = area bounded by $y=\frac{1}{t}$ and t axis. and the vertical lin $t=1$ or $t=x$

Domain log is $(0,\infty)$ Define $\log x \begin{cases} A_x & \text{if } x\geq 1 \\ -A_x & \text{if } 0<x<1 \end{cases}$

e.g. $\log(1)=0$

$\log x > 0$ if $x>1$

$\log x < 0$ if $x<1$

Properties

Strictly increasing $\Rightarrow$ one to one $\Rightarrow$ inverseable

(1) $\log ab = \log a + \log b$

(2) $\log \frac{1}{a} = -\log a$

(3) $\log \frac{a}{b} = \log a - \log b$

(4) $\log x^r = r \log x$ if $r \in \mathbb{Q}$

(understand $x^{\frac{p}{q}} = \sqrt[q]{x^p}$ for $p,q \in \mathbb{Z}$)

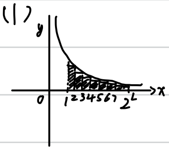
(5) $\log x \to \infty$ as $x\to\infty$ (means for every $C \in \mathbb{R}$, there exist N s.t. $\log x \geq C$ if $x\geq N$)

(6) $\log x \to -\infty$ as $x \to 0^+$ (means for every $C\in\mathbb{R}$, there exist N s.t. $\log x \leq C$ if $x \leq -N$)

Prove (1) $\log x \to \infty$ as $x\to\infty$ (meaning for every M, there exist some N s.t. $x\geq N \Rightarrow \log x \geq M$)

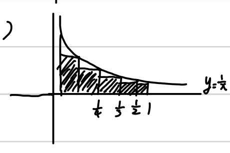
(2) $\log x \to -\infty$ as $x\to 0^+$ (meaning for every $M<0$, there is some $\delta>0$ s.t. if $0<x-0<\delta$, then $\log x < M$)

Proof (1)



$A_x \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{2^L} \geq (L-1)\cdot\frac{1}{2}$, $A_x \to \infty$ as $x\to\infty$

(2)



$A_x = 1\cdot\frac{1}{2} + 2(\frac{1}{2}-\frac{1}{3}) + 3(\frac{1}{3}-\frac{1}{4}) + \cdots\cdots = \frac{1}{2} + \frac{2}{6} + \frac{3}{12} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$

So $A_x \to \infty$ as $x\to 0^+ \Rightarrow \log x = -A_x \to -\infty$

10.11

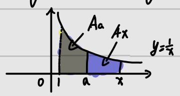
Theorem: $\log x$ is a continuous function

Proof Fix $a \in \text{Domain}(\log) = (0, \infty)$

Case ①: $a > 1$, let $\varepsilon > 0$ Need to find $\delta > 0$ such that $|x-a| < \delta \Rightarrow |\log a - \log a| < \varepsilon$

First Pick $\delta < a-1$ so $|x-a| < \delta \Rightarrow x > 1$

Rough work: $|\log x - \log a| = |A_x - A_a|$ = area under $y = \frac{1}{x}$ between $t=a$ and $t=x$ $\leq$ area of 矩形 with height $\begin{cases} \frac{1}{a}, \text{ if } x>a \\ \frac{1}{x} \text{ if } x<a \end{cases}$



and base $|x-a|$

$= |x-a| \begin{cases} \frac{1}{a} \text{ if } x>a \\ \frac{1}{x} \text{ if } x<a \end{cases}$

?

Looks like squeeze theorm is very natural here

$\Rightarrow \log a - |x-a| \cdot \max\{\frac{1}{x}, \frac{1}{a}\} \leq \log x \leq \log a + |x-a| \cdot \max\{\frac{1}{a}, \frac{1}{x}\}$ for all x "near" a

by squeeze theorem $\log x \to \log a$ as $x \to a$ so $\lim_{x \to a} \log x = \log a$

Case ②: $a = 1$

Case ③: $0 < a < 1$

练习

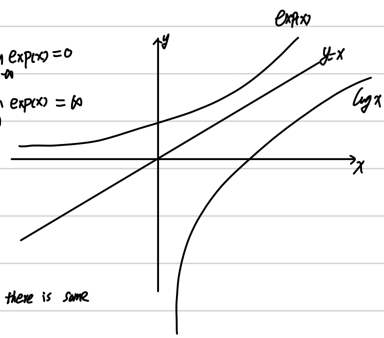
推论

Cor: Range $\log x = (-\infty, \infty)$

Pf apply I.V.T

$\lim_{x \to -\infty} \exp(x) = 0$

$\lim_{x \to \infty} \exp(x) = \infty$



Inverse of log function is call exponetial function $y = \exp x$ (means $\log y = x$)

→ Domain $(-\infty, \infty)$, range $= (0, +\infty)$

Fact $\exp x$ is continuous function since $\log$ is continuous

Since Range $\exp = (0, \infty)$, if $y > 0$ then there is some x s.t. $\exp(x) = y$ (namely $x = \log y$)

$\log(\exp x) = x$

$\exp(\log x) = x$

Fact: $\exp(xr) = (\exp x)^r$ for $r \in \mathbb{Q}$

Follow from $\log(x^r) = r \log x$

Assuming this, let $y = \exp(xr)$

$\log y = \log(\exp xr) = xr = \left(\log(\exp x)\right) \cdot r$

let $z = (\exp x)^r$. Then $\log z = \log((\exp(x))^r) = r \cdot \log(\exp(x)) = \log y$

Since log is one to one, $z = y$

Apply $\exp(xr) = (\exp x)^r$ for $r \in \mathbb{Q}$ with $x = 1$

$$\exp r = (\exp(1))^r$$

Denote by e the number $\exp(1)$, get $\exp r = e^r$, $\forall r \in \mathbb{Q}$

We define $e^x = \exp(x)$ for $x \in \mathbb{R}$

$e \sim 2.71\cdots$ irrational number, Can also define a^x for any $a > 0$ & any $x \in \mathbb{R}$

Know $a = \exp(z)$ for some z (Actually, $z = \log a$)

$a = \exp(\log a) = e^{\log a}$

$a^x = (e^{\log a})^x = e^{x \log a} = \exp(x \cdot \log a)$

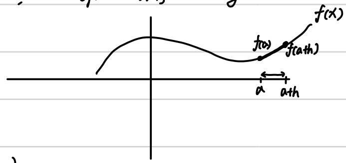
10.21

Differentiation

a can not be end point

Def'n Say f is differentiable at a if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, we define this limit by $f'(a)$

This can be written as $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$



This function f' called the derivative of f

Domain $f' = \{x : f \text{ is differentiable at } x\} \subseteq$ Domain f

$\frac{f(a+h)-f(a)}{h}$ = slope of tangent line from $(a, f(a))$ to $(a+h, f(a+h))$

$f'(a)$ = slope of tangent line to $y=f(x)$ at $x=a$

Tangent line: $y - f(a) = f'(a)\cdot(x-a)$

Examples (1) $f(x) = c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+a)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$$

(2) $f(x) = mx+b$

$$f'(x) = \lim_{h\to 0} \frac{f(x+a)-f(x)}{h} = \lim_{h\to 0} \frac{m(a+x)+b-(mx+b)}{h} = \lim_{h\to 0} \frac{mh}{h} = m$$

(3) $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0} \frac{h\cdot(3x^2+3xh+h^2)}{h} = 3x^2$$

Fact: If f is differentiable at a, then f is continuous at a

Proof Check $\lim_{x\to a} f(x) = f(a)$

Know $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists

$$\lim_{x\to a} (f(x) - f(a)) = \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}\right)\cdot(x-a) = \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}\right)\cdot \lim_{x\to a} (x-a)$$

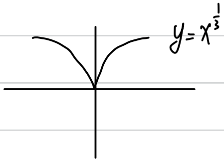
$$= f'(a)\cdot 0 = 0 \Rightarrow f \text{ is continuous at } a$$

Converse is not true

Ex. $f(x) = |x|$ - continuous everywhere, but not diff at 0

Check: $\lim_{x\to 0} \frac{|x|-|0|}{x-0} = \frac{|x|}{x}$

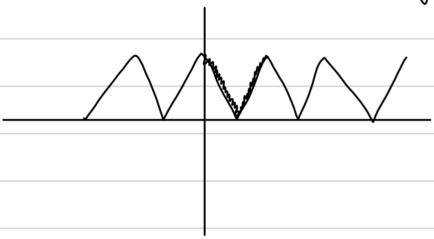
$\lim_{x\to 0^+} \frac{x}{x} = 1$, $\lim_{x\to 0^-} \frac{-x}{x} = -1 \Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ does not exist

Ex. 

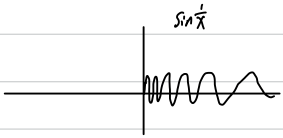
$y' = \frac{1}{3} \cdot x^{-\frac{2}{3}}$ — not defined at $x=0$

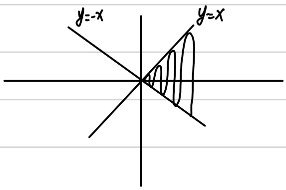
$\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \to 0^-} \frac{1}{x^{\frac{2}{3}}} = +\infty$, does not exist

There exist a function continuous every where and diff no where



Ex. $f(x) = \begin{cases} x \cdot \sin\frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$ — Continuous at 0





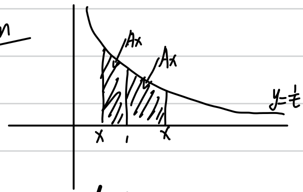
$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h \cdot \sin\frac{1}{h} - 0}{h} = \lim_{h \to 0} \sin\frac{1}{h}$ — does not exist

$\Rightarrow f$ is not diff at 0

Ex $g(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$.

g is diff at 0. but g' is not exist at 0

Logarithm



$$\log x = \begin{cases} A_x & \text{if } x \geq 1 \\ -A_x, & \text{if } x<1 \end{cases}$$

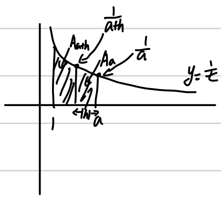
where A_x = area under $y=\frac{1}{t}$ between $t=1, t=x$

$$\lim_{h\to 0} \frac{\log(a+h)-\log a}{h}$$

Case ① $a>0$

$$= \lim_{h\to 0} \frac{A_{a+h}-A_a}{h}$$

Subcase 1(a) $h<0$



$$\frac{|h|}{a} \leq A_a - A_{a+h} \leq \frac{1}{a+h}\cdot|h|$$

$$-\frac{|h|}{a+h} \leq A_{a+h}-A_a \leq -\frac{|h|}{a}$$

$$\frac{h}{a+h} \leq A_{a+h}-A_a \leq \frac{h}{a}$$

$$\frac{1}{a+h} = \frac{h}{(a+h)h} \leq \frac{A_{a+h}-A_a}{h} \leq \frac{h}{a\cdot h} = \frac{1}{a}$$

by squeeze theorem $\lim_{h\to 0} \frac{A_{a+h}-A_a}{h} = \frac{1}{a}$

Case 1b

$$\lim_{h\to 0^+} \frac{A_{a+h}-A_a}{h} = \frac{1}{a}$$

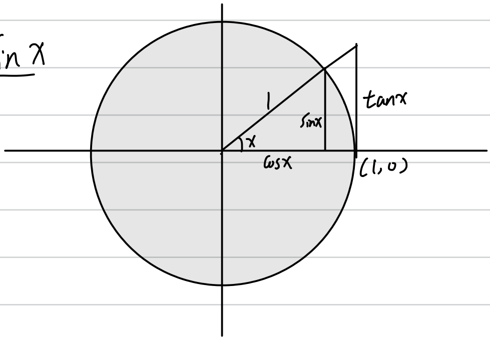
Conclusion: For $a>1$, $\log x$ is diff at a and $\frac{d}{dx}\log x\Big|_{x=a} = \frac{1}{a}$

exercise: Case 2, 3: $a<1$, $a=1$

$$\frac{d}{dx}\log x = \frac{1}{x}, \quad \forall x>0$$

10.23

<u>Sin x</u>



For $x \in [0, \frac{\pi}{4}]$

$0 \le \sin x \le x$

$\lim_{x \to 0^+} \sin x = 0$

Similarly, $\lim_{x \to 0^-} \sin x = 0$

$\therefore \lim_{x \to 0} \sin x = 0$

Area small $\triangle = \frac{1}{2} \cdot \cos x \cdot \sin x$

$\le$ Area of sector of c $\quad \pi \cdot \frac{x}{2\pi} = \frac{x}{2}$

$\le$ Area big $\triangle = \frac{1}{2} \tan x = \frac{1}{2} \cdot \frac{\sin x}{\cos x}$

mulpy by $\frac{2}{\sin x}$ $\Rightarrow$ $\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$

$\cos^2 x = 1 - \sin^2 x \longrightarrow 1$ as $x \to 0$ and $\cos x > 0$ near 0 $\Rightarrow \cos x = +1$ as $x \to 0$

$\Rightarrow \lim_{x \to 0} \frac{x}{\sin x} = 1$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1+\cos x)} = \lim_{x \to 0} \underbrace{\left(\frac{\sin x}{x}\right)}_{\downarrow\, 1} \cdot \underbrace{\left(\frac{\sin x}{1+\cos x}\right)}_{\downarrow\, 0} = 0$$

Derivative of Sine

$$\lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} = \lim_{h \to 0} \frac{\sin a \cdot \cos h + \sin h \cos a - \sin a}{h} = \lim_{h \to 0} \underbrace{\frac{\sin a (\cos h - 1)}{h}}_{\downarrow\, 0} + \underbrace{\frac{\sin h \cos a}{h}}_{\downarrow\, 1 \times \cos a} = \cos a$$

$$\frac{d}{dx} \cos x \Big|_{x=a} = \lim_{h \to 0} \frac{\cos(a+h) - \cos a}{h} = \lim_{h \to 0} \frac{\cos a \cdot \cos h - \sin a \sin h - \cos a}{h} = \lim_{h \to 0} \cos a \underbrace{\left(\frac{\cos h - 1}{h}\right)}_{\downarrow\, 0} - \underbrace{\sin a \cdot \frac{\sin h}{h}}_{\downarrow\, -\sin a \times 1} = -\sin a$$

<u>Cor</u>: Sin. Cos are continuous funcion

Rule of Differentiation

Assume f, g are diff at a

(1) $f \pm g$ is diff at $a \Rightarrow (f \pm g)'(a) = f'(a) \pm g'(a)$

(2) $f \cdot g$ is diff at $(fg)'(a) = f'(a) \cdot g(a) + g'(a) \cdot f(a)$

Pf $(fg)'(a) = \lim_{h \to 0} \frac{fg(a+h) - fg(a)}{h} = \lim_{h \to 0} \frac{(f(a+h) \cdot g(a+h) - f(a+h) g(a)) + (f(a+h) g(a) - f(a) \cdot g(a))}{h}$

$= \lim_{h \to 0} \frac{f(a+h) \cdot (g(a+h) - g(a))}{h} + \frac{g(a) \cdot (f(a+h) - f(a))}{h} = f(a) \cdot g'(a) + g(a) \cdot f'(a)$

Notice f diff at $a \Rightarrow f$ is cont at a, Hence $\lim_{h \to 0} f(a+h) = f(a)$

(3) $(c \cdot f)'(a) = c \cdot f'(a)$

(4) $\frac{d}{dx} x^n = n \cdot x^{n-1}$

Pf use induction on n

True for $n=1$

$x^n = x^{n-1} \cdot x$, by Produce rule $\frac{dx^n}{dx} = (n-1) \cdot x^{n-2} \cdot x + 1 \cdot x^{n-1}$

$= (n-1+1) \cdot x^{n-1} = n \cdot x^{n-1}$

Cor $\frac{d}{dx} (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_1 + 2a_2 + 3a_3 x^2 + \cdots + n \cdot a_n \cdot x^{n-1}$

(5) $(\frac{1}{g})'(a) = -\frac{g'(a)}{g(a)^2}$ if $g(a) \neq 0$ (H.W.)

Cor $\frac{dx^{-n}}{dx} = \frac{d \frac{1}{x^n}}{dx} = \frac{-n \cdot x^{n-1}}{x^{2n}} = -n\, x^{-n-1}$, $x \in \mathbb{N}$

(6) Quotient rule $(\frac{f}{g})'(a) = \frac{f'(a) \cdot g(a) - g'(a) f(a)}{g(a)^2}$

Pf $(\frac{f}{g})'(a) = (f \cdot \frac{1}{g})'(a) = f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot (\frac{1}{g})'(a)$

$= \frac{f'(a)}{g(a)} - \frac{g'(a) \cdot f(a)}{g(a)^2} = \frac{f'(a) g(a) - g'(a) f(a)}{g(a)^2}$

Ex. $\frac{d}{dx} \tan x = \frac{d}{dx} \cdot \frac{\sin x}{\cos x}$

$= \frac{\cos \cdot \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Chain Rule $f: A \to B \subseteq \mathbb{R}$, $g: B \to \mathbb{R}$. If f is diff at a and g is diff at $f(a)$

Then $g \circ f$ is diff at a and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Proof Rough work

$$(g \circ f)' = \lim_{h \to 0} \frac{g \circ f(a+h) - g \circ f(a)}{h} = \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h \cdot (f(a+h) - f(a))} \cdot \frac{f(a+h) - f(a)}{1}.$$

$$= \lim_{h \to 0} \underbrace{\frac{g(\overset{H}{f(a+h)}) - g(\overset{a}{f(a)})}{\underset{H \;-\; a}{f(a+h) - f(a)}}}_{g'(f(a))} \cdot \underbrace{\frac{f(a+h) - f(a)}{x-a}}_{f'(a)}.$$

$H = f(a)$, as $x \to a$, $H \to f(a) = a$

$$\lim_{H \to a} \frac{g(H) - g(a)}{H - a} = g'(a) = g'(f(a))$$

Caratheodory Theorem: If F is diff at a, then there is a function Φ which is continuous at a and $F(x) - F(a) = \Phi(x) \cdot (x-a)$ for all x. In addition, $\Phi(a) = F'(a)$

Pf Take $\Phi(x) = \begin{cases} \frac{F(x) - F(a)}{x-a} & \text{if } x \neq 0 \\ F'(a) & \text{if } x = a \end{cases}$

$$\lim_{x \to a} \Phi(x) = \lim_{x \to a} \frac{F(x) - F(a)}{x-a} = F'(a) = \Phi(a)$$

So Φ is continuous, it satisfies $F(x) - F(a) = \Phi(x)(x-a)$, $\forall x \in \mathcal{D}(f)$ and $\phi(a) = F'(a)$

10.28

Chain Rule:

□: $f: A \to \mathbb{R}$, g: Range $f \to \mathbb{R}$, if f is diff at a and g is diff at $f(a)$ then $g \circ f$ is diff at a and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Proof: f is diff at a, so by Caratheodory theorm, there is some ϕ continuous at a, $\phi(a) = f'(a)$ and
$f(x) - f(a) = \phi(x)(x-a)$ (**)

Similarly, there is a function φ cont at $f(a)$, $\varphi(f(a)) = g'(f(a))$ and $g(z) - g(f(a)) = \varphi(z) \cdot (z - f(a))$
(*)

Want to understand
$$\frac{g \circ f(x) - g \circ f(a)}{x-a} = \frac{g(f(x)) - g(f(a))}{x-a}$$
Letting $z = f(x)$ & plugging in to (*) gives $g(f(x)) - g(f(a)) = \varphi(f(x)) \cdot (f(x) - f(a))$
Plugging **, $= \varphi(f(x)) \cdot \phi(x) \cdot (x-a)$
This gives $\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} = \lim_{x\to a} \varphi(f(x)) \cdot \phi(x)$

$\varphi(f(x)) \to \varphi(f(a))$, $\phi(x) \xrightarrow{x \to a} \phi(a)$

$$= \varphi(f(a)) \cdot \phi(a)$$
$$= g'(f(a)) \cdot f'(a)$$

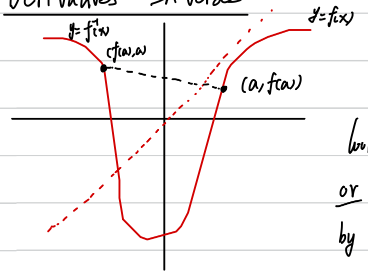
Ex.

(1) $y = (3x^2 + x + 1)^{17}$
$y' = 17(3x^2 + x + 1)^{16} \cdot (6x+1)$

(2) $y = \cos x = \sin(x + \frac{\pi}{2})$
$y' = \cos(x + \frac{\pi}{2}) = -\sin x$

(3) $y = \ln(\cos x)$
$y' = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$

Derivatives Inverses



looks like $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$, at least, if $f'(a) \neq 0$

or $f^{-1} \circ f(x) = x$ ← weakness?

by chain rule $(f^{-1})'(f(x)) \cdot f'(x) = 1$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem let $f: (c,d) \to \mathbb{R}$ be cont, 1-1 function on (c,d)
Suppose f is diff at $a \in (c,d)$ and $f'(a) \neq 0$, Then f^{-1} is diff at $f(a)$ and
$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$
(Write $b = f(a) = f^{-1}(b) = a$, and see $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$)

Proof Put $b = f(a)$, our interest is in $\lim_{h\to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h}$

$b+h \in D(f^{-1}) = R(f)$ so

$b+h = f(a+k)$ for some k

Think of k as a function of h

$f^{-1}(b+h) = a+k$, $\lim_{h\to 0} \frac{a+k-a}{b+h-b} = \lim_{h\to 0} \frac{k}{f(a+k)-f(a)}$

$$= \lim_{h\to 0} \frac{1}{\frac{f(a+k)-f(a)}{k}}$$

$k = \underbrace{f^{-1}(b+h)}_{=a+k} - \underbrace{f^{-1}(b)}_{=a}$

$\lim_{h\to 0} k = \lim_{h\to 0} f^{-1}(b+h) - f^{-1}(b) = 0$ since f^{-1} is continuous at b

because f is a continuous function on (c,d)

Similarly, $h = \underbrace{f(a+k)}_{=b+h} - \underbrace{f(a)}_{=b}$, so $\lim_{k\to 0} h = \lim_{k\to 0} f(a+k) - f(a) = 0$ Since f is continuous

Hence $h \to 0 \Leftrightarrow k \to 0$

$$= \lim_{k\to 0} \frac{1}{\frac{f(a+k)-f(a)}{k}}$$

$$= \frac{1}{f'(a)}$$ so f^{-1} is diff at $f(a)$, & $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$

by quotient rule
lim as $f'(a) \neq 0$

Ex. (1) $y = x^{\frac{1}{n}} = f(x)$, $n \in N$, $n \neq 1$

$f = g^{-1}$ for $g(x) = x^n \Rightarrow g'(x) = n \cdot x^{n-1}$. $g'(x) = 0$ iff (if only if) $x = 0$

$(g^{-1})'(g(a)) = \frac{1}{g'(a)} \Leftrightarrow f'(g(a)) = \frac{1}{na^{n-1}} = f'(a^n)$

$b = a^n$, $b^{\frac{1}{n}} = a \Rightarrow f'(b) = \frac{1}{n \cdot b^{\frac{1}{n}(n-1)}} = \frac{1}{n} \cdot \frac{1}{b^{1-\frac{1}{n}}} = \frac{1}{n} \cdot b^{\frac{1}{n}-1}$

(2) $y = \frac{p}{q}$, $\frac{p}{q} \in Q$

$= (x^{\frac{1}{q}})^p$

(3) $y = \exp x = (\log)^{-1}(x)$

$\frac{d \log x}{dx} = \frac{1}{x} \neq 0$ so exp is diff everywhere , $g(x) = \log x$. $\frac{d \exp x}{dx} = \frac{1}{g'(\exp x)} = \frac{1}{\frac{1}{\exp x}}$

$= \exp x$

$\exp x = e^x$ (def'n)

$\frac{de^x}{dx} = e^x$

Ex. $y = \exp(\cos 2x)$

$y' = \exp(\cos 2x) \cdot (-2 \cdot \sin 2x)$

10.30

Derivatives of Inverses

$f^{-1} \circ f(x) = x \qquad (f^{-1})'(f(x)) = \frac{1}{f'(x)}$ Provided $f'(x) \neq 0$

$(f^{-1})'(f(x)) \cdot f'(x) = 1 \qquad (f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$

(if $f'(x) \neq 0$)

Ex $f(x) = \exp x$

$g(x) = \log x$, $f(x) = g^{-1}(x)$

$f'(z) = (g^{-1})'(z) = \frac{1}{g'(g^{-1}(z))} = \frac{1}{\frac{1}{g^{-1}(z)}} = g^{-1}(z) = \exp z$

$\frac{d}{dz} \exp z \quad \forall z \in R$

$f(x) = \arcsin x$

$f(x) = g^{-1}(x)$ where $g(x) = \sin x$ restricted to $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$g'(x) = \cos x$

$\neq 0$ except at $\pm\frac{\pi}{2}$

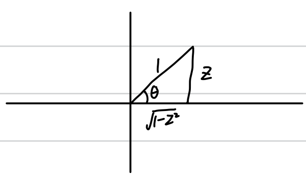
$\Rightarrow f$ is diff at $g(\pm\frac{\pi}{2}) = \pm 1$

$f'(z) = (g^{-1})'(z) = \frac{1}{g'(g^{-1}(z))} = \frac{1}{\cos(\underbrace{\arcsin z}_{\theta})}$, $\sin\theta = z$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

want $\cos\theta$

$\Longrightarrow f'(z) = \frac{1}{\sqrt{1-z^2}}$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$



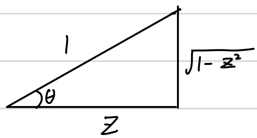
$f(x) = \arccos x \quad f = g^{-1}$ where

$g(x) = \cos x$ where $x \in [0, \pi]$

$f'(z) = \dfrac{1}{g'(f(z))} = \dfrac{1}{-\sin(\arccos x)}$ for $z \neq \pm 1$

$\arccos x = \theta$

$= -\dfrac{1}{\sqrt{1-z^2}}$



$f(x) = \arctan x$

$g(x) = \tan x$ with $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

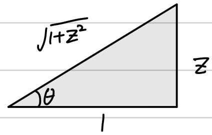
$g'(x) = \sec^2 x \geq 1$ for all x

so $\arctan x$ is diff at all $x \in \mathbb{R}$

$f'(z) = \dfrac{1}{g'(\arctan z)} = \dfrac{1}{\sec^2(\arctan z)} = \cos^2(\arctan z)$

$\arctan z = \theta$

$= \left(\dfrac{1}{\sqrt{1+z^2}}\right)^2 = \dfrac{1}{1+z^2}$



Ex. (1) $y = \exp x^3 \arcsin\sqrt{x}$

$y' = (\exp x^3) \cdot 3x^2 \arcsin\sqrt{x} + \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2} x^{-\frac{1}{2}} \cdot \exp(x^3)$

(2) $y = \dfrac{1}{\arctan(\log(x+1))}$

$y' = \dfrac{-\frac{1}{1+(\log(x+1))^2} \cdot \frac{1}{x+1}}{(\arctan(\log(x+1)))^2}$

Implice Differentiation

e.g. $x^2 + y^2 = 1$, $y \geqslant 0$

$\Rightarrow y = \sqrt{1-x^2}$

e.g. $y^5 + y + x = 0$

$5 \cdot y^4 \cdot y' + y' + 1 = 0 \Rightarrow y' = -\frac{1}{5y^4+1}$

Ex. Find $\frac{dy}{dx}$ at Point $(1,1)$

for $x^7y + y^7x = 2$

$7x^6y + x^7y' + 7y^6 \cdot y'x + y^7 = 0$

$7 + y' + 7y' + 1 = 0$

$8y' = -8$ at $(1,1)$ so $y' = -1$

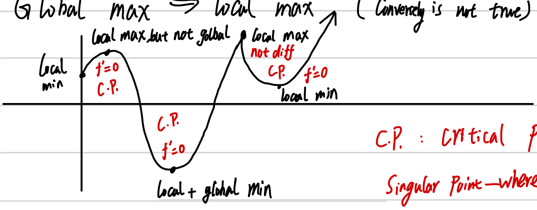
Significance of the Derivative

Optimization

<u>Def'n</u>: a point x is a local maximum for f if there is some $\delta > 0$ if $y \in (x-\delta, x+\delta)$ and $y \in$ Domain f, then $f(y) \le f(x)$

x is a global maximum if $f(y) \le f(x)$ for all $y \in$ Domain f

Global max $\Rightarrow$ local max (Conversely is not true)



E.V.T is about global min/max

C.P. : critical point - where $f'=0$

Singular Point — where f' diff not exist

11.1

Critical Points Theorem

If f has a local max or min at $x \in (a,b) \subseteq (f)$ and if f is diff at x, then $f'(x) = 0$ or f is not diff

Proof f is diff at x (local max) so $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exist. Get $\delta > 0$ from def'n if if local max. if $z \in (x-\delta, x+\delta)$, then

$$f(x) \geq f(z)$$

If $|h| < \delta$, then $x+h \in (x-\delta, x+\delta)$

$\therefore$ $f(x+h) - f(x) \leq 0$

If $0 < h < \delta$, the $\frac{f(x+h) - f(x)}{h} \leq 0$, So $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0$

If $-\delta < h < 0$, then $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \geq 0$

Then $\underbrace{\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}}_{= f'(x)} = 0$ (Since we know the limit exists)

Ex. let $f(x) = x - x^{\frac{2}{3}}$ on $[-1, 8]$. Find the global max/min if they exist

Answers We know the global max/min exist because of E.V.T. a fact that f is cont on closed interval $[-1, 8]$, we'll find the all local max/min from the determine the global max/min

Candidates End Point $x = -1, 8$

Critical Point Slve $f'(x) = 0 \Rightarrow 1 - \frac{2}{3} x^{-\frac{1}{3}} = 0 \Rightarrow x = \frac{8}{27}$

($f'(x)$ 不存在) S.P. $x = 0$

Evaluate these Points : $f(-1) = -2$, $f(0) = 0$, $f(\frac{8}{27}) = -\frac{4}{27}$, $f(8) = 4$

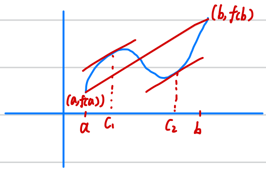
min max

Concluding Sentence

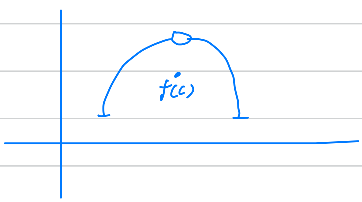
Mean Value Theorem

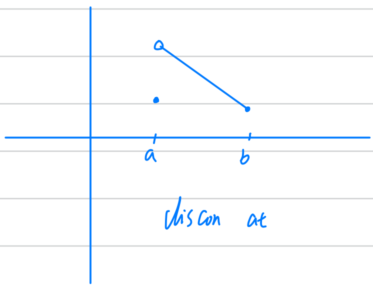
If f is Continuous on $x \in [a,b]$ and differentiable on (a,b), then There is $c \in (a,b)$ s.t.

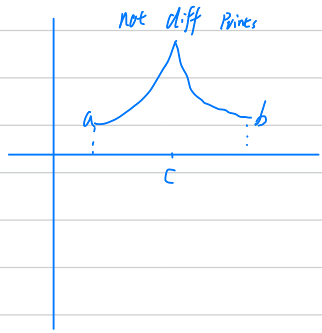
average rate of change of f on $[a,b]$ $\quad \frac{f(b)-f(a)}{b-a} = f'(c) \quad$ instantances rates of changes



Examples to see necessarity of hypotheses







First we will Prove

Rolle's Theorem : If f is Contin on $[a,b]$, diff on (a,b) and $f(a) = f(b)$, then there is $c \in (a,b)$ s.t. $f'(c) = 0$

Proof Since f is Con at $[a,b]$ it has a global max/min by E.V.T. If one of these occurs at some $c \in (a,b)$, then Since f is diff there. $f'(c) = 0$ by C.P Theorem, and we done

Otherwise, the global max/min occur at a, b $\circ$ $f(a) = f(b)$ and that implies f is Constant

so $f'(c) = 0$ for every $c \in (a,b)$

Proof of M.V.T

Define $L(x)$ = Secant line there $(a, f(a))$, $(b, f(b))$

let $g(x) = f(x) - L(x)$, g is cont on $[a,b]$, diff on (a,b)

und $g(b) = g(a) = 0$

By Rolle's Theorem, there exist $c \in (a,b)$ where $g'(c) = 0$

But $g'(x) = f'(x) - L'(x)$

$= f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)$ (derivative of a linear function is its slope)

$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$

Cor If $f'(x) = 0$ at every $x \in$ Interval I, then f is constant on I

Proof let $a<b$, $a,b \in I$, f is con on $[a,b]$ und diff on (a,b)

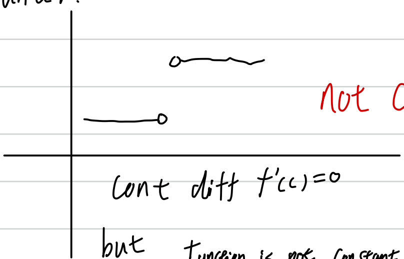
because f is diff on I

By M.V.T, there exist $c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$, Since $c \in I \Rightarrow f'(c) = 0$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f$ is constant at the interval

Counter. e.x.



Not Complete interval

Cont diff $f'(c) = 0$

but function is not Constant

11.4

MVT : f cont on $[a,b]$ and diff on (a,b), then there exises $c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$

Cor: If $f'(x)=0$ for every $x \in$ interval I then $f =$ constant

Cor: If $f'(x) = g'(x)$ for every $x \in I$, then $f(x) = g(x) +$ constant

Pf: $(f-g)'=0$ on I

Ex. Prove $\arcsin x + \arccos x = \frac{\pi}{2}$

Answer. Diff LHS: $\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$ for every $x \in (-1,1)$

Therefore $\arcsin x + \arccos x =$ constant

Evaluate at $x=0$: $0 + \frac{\pi}{2} =$ constant

Prove Properties of log function

(1) $\log ab = \log a + \log b$, for $a, b > 0$

(2) $\log \frac{1}{a} = -\log a$

(3) $\log a^r = r \log a$ if $a>0$, $r \in \mathbb{Q}$

Pf (1): let $f(x) = \log xb - (\log x + \log b)$

If $x=1$, $f(1) = 0$

Check $f'(x) = \frac{1}{xb} \cdot b - \frac{1}{x} = 0$ for all $x>0$

Therefore $f =$ Constant $= f(1) = 0$

(3) let $f(x) = \log x^r - r \log x$

$f'(x) = \frac{1}{x^r} \cdot r \cdot x^{r-1} - \frac{r}{x} = 0$ for all $x>0$

$f(1) = 0$

$\therefore f(x) = 0$ for all $x > 0$

(4) $\exp(xr) = (\exp x)^r$ for all $x \in \mathbb{R}, r \in \mathbb{Q}$

Pf: let $y_1 = \exp(xr)$

$y_2 = (\exp x)^r$

Take log. $\log y_1 = \log(\exp xr) = xr$

$\log y_2 = \log(\exp x)^r = r \cdot \log(\exp x) = rx$

Since log is 1-1 then implies $y_1 = y_2$

Recall $e = \exp(1)$: $\exp r = \exp(1 r) = (\exp 1)^r = e^r$ for $r \in \mathbb{Q}$

Def'n: For any $x \in \mathbb{R}$, define $e^x = \exp(x)$

Consistent when $x \in \mathbb{Q}$ when e^x is already known

Def'n for $a > 0$, define $a^x = \exp(x \log a)$

Notice if $x \in \mathbb{Q}$, then $\exp(x \log a) = \exp(\log a^x) = a^x$, So our new def'n extends the already known of a^x for $x \in \mathbb{Q}$

Notice $\log(a^x) = \log(\exp x \cdot \log a) = x \log a$ for all $x \in \mathbb{R}$ and $a > 0$

Ex. $(a^b)^c = a^{bc}$

$a^b \cdot a^c = a^{b+c}$

$$\underbrace{\exp(x+y) = e^{x+y}}_{\text{def'n}} \overset{\text{to prove}}{=} \underbrace{e^x e^y = (\exp x) \cdot (\exp y)}_{\text{def'n}}$$

Finish Differentiation Rules

(1) $y = x^r$, $x > 0$, $r \in \mathbb{R}$

$= \exp(r \log x)$

$y' = \exp(r \log x)\left(\frac{r}{x}\right)$

$= x^r \cdot \frac{r}{x} = r \cdot x^{r-1}$

(2) $y = 2^x = \exp(x \log 2)$

$y' = \exp(x \log 2) \cdot (\log 2) = 2^x \cdot \log 2$

(3) $y = x^{\tan x} = \exp(\ln x^{\tan x}) = \exp(\tan x \cdot \ln x)$

$y' = \exp(\tan x \cdot \ln x) \cdot \left(\sec^2 x \cdot \ln x + \frac{1}{x} \cdot \tan x\right)$

$= x^{\tan x}\left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right)$

(4) $y = (\arcsin x)^x = \exp(x \cdot \log \arcsin x)$

$y' = \exp(x \cdot \log \arcsin x) \cdot \left(\log \arcsin x + x \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}\right)$

$= (\arcsin x)^x \cdot \left(\log \arcsin x + x \cdot \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}\right)$

More application of M.V.T.

Recall: Say f is (strictly) increasing on interval I if whenever $x<y$, $x,y \in I$

Then $f(x) \le f(y)$ ($f(x) < f(y)$)

Cor: If $f'(x) \geq 0$ (>0) for every $x \in (a,b)$ and f is cont on $[a,b]$ the f is (strict) increasing on $[a,b]$

Pf: let $a \le x < y \le b$, by MVT. there is $c \in (x,y)$ s.t. $f'(c) = \frac{f(y)-f(x)}{y-x} \geq 0 \Rightarrow f(y) \geq f(x)$

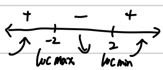
Partial Converse: If f is diff on (a,b) and f is increasing on (a,b) then $f'(x) \geq 0$ at every $x \in (a,b)$

(Note: $y = x^3$ is strictly increasing, but $f'(x) = 0$ at $x=0$)

Pf: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \geq 0$ as f is $\uparrow$

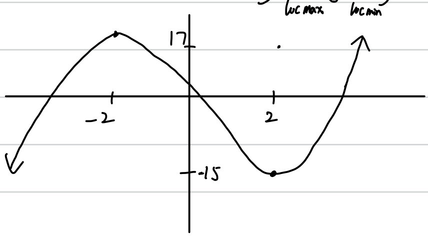
Ex. $f(x) = x^3 - 12x + 1 \Rightarrow f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$

C.P. $x = 2, -2$. Diff everywhere



f'

$\lim_{x \to \pm\infty} f(x) = \pm\infty$ because deg of Polynomial is odd with leading coeff $= +1$



11. 6

First Dervative Test

Assume f is Continuous on $[a,b]$ and $c \in (a,b)$

(1) If $f' \geq 0$ on (a,c) and $f' \leq 0$ on (c,b), this c is local maximum

(2) If $f' \leq 0$ on (a,c) and $f' \geq 0$ on (c,b), this c is local minimum

(3) If f' is on both sides ($f>0$ / $f<0$ on both sides), then c is neither local max/min

Pf ~ Immediate from increasing funcion theorem

Ex. $f(x) = x \cdot e^{-x}$ on $[0, +\infty)$ and determine if there are any global max/min

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$ Candidates: E.P. $x=0$

C.P. $x=1$ (Since $e^{-x} > 0$ $\forall x$)

$\Rightarrow f(0)=0$ - global min : $f(x) \geq 0$ for all $x \in [0,\infty)$, $f(x) >$ for all $x \in (0,+\infty)$

Ex. $y = x^{\frac{5}{3}} + 5x^{\frac{2}{3}}$

Analyze extrema : Continuous everywhere, differenable everywhere except 0

$y' = \frac{5}{3}x^{\frac{2}{3}} + \frac{10}{3}x^{-\frac{1}{3}} = \frac{5}{3}x^{\frac{2}{3}} + \frac{10}{3x^{\frac{1}{3}}} = \frac{5x+10}{3 \cdot x^{\frac{1}{3}}}$

C.P. $x=-2$

S.P. $x=0$

+   −   +

-2 local max 0 local min

$\uparrow$ $\downarrow$ $\uparrow$

$y \to +\infty$ as $x \to +\infty$

$y \to -\infty$ as $x \to -\infty$

] no global min/max

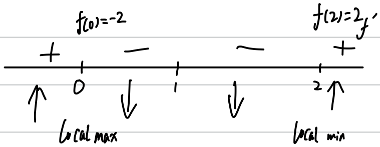
Graph $f(x) = \frac{x^2-2x+2}{x-1}$

Domain: $x \neq 1$, Continuous on its domain

Hence V.A $x=1$ (Vertical asymptote: $x=a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$)

$\lim_{x\to 1^+} f = +\infty$, $\lim_{x\to 1^-} f = -\infty$

$f'(x) = \frac{x(x-2)}{(x-1)^2}$: C.P: $x = 0, 2$



$f(0) = -2$

$f(2) = 2$

Horizontal Asympote: $y=b$ if $\lim_{x\to\pm\infty} f(x) = b$

Oblique Asymptote: $y = mx+b$, $m \neq 0$, $\lim_{x\to\pm\infty} [f(x)-(mx+b)] = 0$

If $f = \frac{\text{Ploy}: P}{\text{Ploy}: Q}$ get O.A if $\deg P = 1 + \deg Q$

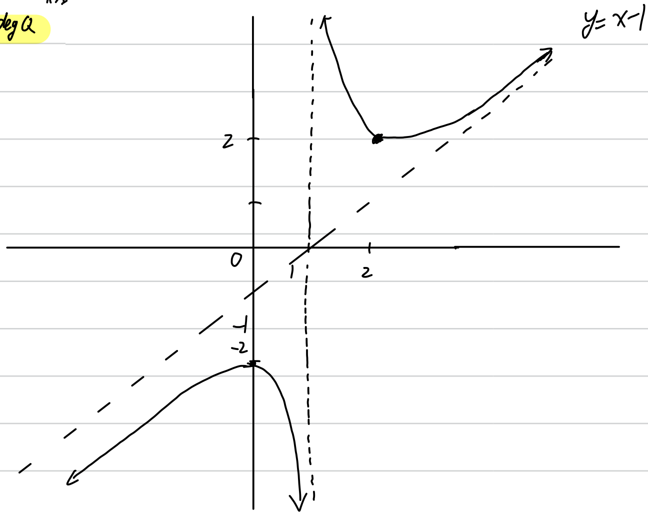
Get H.A if $\deg P \leq \deg Q$

To find O.A. long division on f

$f(x) = x-1+\frac{1}{x-1}$

$\lim_{x\to\pm\infty} f(x)-(x-1) = \lim_{x\to\pm\infty} \frac{1}{x-1} = 0$

Therefore $y = x-1$ is the O.A.



Find asymptotes

Ex. $f(x) = \frac{1}{x^2-x} = \frac{1}{x(x-1)}$

V.A. $x = 0, 1$

H.A $y = 0$

Ex. $f(x) = \frac{x^2-1}{x^2-4} = \frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}}$

V.A. $x = \pm 2$

H.A $y = 1$

even function $f(x) = f(-x)$

Ex. $y = \frac{2x^3+x^2+1}{x^2}$

V.A: $x = 0$

O.A: $y = 2x+1$

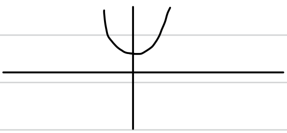
Second Derivative

Def'n say f is <u>concave up</u> on Interval I

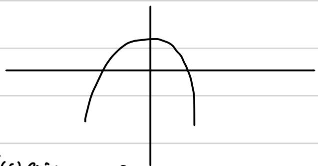
If $f'(x)$ is strictly increasing on I

say f is <u>concave down</u> on I if $f'(x)$ is strictly decreasing on I

Concave up



Concave down

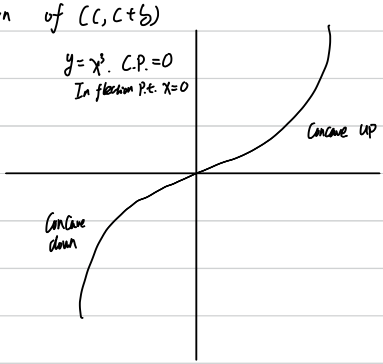


Call Inflection Point if $f'(c)=0$, if $f'(c)$ exises at c

and the concavity of f changes at c

$\exists\ \delta>0$ s.t. f is concave up on $(c-\delta, c)$

and concaved down of $(c, c+\delta)$



11.8

Second Derivate

Say f is Concave up (on I) if $f' \nearrow$ in I : $\cup$

Concave down if $f' \searrow$ in I : $\cap$

c - Inflection Point if ($f'(c)\ \exists$) and Concavity changes at c

Theorem: (1) If $f'' \geq 0$ (≤ 0) on I then f is Concave up (down) on I

(2) If f have an I.P. at c and $f''(c)$ exists, then $f''(c) = 0$

Pf: (1) $f'' \geq 0$ on $I \Rightarrow f'$ is increasing on I by increasing function theorem

$\therefore f$ is Concave up

(2) Suppose f' increasing on left of c (say Interval: I_L) and f' decrease on right of c (Say Interval: I_R)

Then if $x \in I_L$, then $f'(x) \leq f'(c)$

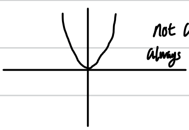
$x \in I_R$, then $f'(x) \leq f'(c)$

So $f'(c)$ is local max of f'

f' is diff at c, so $f''(c) = 0$ by. C.P. Thm

Converse of (2) is not true

e.g. $f(x) = x^4$, $f''(0) = 0$



not an I.P.

always concave up

Second Derivate Test

Suppose $f'(c)=0$

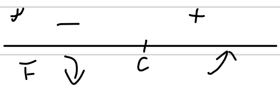
(1) If $f''(c)>0$ then f have a local min at c

(<0) (max)

(2) If $f''(c)=0$, then anything is possible

Proof (1) $f''(c)=\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h} \Rightarrow \lim_{h\to 0}\frac{f'(c+h)}{h}\geq 0$ for all small h

If $h>0$, $f'(c+h)\geq 0$ and $h<0$, $f'(c+h)\leq 0$



$\Rightarrow$ c is local min of F by first derivative test

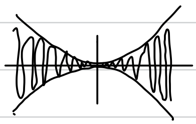
(2) $f(x)=x^4$, $c=0$ is a local min and $f''(c)=0$

$g(x)=-x^4$. - - - - - max . - - -

$h(x)=x^3$, $f'(0)=f''(0)=0 \Rightarrow$ neither local max/min ex. where $f'(0) = f''(0) = 0$, not local max/min on I.P.

$$f(x)\begin{cases} x^4\cdot\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

No local min/max at 0 since $x^4\cdot\sin\frac{1}{x}$ takes on both >0 / <0 value arbitrarily close to 0



$f'(x) = 4x^3\cdot\sin\frac{1}{x} - x^2\cdot\cos\frac{1}{x}$, $x\neq 0$

$f'(0) = 0$

$f''(0) = \lim_{h\to 0}\frac{f'(h)-f'(0)}{h} = \lim_{h\to 0} 4x^2\sin\frac{1}{x} - x\cos\frac{1}{x} = 0$

$x\neq 0$ $\quad f''(x) = 12x^2\cdot\sin\frac{1}{x} + 4x^3\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right) - 2x\cos\frac{1}{x} + x^2\sin\frac{1}{x}\left(-\frac{1}{x^2}\right)$

$= 12x^2\cdot\sin\frac{1}{x} - 6x\cos\frac{1}{x} - \sin\frac{1}{x}$

Remark 1. $\lim_{x\to 0} 12x^2\sin\frac{1}{x} = 0 = \lim_{x\to 0} 6x\cos\frac{1}{x}$. Claim: Take any interval $(0,\varepsilon)$, $\varepsilon>0$. Then there are interval $I_1, I_2\subset(0,\varepsilon)$. s.t. $f''(x)>0$ on I_1 (CU) and $f''(0)<0$ on I_2 (CD)

This claim prove 0 is not I.P.

First pick $0<\varepsilon_1<\varepsilon$ s.t. $|12x^2\cdot\sin\frac{1}{x}|<\frac{1}{4}$ and $|6x\cdot\cos\frac{1}{x}|<\frac{1}{4}$ $\forall x\in(0,\varepsilon_1)$

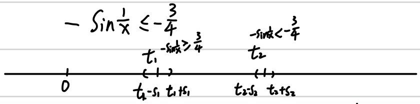
Remark 2. There are points $t_1, t_2\in(0,\varepsilon_1)$ where $-\sin\frac{1}{t_1}=1$ and $-\sin\frac{1}{t_2}=-1$

Remark 3. $f''(x)$ is continuous at t_1 and t_2 and $-\sin\frac{1}{x}$ cont at t_1, t_2

Pick $\delta_1,\delta_2>0$ s.t. $(t_j-\delta_j, t_j+\delta_j)\subset(0,\varepsilon_1)$ for both $j=1,2$

and $-\sin\frac{1}{x}\geq\frac{3}{4}$ on $(t_1-\delta_1, t_2-\delta_2)$

$-\sin\frac{1}{x}\leq-\frac{3}{4}$



Let $x\in(t_1-\delta_1, t_1+\delta_1)$ then $f''(x)\geq-\frac{1}{4}-\frac{1}{4}+\frac{3}{4}=\frac{1}{4}$. Call $I_1\,(t_1-\delta_1, t_1+\delta_1)$

Let $x\in(t_2-\delta_2, t_2+\delta_2)\equiv I_2$, then $f''(x)\leq\frac{1}{4}+\frac{1}{4}-\frac{3}{4}=-\frac{1}{4}<0$

This prove the claim (so within every interval $(0,\varepsilon)$, concavity switches)

11.11

L'Hôpital's Rule (1) $\frac{0}{0}$ $\quad (\varepsilon < 1)$

Assume f, g are differentiable on interval $I = [a-\delta, a+\delta]$, except possibly at a.

Assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ and either $\lim_{x\to a} f(x) = 0$ or ∞, Suppose g, g' are non-zero on I, except perhaps at a

If $\lim_{x\to a} \frac{f(x)}{g(x)} = L \in R$, then $\lim_{x\to a} \frac{f}{g} = L$

引入: Cauchy Mean Value Thm: If f, g are continuous on $[a,b]$ and diff on (a,b). Then there is some $c \in (a,b)$ s.t. $(f(b)-f(a))\, g'(c) = f'(c)\,(g(b)-g(a))$

Proof let $h(x) = f(x)\cdot(g(b)-g(a)) - g(x)\cdot(f(b)-f(a))$. h is continuous on $[a,b]$ and h is diff on (a,b)

so MVT apply to $h \Rightarrow h(b)-h(a) = h'(c)\cdot(b-a)$ for some $c \in (a,b)$ except $h(b) = h(a) = f(b)\cdot g(a) + g(b)\cdot f(a)$

Therefore, $h'(c) = 0$, $h'(x) = f'(x)\,(g(b)-g(a)) - g'(x)\cdot(f(b)-f(a))$

$h'(c) = 0 \Rightarrow f'(c) = (g(b)-g(a)) = g'(c)\cdot(f(b)-f(a))$

Proof. Case ① $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

Recall: f, g were cont and diff on $I = [a-\delta_0, a+\delta_0]$ for some $\delta_0 > 0$ except possibly at a

(Re) define f and g at a by setting: $f(a) = g(a) = 0$. This does not change f' or g' at points $x \neq a$

our new f, g are still diff on I except possibly at a

our new f, g are cont on all of I since $\lim_{x\to a} f = f(a) = 0$

To see that $\lim_{x\to a} \frac{f}{g} = L$, we have take any $\varepsilon > 0$ and find $\delta > 0$ s.t. $0 < |x-a| < \delta$, then $\left|\frac{f}{g}(x) - L\right| < \varepsilon$

Know $\lim_{x\to a} \frac{f'}{g'} = L$, let $\varepsilon > 0$ Then there exist $\delta_1 > 0$ s.t. $0 < |z-a| < \delta_1 \Rightarrow \left|\frac{f'}{g'}(z) - L\right| < \varepsilon$ $*$, let $\delta = \min\{\delta_0, \delta_1\} > 0$

let $0 < |x-a| < \delta$ i.e. $x \in I$ $\quad a-\delta_0 \quad a-\delta \quad a \quad x \quad a+\delta \quad a+\delta_0$

Suppose $x > a$, we have f, g cont on $[a, x]$ and diff on (a, x) so CMVT applied

Hence there exist $c \in (a, x)$ s.t. : $(f(x) - f(a))\cdot g'(c) = f'(c)\cdot(g(x)-g(a))$

By assuption of theorem, $g'(c) \neq 0$ since $c \neq a$

Also $g(x) - g(a) \neq 0$ since $g(a) = 0$ and $g(z) \neq 0$ for any $z \in I$ except $z = a$

Then, $\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f(x)-f(a)}{g(x)-g(a)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right|$ for some $c \in (a, x)$

Hence $0 < |c-a| < \delta \le \delta_1 < \varepsilon$ by $*$ Done Case ①

11.13

L'Hôpital's Rule (2) $\frac{\infty}{\infty}$ $(\varepsilon < 1)$

Assume f, g are differentiable on interval $I = [a-\delta, a+\delta]$, except possibly at a.

Assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ and either $\lim_{x\to a} f(x) = 0$ or ∞, Suppose g, g' are non-zero on I, except perhaps at a

If $\lim_{x\to a} \frac{f(x)}{g(x)} = L \in \mathbb{R}$, then $\lim_{x\to a} \frac{f}{g} = L$

引入: Cauchy Mean Value Thm: If f, g are continuous on $[a,b]$ and diff on (a,b). Then there is some $c \in (a,b)$ s.t. $(f(b)-f(a))\, g'(c) = f'(c)\,(g(b)-g(a))$

Case ② $\lim_{x\to a} f = \lim_{x\to a} g = \infty$

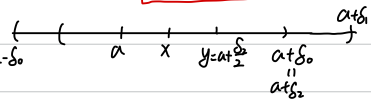
Recall: This means for every $C \in \mathbb{R}$, there exist $\delta > 0$ s.t. if $0<|x-a|<\delta$, then $f(x) \geq C$

Look at $\lim_{x\to a^+} \frac{f}{g}$

let $\varepsilon > 0$. Have to show there exist $\delta > 0$ s.t. $0<|x-a|<\delta \Rightarrow |\frac{f}{g}(x) - L| < \varepsilon$

Given $\lim_{x\to a} \frac{f'}{g'} = L$, So we know there exists $\delta_1 > 0$ s.t. if $0<|t-a|<\delta_1$, then $|\frac{f'}{g'}(t) - L| < \varepsilon$ ⊛

let $\delta_2 = \min(\delta_0, \delta_1) > 0$ $\boxed{\text{Take } y = a + \frac{\delta_2}{2}}$ Then $y < a+\delta_0$ and $y < a+\delta_1$.

Consider $x \in (a, y)$  $[x,y] \subset (a, a+\delta_0)$, Thus f, g are cont and diff on $[x,y]$

Apply the C.M.V.T to get $c \in (x,y)$ s.t. $(f(x)-f(y))\cdot \underset{\neq 0}{g'(c)} = f'(c)\cdot \underset{\neq 0}{(g(x)-g(y))}$ (by the assumption of MVT)

$$\Longrightarrow \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)} \qquad 0<x<c<y=a+\frac{\delta_2}{2}<\delta_1 \qquad \text{So ⊛ applies with } t=c$$

Gives: $\varepsilon > |\frac{f'(c)}{g'(c)} - L| = |\frac{f(x)-f(y)}{g(x)-g(y)} - L|$ This is true for every $a<x<y$

Equivalently, $L-\varepsilon < \frac{f(x)-f(y)}{g(x)-g(y)} < L+\varepsilon$

Since $\lim_{x\to a} g = +\infty$, we must have $g(x) > 0$ for "x close enough to a"

Also, Since $g(y) \in \mathbb{R}$ is fixed, $g(x) > g(y)$ for "x close enough to a"

Say these are true if $0 < x-a < \delta_3 < \frac{\delta_2}{2}$

$$\frac{(g(x)-g(y))\cdot(L-\varepsilon)}{g(x)} < \frac{f(x)-f(y)}{g(x)} < \frac{(g(x)-g(y))\cdot(L+\varepsilon)}{g(x)} \Longrightarrow \left(1-\frac{g(y)}{g(x)}\right)\cdot(L-\varepsilon) < \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < \left(1-\frac{g(y)}{g(x)}\right)\cdot(L+\varepsilon)$$

$$\Longrightarrow \frac{f(y)}{f(x)} + \left(1-\frac{g(y)}{g(x)}\right)\cdot(L-\varepsilon) < \frac{f(x)}{g(x)} < \frac{f(y)}{g(x)} + \left(1-\frac{g(y)}{g(x)}\right)\cdot(L+\varepsilon)$$

$g(x) \to \infty$

$f(y), g(y) \to$ a real number

$\left|\frac{g(y)}{g(x)}\right| < \varepsilon$ if "x is near a", $\left|\frac{f(y)}{g(x)}\right| < \varepsilon$ if "x is near a"

because $\lim_{x\to a} g = \infty$, That means there exist $0 < \delta_4 < \delta_3$ s.t. $0 < x-a < \delta_4$

then RHS: $\frac{f(y)}{g(x)} + (1-\frac{g(y)}{g(x)})\cdot(L+\varepsilon) = L + \left[\frac{f(y)}{g(x)} + (1-\frac{g(y)}{g(x)})\cdot\varepsilon - L\cdot(\frac{g(y)}{f(x)})\right]$ ※※

※※ $\leq \varepsilon + \varepsilon\cdot\varepsilon + |L|\cdot\varepsilon \leq (|L|+2)\cdot\varepsilon$, So $\frac{f(x)}{g(x)} \leq L + (|L|+2)\cdot\varepsilon$

$LHS = \frac{f(y)}{g(x)} + (1-\frac{g(y)}{g(x)})\cdot(L-\varepsilon) = L + (\frac{f(y)}{g(x)} - (1-\frac{g(y)}{g(x)})\cdot\varepsilon - \frac{g(y)}{g(x)}\cdot L)$

$\geq L - \left|\frac{f(y)}{g(x)} + (1-\frac{g(y)}{g(x)})\cdot(L-\varepsilon) = L + (\frac{f(y)}{g(x)} - (1-\frac{g(y)}{g(x)})\cdot\varepsilon - \frac{g(y)}{g(x)}\cdot L)\right| \geq L - (\varepsilon + (1+\varepsilon)\cdot\varepsilon + \varepsilon\cdot|L|) \geq L - (|L|+3)\cdot\varepsilon$

Takes $\delta = \delta_4$ If $0 < x-a < \delta$ then $L - (|L|+3)\cdot\varepsilon < \frac{f(x)}{g(x)} < L + (|L|+3)\cdot\varepsilon$

Remarks (1) Here we didn't use $\lim_{x\to a} f = \infty$ It's not necessary

(2) But $\frac{0}{0}$ form is needed

Ex. $\lim_{x\to 1}\frac{x}{x-1}$ – get wrong answer if you apply L'H rule

(3) Same Proof shows L'H's Rule works for one-sided limit

(4) Value of $a = \pm\infty$: $\lim_{x\to+\infty}\frac{f}{g} = \lim_{y\to 0^+}\frac{f(\frac{1}{y})}{g(\frac{1}{y})} = \lim_{y\to 0^+}\frac{f'(\frac{1}{y})\cdot\cancel{(-\frac{1}{y^2})}}{g'(\frac{1}{y})\cdot\cancel{(-\frac{1}{y^2})}} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$

(5) Also valid if $L = \pm\infty$ (HW)

(6) We can have $\lim_{x\to a}\frac{f'}{g'}$ does not exist, but $\lim_{x\to a}\frac{f}{g}$ does

Ex $\lim_{x\to\infty}\frac{x+\sin x}{x} = \cancel{\lim_{x\to\infty}\frac{1+\cos x}{1}}$
$(\frac{\infty}{\infty})$ does not exist

but $\left|\frac{x+\sin x}{x} - 1\right| = \left|\frac{\sin x}{x}\right| \leq \frac{1}{x} \to 0$ as $x \to \infty$

11.15

Examples. (1) $\lim_{x\to 0} \frac{\tan x - x}{\sin^3 x}$ ($\frac{0}{0}$ form – L'HR)

$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3\sin^2 x \cos x}$

$= \lim_{x\to 0} \frac{\tan^2 x}{3\sin^2 \cos x}$ ($\sec^2 x - 1 = \tan^2 x$)

$= \lim_{x\to 0} \frac{1}{3\cos^3 x}$ (化简)

$= \frac{1}{3}$

(2) $\lim_{x\to\infty} x^2 e^{-x}$ $(0\cdot\infty)$ e^x win ^_^

$= \lim_{x\to\infty} \frac{x^2}{e^x}$ or $\lim_{x\to\infty} \frac{e^{-x}}{x^{-2}}$ $(\frac{\infty}{\infty})$ or $(\frac{0}{0})$

bad way

$= \lim_{x\to\infty} \frac{2x}{e^x}$

$= \lim_{x\to\infty} \frac{2}{e^x} = 0$

(3) $\lim_{x\to\infty} \frac{(\log x)^3}{x^2}$ $(\frac{\infty}{\infty})$ Log lose ^_^

$= \lim_{x\to\infty} \frac{3(\log x)^2}{2x^2}$ $(\frac{\infty}{\infty})$

$= \lim_{x\to\infty} \frac{6\cdot\log x\cdot\frac{1}{x}}{4x}$

$= \lim_{x\to\infty} \frac{6\cdot\log x}{4x^2}$

$= \lim_{x\to\infty} \frac{6\cdot\frac{1}{x}}{8\cdot x} = \lim_{x\to\infty} \frac{6}{8x^2} = 0$

(4) $\lim_{x\to\infty} x\cdot e^{\frac{1}{x}} - x = \lim_{x\to\infty} x\cdot(e^{\frac{1}{x}} - 1)$

$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$ $(\frac{0}{0})$

$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\cdot(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1$

Alternate

let $y = \frac{1}{x}$

$x\to\infty$, $y\to 0^+$

$\lim_{x\to\infty} \frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{y\to 0^+} \frac{e^y - 1}{y} = \lim_{y\to 0^+} \frac{e^y}{1} = 1$

(5) $\lim_{x\to 0^+} x^{\sqrt{x}}$ (0^0)

$= \lim_{x\to 0^+} e^{\sqrt{x}\log x}$

$= \lim_{x\to 0^+} e^0 = 1$

Find $\lim_{x\to 0^+} \sqrt{x}\cdot\log x = \lim_{x\to 0^+} \frac{\log x}{x^{-\frac{1}{2}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$

argue $\lim_{x\to 0^+} e^{\sqrt{x}\cdot\log x} = e^{(\lim_{x\to 0^+} \sqrt{x}\cdot\log x)} = e^{f(0)} = e^0 = 1$

Let $f(x) = \begin{cases} \sqrt{x}\log x, & x>0 \\ 0, & \text{else} \end{cases}$

$\lim_{x\to 0^+} f = 0 = f(0)$

this function is Cont at 0

(6) $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$, $n\in\mathbb{N}$ (1^∞)

Consider $\lim_{x\to\infty} (1+\frac{1}{x})^x = \lim_{x\to\infty} e^{x\cdot\log(1+\frac{1}{x})}$ $(\infty\cdot 0)$

$\lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{y\to 0^+} \frac{\log(1+y)}{y} = \lim_{y\to 0^+} \frac{\frac{1}{1+y}}{1} = 1$

$\lim_{x\to\infty} (1+\frac{1}{x})^x = \lim_{x\to\infty} e^{x\cdot\log(1+\frac{1}{x})} = e^{\lim_{x\to\infty} x\cdot\log(1+\frac{1}{x})}$

☆ $\lim_{x\to\infty} f(x)^{g(x)}$

$= \lim_{x\to\infty} e^{g(x)\cdot\log f(x)}$ $\to = h(x)$

Assuming $\lim_{x\to\infty} h(x) = L \in \mathbb{R}$

then $\lim_{x\to\infty} f(x)^{g(x)} = e^L$

Pf let $\varepsilon > 0$ want to show $\exists N$ s.t. $\forall x \geq N$

$|e^{h(x)} - e^L| < \varepsilon$ e^x is continuous at L

So we can get $\delta > 0$ s.t. if $|z - L| < \delta$, the $|e^z - L| < \varepsilon$

Use the fact that $\lim_{x\to\infty} h(x) = L$ to get $N\in\mathbb{R}$

s.t. $x \geq N \Rightarrow |h(x) - L| < \delta$

Now if $x\geq N$, then $|h(x) - L| < \delta \Rightarrow |e^{h(x)} - e^L| < \varepsilon$

That Prove $\lim_{x\to\infty} e^{h(x)} = e^L = e^{\lim_{x\to\infty} h(x)}$

11.18

Taylor Polynomials

f is n times differentiable at a Taylor Polynomial of f of deg $k \le n$, centered at a
$= P_{n,a}(x) = a_0 + a_1 \cdot (x-a) + \cdots + a_k (x-a)^k$, where $a_0 = f(a)$, $a_j = \frac{f^{(j)}(a)}{j!}$ for $j = 1, 2, 3, \cdots$

$P_{n,a}(x) = f(a) + f'(a) \cdot (x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$P_{n,a}(a) = f(a)$

$P'_{n,a}(x) = f'(a) + 2(x-a) \cdot \frac{f''(a)}{2!} + 3 \cdot (x-a)^2 \cdot \frac{f'''(a)}{3!} + \cdots + \frac{n \cdot (x-a)^{n-1}}{n!} \cdot f^{(n)}(a)$

$P'_{n,a}(a) = f'(a)$

$P''_{n,a}(x) = f''(a) + \frac{6}{3!} \cdot (x-a) f'''(a) + \cdots + \frac{n \cdot (n-1) \cdot (x-a)^{n-2}}{n!} \cdot f^{(n)}(a)$

$P''_{n,a}(a) = f''(a)$

Remarks:

1. $P^{(j)}_{n,a}(a) = f^{(j)}(a)$ for $j = 1, 2, 3, \cdots$
2. $P_{1,a}(x) =$ tangent line
3. $P_{n,a}(x) = P_{n-1,a}(x) + \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$
4. If f is a polynomial of deg n, then $f(x) = P_{n,a}(x)$

Example. Talor polynomial centered at 0 : $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$

$$\left.\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \\ f''''(x) = \sin x & f''''(0) = 0 \end{array}\right\} = 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots$$

Recall: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Questions: Always $f(a) = P_{n,a}(a)$ $\forall n$

(1) Is $P_{n,a}(x)$ a good approximation to f at other x? No!

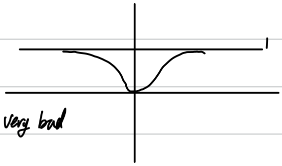
at least x near a? at least for

ex. $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

f is infinitely diff everywhere

$f^{(j)}(0) = 0$ $\forall j$

all $P_{n,0}(x) = 0$ $\forall n$



Clearly $P_{n,0}(x)$ is a very bad approximation to f

(2) Does $P_{n,a}(x) \to f(x)$ in some sense?

(3) How rapidly? size of error? too big or too small?

Theorem Suppose f is n times diff at a, then $\lim_{x\to a}\frac{f(x)-P_{n,a}(x)}{(x-a)^n}=0$

$\lim_{x\to a} f(x)=f(a)=P_{n,a}(a)=\lim_{x\to a}P_{n,a}(x)$

H.W. If Q is a Polynomial of deg n, f $n+1$ times diff and $\lim_{x\to a}\frac{f(x)-Q(x)}{(x-a)^n}=0 \Rightarrow Q=P_{n,a}$

Proof the Theorem

$$\lim_{x\to a}\frac{f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(a)\cdot(x-a)^k}{k}}{(x-a)^n}=\lim_{x\to a}\frac{f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)\cdot(x-a)^k}{k!}}{(x-a)^n}-\frac{\frac{f^{(n)}(a)\cdot\cancel{(x-a)^n}}{n!}}{\cancel{(x-a)^n}}$$

Prove $\lim_{x\to a}\frac{f(x)-Q_n(x)}{(x-a)^n}=\frac{f^n(a)}{n!}$

$Q_n(a)=f(a)$, So $\lim_{x\to a} f(x)-Q_n(x)=f(a)-Q_n(a)=0$

$\lim_{x\to a}\frac{f'(x)-Q_n'(x)}{n\cdot(x-a)^{n-1}}$, Really, $Q_n=P_{n-1,a}$

$Q_n^{(i)}(a)=P_{n-1}^{(i)}(a)$

$=f^{(i)}(a)$, $i=1,2,3\cdots$ So $\lim_{x\to a} f'(x)=f'(a)=Q_n'(a)=\lim_{x\to a}Q_n'(x)$ ($\frac{0}{0}$ form again)

Since Q_n has deg $(n-1)$, So $(n-1)$ times diff is a Constant

Repeat.... Eventually, we get to $\lim_{x\to a}\frac{f^{(n-1)}(x)-Q_n^{(n-1)}(x)}{n!(x-a)}$, $Q_n^{(n-1)}(x)=f^{(n-1)}(a)$

So $\lim_{x\to a} f'(x)=f'(a)=Q_n'(a)=\lim_{x\to a}Q_n'(x)$, $\lim_{x\to a} f^{(n-1)}(x)=f^{(n-1)}(a)$ take L'H Rule again $\lim_{x\to a}\frac{f^{(n)}(x)-Q^{(n)}(x)}{n!}$, $Q^{(n)}=0$

But we don't know $f^{(n)}(x)$ is continuous at a, So we can't justify $\lim_{x\to a}f^n(x)=f^n(a)$

Get $\lim_{x\to a}\frac{f^{(n-1)}(x)-Q_n^{(n-1)}(x)}{n!(x-a)}=\frac{1}{n!}\cdot\lim_{x\to a}\frac{f^{n-1}(x)-f^{n-1}(a)}{x-a}=\frac{1}{n!}\cdot f^n(a)$ since we are told $f^n(a)$ exists

Apply L'H Rule, we get the result.

11.20

Theorm If f is n-times diff at x, then $\lim_{x\to a}\frac{f(x)-P_{n,a}(x)}{(x-a)^n}=0$, where $P_{n,a}(x)=f(a)+\sum_{k=1}^{n}\frac{f^{(k)}(a)\cdot(x-a)^k}{k!}$

General nth derivative test

Suppose $f'(a)=f''(a)=\cdots=f^{(n-1)}(a)=0$ and $f^{(n)}(a)\neq 0$

(1) If n is even and $f^{(n)}(a)>0$, then f has a local min at a ...... $f^{(n)}(a)<0$, ... local max at a

(2) If n is odd, then f has neither a local min or max at a

Proof If $f(a)\neq 0$, replace f by $f(x)-f(a)$, Then the derivative are unchanged, but now $f(a)=0$

So wlog we can assume $f(a)=0$, $\lim_{x\to a}\frac{f(x)-P_{n,a}(x)}{(x-a)^n}=0=\lim_{x\to a}\frac{f(x)-\frac{f^{(n)}(a)(x-a)^n}{n!}}{(x-a)^n}=\lim_{x\to a}\frac{f(x)}{(x-a)^n}-\frac{f^{(k)}(a)}{n!}$

- $P_{n,a}(x)=f(a)+f'(a)\cdot(x-a)+\cdots\cdots+\frac{f^{(n)}(a)\cdot(x-a)^n}{n!}=\frac{f^{(n)}(a)\cdot(x-a)^n}{n!}$

(1) Suppose n is even and $f^{(n)}(a)>0$ $\therefore \frac{f(x)}{(x-a)^n}>0$ for x "near" a

Since $(x-a)^n>0$ $\forall x\neq 0$ therefore $f(x)>0$ for x "near" a (i.e. on some interval $(a-\delta,a+\delta)\setminus\{a\}$) Hence a is a local min

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that is

Similarly, if $f^{(n)}(a)<0$, then $f(x)<0=f(a)$ if x is "near" a

2. Suppose n is odd and $f^{(n)}(a)>0$ Again $\frac{f(x)}{(x-a)^n}>0$ for x "near" a

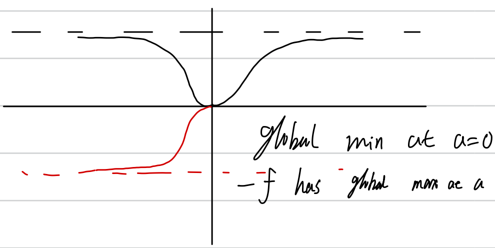
If $x>0$, then $(x-a)^n>0 \Rightarrow f(x)>0=f(a)$

But if $x<a$, then $(x-a)^n<0\Rightarrow f(x)<0=f(a)$. Similar argument if $f^{(n)}(a)<0$

What if $f^{(n)}(a)=0$ $\forall n$?

ex. $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

Face $f^{(n)}(0)=0$ $\forall n$



Define $g=\begin{cases} e^{-\frac{1}{x^2}}, & x>0\\ 0, & x=0\\ -e^{-\frac{1}{x^2}}, & x<0\end{cases}$

$g^{(n)}(a)=0$ $\forall n$ g has neither min or max at 0

Taylor's Theorems

Suppose $f, f', \cdots, f^{(n+1)}$ are defined on $[a, x]$ $\quad$ x is fixed

Then $f(x) - P_{n,a}(x) = \dfrac{f^{(n+1)}(c) \cdot (x-a)^{n+1}}{(n+1)!}$ for some $c \in (a, x)$

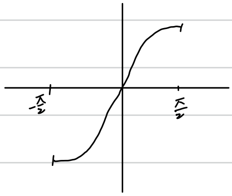
Similar statment for $x < a$

Ex. $f(x) = \sin x$

$$\left| f(x) - P_{n,0}(x) \right| \leq 1 \cdot \frac{|x-0|^{n+1}}{(n+1)!} \longrightarrow 0 \text{ as } n \to \infty \text{ for all } x$$

of Course, here we might well assume $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then given $\left| \sin x - P_{n,0}(x) \right| \leq \dfrac{\pi^{n+1}}{(n+1)!}$



Even $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is enough with a little of trigonometry

Take $n=50$, get 70 decimal place of accuracy

E.g. $n=1$

$$\left| f(x) - (f(a) + f'(a)(x-a)) \right| \leq \frac{|f''(c)| \cdot |x-a|^2}{2}$$

Proof Fix x For each $t \in [a, x]$

Define $R(t) = f(x) - \Big(\underbrace{f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)\,(x-t)^k}{k!}}_{= P_{n,t}(x) \text{ (for } f)} \Big)$

$(t=a)$ $\quad R(a) = f(x) - P_{n,a}(x)$

$(t=x)$ $\quad R(x) = f(x) - (f(x) + 0) = 0$

Define $g(t) = \dfrac{(x-t)^{n+1}}{(n+1)!}$, $\quad g(a) = \dfrac{(x-a)^{n+1}}{(n+1)!}$ $\qquad g(x) = \dfrac{(x-x)^{n+1}}{(n+1)!} = 0$

Want to prove $R(a) = f^{(n+1)}(c) \cdot g(a)$ for some $c \in (a, x)$ $\quad \Leftrightarrow \quad R(a) - R(x) = f^{(n+1)}(c) \cdot (g(a) - g(x))$

Cauchy Mean Value Thm under suitable assumptions $(F(x) - F(a)) \cdot G'(c) = F'(c) \cdot (G(x) - G(a))$ for some $c \in (a, x)$

11.22

Taylor's Theorem

Suppose $f, f', \cdots, f^{(n+1)}$ are defined on $[a, x]$

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$P_{n,a}$ = Taylor Polynomial of deg n, Centered at a, for f. Then $f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)\cdot(x-a)^{n+1}}{(n+1)!}$ for some $c \in (a, x)$

Pf Fix x, Define $R(t) = f(x) - f(t) - \sum_{1}^{n} \frac{f^{(k)}(t)\ (x-t)^k}{k!}$, $R(a) = f(x) - P_n, a(x)$, $R(x) = 0$

Define $g(t) = \frac{(x-t)^{n+1}}{(n+1)!}$, $g(x) = 0$, $g(a) = \frac{(x-a)^{n+1}}{(n+1)!}$, Want $R(a) - R(x) = f^{n+1}(c)\cdot(g(a) - g(x))$

Use Cauchy MVT - here we have assumption on

high order diff of f to see the hypothese of CMVT are Satisfied

$g'(t) = -\frac{(n+1)(x-t)^n}{(n+1)!} = -\frac{(x-t)^n}{n!}$

$R(t) = ?$

$\frac{d}{dt}\left(\frac{f^{(k)}(t)\cdot(x-t)^k}{k!}\right) = \frac{f^{(k+1)}(t)\cdot(x-t)^k}{k!} - \frac{\cancel{k}(x-t)^{k-1}}{\cancel{k}(k-1)!}\cdot f^{k}(t)$

$R'(t) = -f'(t) - \sum_{k=1}^{n}\left(\frac{f^{(k+1)}(t)(x-t)^k}{k!} - \frac{f^{(k)}(t)\cdot(x-t)^k}{(k-1)!}\right) = -f(t) - \left(\frac{f^{(n+1)}(t)\ (x-t)^n}{n!} - \frac{f^{n}(t)\ (x-t)^{n-1}}{(n-1)!} + \frac{f^{(n)}(t)\ (x-t)^{n-1}}{(n-1)!} - \frac{f^{(n-1)}(t)\cdot(x-t)^{n-2}}{(n-2)!} + \cdots + \frac{f^{(2)}(t)\ (x-t)^1}{1!} - \frac{f'(t)\ (x-t)^0}{0!}\right)$

$= -f(t) - \frac{f^{(n+1)}(x)\ (x-t)^n}{n!} + f(t)$

$= \frac{f^{(n+1)}(t)(x-t)^n}{n!}$

Now we see that there exist $c \in (a, x)$ s.t. $(R(a) - R(x))\, g'(c) = R'(c)\ (g(a) - g(x))$ by CMVT

$\cancel{-}\,(R(a) - R(x))\cdot\frac{\cancel{(x-c)^n}}{\cancel{n!}} = \cancel{-}\,\frac{f^{(n+1)}(c)\cdot\cancel{(x-c)^n}}{\cancel{n!}}\ (g(a) - g(x)) \implies R(a) - R(x) = g(a) - g(x)$

Ex

Evaluate $\lim_{x\to 0} \frac{x^2 \sin x^3 - x^5}{x^{11}}$

$\sim x^3$ $\quad \lim_{x\to 0}\frac{\sin x}{x} = 0$

☆: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = z - \frac{z^3}{3!} + E(z)$ where $E(z)$ = error that we understand from Taylor's Theorem

$E(z) = \frac{|\sin^4(c)|\cdot|z|^4}{4!} \leq \frac{z^4}{4!}$

$\lim_{x\to 0}\frac{x^2 \sin x^3 - x^5}{x^{11}} = \lim_{x\to 0}\frac{x^2(x^3 - \frac{x^9}{3!} + E(x^3)) - x^5}{x^{11}}$

$= \lim_{x\to 0}\frac{-\frac{x^{11}}{3!}}{x^{11}} + \frac{x^2\cdot E(x^3)}{x^{11}}$

$= \lim_{x\to 0} -\frac{1}{3!} + \frac{x^2 E(x^3)}{x^{11}}$ where $\frac{|x^2\cdot E(x^3)|}{|x^{11}|} \leq \frac{x^2\cdot x^{12}}{4!\cdot x^{11}} = \frac{x^2}{4} \to 0$ as $x \to 0$

$= -\frac{1}{6}$

Prove e is irrational

let $f(x) = e^x$ $\quad f(0) = 1$

$f^{(k)}(x) = e^x$ $\quad f^{(k)}(0) = 1$

$$P_{n,0}(x) = 1 + \sum_{k=1}^{n} 1 \cdot \frac{x^k}{k!}$$

$f(x) - P_{n,0}(x) = E_n(x)$ where $E_n(x) = \dfrac{f^{(n+1)}(c_n) \cdot (x-0)^{n+1}}{(n+1)!}$ for some $c_n \in (0, x)$

$$= \frac{e^{c_n} \cdot x^{n+1}}{(n+1)!}$$

$e = f(1) = \exp(1)$

$f(1) - P_{n,0}(1) = E_n(1) = \dfrac{e^{c_n} \cdot 1^{n+1}}{(n+1)!}$ where $c_n \in (0,1)$

$\overset{\shortparallel}{e}$ $\quad \leq \dfrac{4}{(n+1)!}$ 放缩 and $E_n(1) > 0$

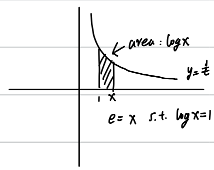
$$e = 1 + 1 \cdot \frac{1}{1!} + 1 \cdot \frac{1^2}{2!} + \cdots + 1 \cdot \frac{1^n}{n!} + E_n(1)$$

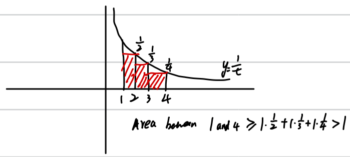
suppose $e = \frac{p}{q}$ for $p, q \in \mathbb{N}$, choose $n > \max\{q, 4\}$

$$\underbrace{n!\, \overset{\shortparallel}{e}}_{\frac{p}{q}\ \in \mathbb{Z}} = n! \underbrace{\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!}\right)}_{\in \mathbb{Z}} + n!\, E_n(1)$$

Hence $n!\, E_n(1) \in \mathbb{Z}$ and $n!\, E_n(1) > 0$ $\quad n > \max\{q, 4\}$

Then, $n!\, E_n(1) \geq 1$, but $n!\, E_n(1) \leq \dfrac{n!\, 4}{(n+1)!} = \dfrac{4}{n+1} < 1$, Contradiction!



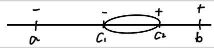


Final Exam Content to here

11.25

Newton's Method

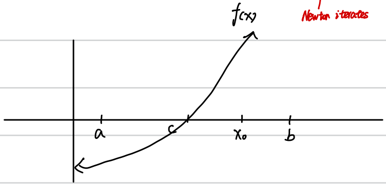
Suppose f continuous on $[a,b]$ and $f(a) < 0 < f(b)$, by I.V.T $\exists\, c \in (a,b)$ s.t. $f(c) = 0$



Suppose $f' > 0$ on $[a,b]$ $\Rightarrow$ f is strictly increasing and therefore the root is unique.

Plan Pick $x_0 \in [c,b]$, inductively define $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton iterates



equation of tangent line to $y = f(x)$, though $(x_0, f(x_0))$

$y - f(x_0) = f'(x_0)\cdot(x_1 - x_0)$

Crosses x axis when $y=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x_{n+1} = Point where the tangent line at x_n crosses x-axis

Theorem Suppose $f: [a,b] \to \mathbb{R}$, f, f', f'' are continuous $f(a) < 0 < f(b)$. $f', f'' > 0$ on $[a,b]$

Assume $f(c) = 0$ for $c \in [a,b]$. Define $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n \in \mathbb{N}$ where $x_0 \in [c,b]$ Then $x_n \in [c,b]$, x_n is decreasing and $x_n \to c$

Proof First, check $c < x_1 < x_0$. know $f(x_0) > 0$ since $c < x_0$ and $f' > 0$ on $(c, x_0]$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} < x_0$, By MVT $\frac{f(x_0) - f(c)}{x_0 - c} = f'(t_0)$ for some $t_0 \in (c, x_0)$ $\Rightarrow$ $f(x_0) = f'(t_0)\cdot(x_0 - c)$, $c = x_0 - \frac{f(x_0)}{f'(t_0)}$

$f'' > 0 \Rightarrow f'$ is increasing $\Rightarrow f'(t_0) < f'(x_0) \Rightarrow c < x_0 - \frac{f(x_0)}{f'(x_0)} = x_1 \Rightarrow x_1 \in (c, x_0)$

Proceed inductively, Suppose $b \geq x_0 > x_1 > \cdots > x_n > c$, check $x_n > x_{n+1} > c$

The fact that $x_n \in (c,b) \Rightarrow f'(x_n) \neq 0$ so x_{n+1} is well defined. Argument to valid, $x_n \in (c_1, x_n)$ is the same as x_1 case

So (x_n) is decreasing and bounded below. Hence by MCT, $x_n \to p$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, f, f' continuous, $f(x_n) \longrightarrow f(p)$ and $f'(x_n) \to f'(p) \neq 0$ as $p \in [a,b]$

$\downarrow$ $\quad\quad$ $\downarrow$

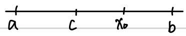
$p \quad\quad p - \frac{f(p)}{f'(p)}$, $\frac{f(p)}{f'(p)} = 0$

$\therefore f(p) = 0$, so $p = c$

Theorem continued

Let $M_1 = \max\{|f''(x)| : x \in [a,b]\}$

$M_2 = \min\{|f'(x)| : x \in [a,b] = f'(a)\}$

Put $M = \frac{M_1}{M_2}$. then $|x_n - c| \le \frac{1}{M}\left(M\cdot|x_0-c|\right)^{2^n} \le \frac{1}{M}\cdot\left(M\cdot(b-a)\right)^{2^n}$, 

, if $M(b-a)<1$, this is very strong

Pf: Recall $c = x_n - \frac{f(x_n)}{f'(t_n)}$ for some $t_n \in (c, x_n)$, $|x_{n+1} - c| = \left|x_n - \frac{f(x_n)}{f'(x_n)} - \left(x_n - \frac{f(x_n)}{f'(t_n)}\right)\right|$

$$= \left|\frac{f(x_n)}{f'(t_n)} - \frac{f(x_n)}{f'(x_n)}\right|$$

$$= \left|\frac{f(x_n)\cdot(f'(x_n) - f'(t_n))}{f'(t_n)\cdot f'(x_n)}\right|$$

— by MVT $= \left|\frac{f(x_n)\cdot f''(u_n)\cdot(x_n-t_n)}{f'(t_n)\cdot f'(x_n)}\right| \le \frac{M_1\cdot|x_n-c|^2}{M_2} = M\cdot|x_n-c|^2$

for some $u_n \in (t_n, x_n)$

(*) $\frac{|x_n-c|\cdot|f''(u_n)|\cdot|x_n-t_n|}{|f'(x_n)|}$ $= M\cdot|x_{n+1}-c| \le (M\cdot|x_n-c|)^2$

$$\le \left[(M\cdot|x_{n-1}-c|)^2\right]^2$$

$$= (M\cdot|x_{n-1}-c|)^{2^2}$$

$$\le (M\cdot|x_{n-2}-c|^2)^{2^2}$$

$$\vdots$$

$$= (M\cdot|x_0-c|)^{2^{n+1}}$$

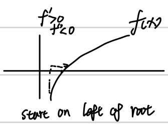
Ex. Approx to $\sqrt{2}$ to 8 decimal places, $\sqrt{2} \in [1.4, 1.5]$

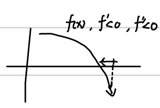
$x_0 = 1.5$

$|x_0 - \sqrt{2}| < 0.1$

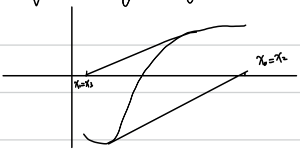
$f(x) = x^2 - 2$, $f' = 2x$, $f'' = 2$, $M = \frac{1}{1.4}$

$n = 3$, $|x_n - \sqrt{2}| < 9.5 \times 10^{-10}$, $x_2 = 1.41421|5687$





But things can go wrong if, for example, f' is not constant sign



then have $x_n \to P$, $P \neq$ root

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$P \quad P \quad \frac{f(P)}{f'(P)} \quad \text{if } f'(P) \neq 0$$

What if f' is not cont on P

so if f' does not exist at p can be a Problem

or if $f'(P) = 0$ can be a Problem

11.27

Size of Infinite

If E is finite then $E \overset{\text{bijection}}{\longleftrightarrow} \{1, 2, \cdots, n\}$ where $n = \#$ element of E

<u>Def'n</u> let A, B be two sets. Say **Cardinality** of A = Cardinality of B (write $|A| = |B|$) if there is a bijection $f: A \to B$

Say E is **Countable** if there is a bijection $f: N \to E$ (ie $|E| = |N|$)

Say E is **uncountable** if it is neither finite nor countable

Countable set have same cardinality

Countable sets are "smallest" infinite sets a countable subset. Every infinite set contains a countable subset

Set difference 在E中把$\{x_1\}$去掉

<u>Pf</u> let E be infinite. Pick $x_1 \in E$, look at $E \setminus \{x_1\}$ this is not empty, so pick $x_2 \in E \setminus \{x_1\}$

If $x_1, \cdots, x_n$ are chosen in E and distinct then $E \setminus \{x_1, \cdots, x_n\}$ is not empty. This given $\underbrace{\{x_n : n \in N\}}_{= X} \subseteq E$ $\quad X \longleftrightarrow N$, $x_n \overset{}{\longleftrightarrow} n$ bijection

Hence $X \subseteq E$ is countable

<u>Ex.</u> $2N$ is Countable

Take $f: N \to 2N$

$n \to 2n$

This is bijection

<u>Ex.</u> Z – Countable

Z: $0, 1, -1, 2, -2, \cdots$

$\downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow$

N: $1, 2, 3, 4, 5$

This is bijection

Ex. $N \times N = \{(x, y) . x, y \in N\}$ – Countable

	1	2	3	4
1	(1,1) →1	(1,2) →2	(1,3) →6	---- 7
2	(1,2) →3	5	8	
3	⋮ 4	9		
4	⋮ 10			

Q^+

$\frac{p}{q}$

$p, q \in N$

(p,q) co-prime

$\frac{p}{q}$	1	2	3	4	5	6
1	$\frac{1}{1}$ 1	$\frac{1}{2}$ 2	$\frac{1}{3}$ 5	$\frac{1}{4}$ 6	$\frac{1}{5}$	$\frac{1}{6}$
2	$\frac{2}{1}$ 3	x	$\frac{2}{3}$	x		
3	$\frac{3}{1}$ 4	$\frac{3}{2}$	x	$\frac{3}{4}$		

Cantor Diagonal argument

bijection to N. So Q^+ is countable and similarly for Q

Countable sets can always be written as $\{r_n\}_{n=1}^{\infty}$ because if $f: N \to E$ is bijection then $E = \{f(n) : n \in N\}$ and converse is true too

Thm R is not countable

First show $(0,1)$ is uncountable

Pf suppose $(0,1)$ is countable

say $(0,1) = \{r_j\}_{j=1}^{\infty}$

(Cantor) Define r as follows: Put $a_1 = \begin{cases} 5 & \text{if the first digit of } r_1 \neq 5 \\ 4 & \text{if the first digit of } r_1 = 5 \end{cases}$

$a_2 \begin{cases} 5 & \text{if 2nd digit } r_2 \neq 5 \\ 4 & \text{if} \cdots\cdots \ r_2 = 5 \end{cases}$

避开 0 和 9

因: $0.49999\cdots = 0.5$

$\vdots$

$a_n \begin{cases} 5 \cdots\cdots & r_n \neq 5 \\ 4 \cdots\cdots & r_n = 5 \end{cases}$

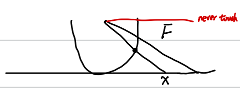
let $r = 0.a_1a_2a_3\cdots$ $\quad r \in (0,1)$

$r \neq r_j$ since jth digits disagree r has a unique decimal representation

$\therefore (0,1) \neq \{r_j\}_1^{\infty}$ so $(0,1)$ is not countable

$|R| = |(0,1)|$, $f(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}$: $R \to (0,1)$ bijection

Pf2



Fact $\cup$ of 2 countable sets is countable

Pf $A = \{r_j\}_1^\infty$, $B = \{t_j\}_1^\infty$, $A \cup B = \{r_1, t_1, r_2, t_2, \dots\}$

Cor Irration numbers are uncble because if they were countable, then $\mathbb{Q}$ is countable Irrationals $\cup \mathbb{Q} = \mathbb{R}$ would be countable and that's false

Schroeder – Bernstein Thm (very hard)

If there is an injection $A \to B$ and an injection from $B \to A$ then there is a bijection from $A \to B$ ie $|A| = |B|$

$|[0,1)| = |\mathbb{R}| = |(0,1)|$

Cor If $A \subseteq C \subseteq B$. $|A| = |B|$, then $|C| = |A| = |B|$

Pf $f: A \to B$ bijection

$j: C \to B$ injection (map $x \in C$ to $x \in B$) identity map

$f^{-1} \circ j: C \to B \to A$. And identity $A \to C$ is an injection By S-B thm $|A| = |C| = |B|$

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Cardinality $|A| = |B|$ if there is a bijection : $A \longrightarrow B$

Countable : $|X| = |N|$ ex. $\mathbb{Z}$, $\mathbb{Q}$

Uncountable: infinite sets that are not countable. ex: $\mathbb{R}$, $\mathbb{Q}^c$ - irrationals

$[a,b]$, (a,b), $[a,b)$, $(a,b]$ same cardinality - $\mathbb{R}$

Cantor's Thm If X is any non-empty set, then $|\{\text{all subset of } X\}| \neq |X|$

$\underset{P(X)\text{ - Power set of } X}{\parallel}$

Example $X = \{1, 2, \cdots, n\}$ $|X| = n$, $|P(X)| = 2^n$ $\sum_{k=0}^{n} \binom{n}{k}$

$|X| \leq |P(X)|$ $j: X \longrightarrow P(X)$ injection
$x \longmapsto \{x\}$

To prove Cantor's theorem, one has to prove there is no bijection : $X \longrightarrow P(X)$

$$|\mathbb{R}| < |P(\mathbb{R})| < |P(P(\mathbb{R}))|$$

Ex. $|P(N)| = |\mathbb{R}|$

Idea: $A \subseteq N$ define $f_A : N \longrightarrow \{0,1\}$

$$f_A(n) = \begin{cases} 0, & \text{if } n \notin A \\ 1, & \text{if } n \in A \end{cases}$$

$f_A = (f_A(n))_{n=1}^{\infty}$

$A \longleftrightarrow f_A$ bijection $P(N) \rightarrow \{f_A : A \subseteq N\}$

$(0,1) \longrightarrow P(N)$

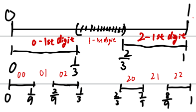
$x = \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} \cdots$ $x_j \in \{0,1\}$ binary representation of x

$x \longmapsto (x_1, x_2, \cdots) = f_A(n)$ where $A = \{n : x_n = 1\}$

j is 1-1 and hits all of $P(N)$, except for countably many (Problem arises with non uniqueness of binary rep'n)

Set of all $f: N \rightarrow \{0,1\}$ call $\{0,1\}^N$, Notation "suggests" $|P(N)| = |\{0,1\}|^N = |\{0,1\}|^N = 2^{|N|}$

Cantor Set



C_0

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$ $\quad 2^2 = 4$ interval of length $\frac{1}{9} = (\frac{1}{3})^2$

C_n = Union 2^n closed interval, length $(\frac{1}{3})^n$ with gaps between of length $\geq (\frac{1}{3})^n$

$C_0 \supseteq C_1 \supseteq C_2 \cdots \supseteq C_n \supseteq \cdots$. Cantor set $C = \bigcap_{n=1}^{\infty} C_n = \{ x \in [0,1] : x \in C_n \text{ for every } n \}$

$0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9} \cdots$ End Points of Cantor interval all belong to C - there are all in $\mathbb{Q}$. this is countable set.

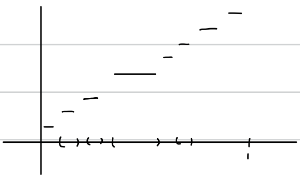
Largest interval in C_n is length $(\frac{1}{3})^n \Rightarrow$ no interval in C

Base 3 : $\frac{x_1}{3} + \frac{x_2}{9} + \frac{x_3}{27} \cdots \quad x_j \in \{0,1,2\}$

$C = \{ \sum_{j=1}^{\infty} \frac{a_j}{3^j} : a_j = \{0,1\} \}$

Bijection : $C \to [0,1]$, $\sum \frac{a_j}{3^j} \to \sum \frac{\frac{a_j}{2}}{2^j}$ Notice $\frac{a_j}{2} \in \{0,1\}$

$\therefore |C| = |[0,1]|$ so C is uncountable



Endpoints of Cantor intervals These are all in $\mathbb{Q}$