Class Notes

数分 II

Math 148

Calculus II

Advanced Level

Winter 2020

section 2

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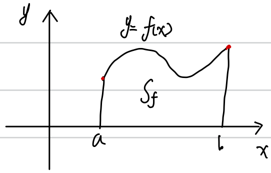
1.6

Integration

Motivation : Area

Let $a<b$ in $\mathbb{R}$, and let $f:[a,b] \to [0,\infty)$

Let $S_f = \{(x,y): 0 \le y \le f(x), x \in [a,b]\}$ ("subgraph")



Question: How do we define area: area (S_f)?

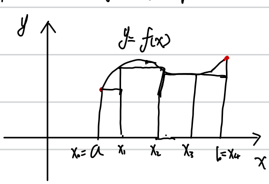
idea:

(i) (Postulate) area of rectangle = height × width

i.e. $x_1 < x_2$, $y \geq 0$

area $([x_1, x_2] \times [0, y]) = y \cdot (x_2 - x_1)$

(ii) Approximate S_f by rectangles from below

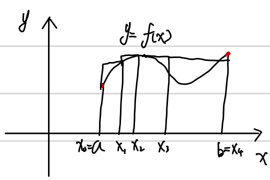


$j = 1,2,3,4$

$m_j = \inf\{f(x): x \in [x_{j-1}, x_j]\}$

$\sum_{j=1}^{4} m_j (x_j - x_{j-1}) \le \text{area}(S_f)$

(ii') Approximate S_f by rectangles from above



$j = 1,2,3,4$

$M_j = \sup\{f(x): x \in [x_{j-1}, x_j]\}$

$\text{area}(S_f) \le \sum_{j=1}^{4} M_j (x_j - x_{j-1})$

(iii) if we can arrange lower sum ≈ upper sum, then we can have a "good" approximation for S_f

Towards defining the integral We let $a<b$ in $\mathbb{R}$: $f:[a,b]\to\mathbb{R}$

Definition : A Portion of $[a,b]$ is any finite set of points
$$P=\{x_0,x_1,\cdots,x_n\} \text{ s.t. } a=x_0<x_1<\cdots<x_n=b$$

often, we write $P=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$

A refinement of P is any Portion Q of $[a,b]$ s.t. $P\subseteq Q$

Now, fix a Portion P of $[a,b]$ and let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$

i.e. $\sup_{x\in[a,b]}|f(x)|\le M<\infty$. Write $P=\{a=x_0<x_1<\cdots<x_n=b\}$. We let

for $j=1,\cdots,n$. $m_j=m_j(P)=\inf\{f(x): x\in[x_{j-1},x_j]\}$

$M_j=M_j(P)=\sup\{f(x): x\in[x_{j-1},x_j]\}$

Notice that $-M\le m_j\le M_j\le M$ for each j, and these "inf"/ "sup" exist (completeness of $\mathbb{R}$)

We then define (after Riemann-Darboux) for P and f as above

lower sum: $L(f,P)=\sum_{j=1}^{n} m_j\,\underbrace{(x_j-x_{j-1})}_{\text{width of }[x_{j-1},x_j]}$

upper sum : $U(f,P)=\sum_{j=1}^{n} M_j\,(x_j-x_{j-1})$

Recall (i) if f is not bounded, at least one of $L(f,P)$ or $U(f,P)$ cannot be defined.

(ii) We have $L(f,P)\le U(f,P)$ Indeed, for each $j=1,\cdots,n$ $m_j\le M_j$ (exactly from defn)

$$L(f,P)=\sum_{j=1}^{n} m_j(x_j-x_{j-1})\le\sum_{j=1}^{n} M_j(x_j-x_{j-1})=U(f,P)$$

Lemma: if P is a Partion of $[a,b]$, $f:[a,b]\to\mathbb{R}$ is bounded and Q is a refinement of P, then

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$$L(f,P)\le L(f,Q) \quad \text{and} \quad U(f,Q)\le U(f,P)$$

Proof: Case #0 : $Q=P$ (obvious)

Case #1 : $Q=P\cup\{q\}$ where $q\notin P$

write $P=\{a=x_0<x_1<\cdots<x_n=b\}$ so $Q=\{a=x_1<\cdots<x_{k-1}<q<x_k<\cdots<x_n=b\}$

Then $m_k(P)=\inf\{f(x): x\in[x_{k-1},x_k]\}$ $\quad [x_{k-1},x_k]=[x_{k-1},q]\cup[q,x_k]$

$$=\min\{\inf\{f(x): x\in[x_{k-1},q]\},\ \inf\{f(x): x\in[q,x_k]\}\}$$

$$=\min\{m_k(Q),\ m'_k(Q)\}$$

Thus $L(f,P)=\sum_{j=1}^{n} m_j(P)\cdot(x_j-x_{j-1}) = \sum_{j=1}^{k-1} m_j P(x_j-x_{j-1}) + m_k(P)\underbrace{(x_k-x_{k-1})}_{x_k-q+q-x_{k-1}} + \sum_{j=k+1}^{n} m_j(P)(x_j-x_{j-1})$

(empty sum $=0$)

$$\le \sum_{j=1}^{k-1}\underbrace{m_j(P)}_{m_j(Q)=\inf\{f(x):\ x\in[x_{j-1},x_j]\}}(x_j-x_{j-1}) + m_k(Q)(q-x_{k-1}) + m'_k(Q)(x_k-q) + \sum_{j=k+1}^{n} m_j(P)(x_j-x_{j-1})$$

$$=L(f,Q)$$

Case #2 : $Q=P\cup\{q_1,q_2,\cdots q_m\}$ $\quad q_1\ne q_2\ne\cdots\ne q_m,\ q_k\notin P$

by Case #1 we have

$$L(f,P)\le L(f,P\cup\{q_1\})\le L(f,P\cup\{q_1,q_2\})\le\cdots\le L(f,P\cup\{q_1,q_2,\cdots q_m\})=L(f,Q)$$

The case $U(f,Q)\le U(f,P)$ is similar

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Last time

$f:[a,b]\to\mathbb{R}$ bounded, $P=\{a=x_0<x_1<\cdots<x_n=b\}$

$m_j = \inf\{f(x): x_j\in[x_{j-1},x_j]\}$

$M_j = \sup\{f(x): x_j\in[x_{j-1},x_j]\}$

$L(f,P)=\sum_{j=1}^{n} m_j(x_j-x_{j-1})$, $U(f,P)=\sum_{j=1}^{n} M_j(x_j-x_{j-1})$

Easy Fact: $L(f,P)\le U(f,P)$

Lemma: If $P\subseteq Q$ (refinement), then $L(f,P)\le L(f,Q)$, $U(f,Q)\le U(f,P)$

Corollary: Let P, Q be any partitions of $[a,b]$ and $f:[a,b]\to R$ be bounded. Then $L(f,P)\le U(f,Q)$

Proof: We have $P, Q\subseteq P\cup Q$, i.e. $P\cup Q$ refines each of P and Q

Thus $L(f,P)\underset{\uparrow\ \text{Lemma}}{\le} L(f,P\cup Q)\underset{\uparrow\ \text{easy fact}}{\le} U(f,P\cup Q)\le U(f,Q)$

Definition: Given bounded $f:[a,b]\to R$ we define

Lower integral : $\underline{\int_a^b} f = \sup\{L(f,P): P \text{ is a partition of } [a,b]\}$

upper integral : $\overline{\int_a^b} f = \inf\{U(f,Q): Q \text{ is a partition of } [a,b]\}$

Note: $\underline{\int_a^b} f = \sup\{L(f,P): P \text{ is a partition of } [a,b]\} \overset{\text{Collary}}{\le} \inf\{U(f,Q): Q \text{ is a partition of } [a,b]\} = \overline{\int_a^b} f$

So $\underline{\int_a^b} f\le\overline{\int_a^b} f$ both exist, by completeness of $\mathbb{R}$

☆ We can say that f is **integralable** on $[a,b]$ provided that $\underline{\int_a^b} f=\overline{\int_a^b} f$. In this case we write $\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$

Notation: write $\int_a^b f = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(☺)\,d☺$

E.g. $\int_0^1 x^2\,dx$

Ex: (not every bounded function is integralable)

Define $\chi_Q(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in R/Q \end{cases}$

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Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be any Partition of $[0,1]$

we have that Q is dense in R, so there is $q_j \in Q \cap (x_{j-1}, x_j)$, $j = 1, 2, \ldots, n$

we have that $R \backslash Q$ is dense in R, so there is $r_j \in R \backslash Q \cap (x_{j-1}, x_j)$, $j = 1, 2, \ldots, n$

inf of $\chi_Q(x)$ on $[x_{j-1}, x_j]$

$$0 \leq L(\chi_Q, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \leq \sum_{j=1}^{n} \chi_Q(r_j)(x_j - x_{j-1}) = 0$$

as $\chi_Q \geq 0$

$$\Rightarrow \underline{\int_0^1} \chi_Q = 0$$

Like wise

$$1 \geq U(\chi_Q, P) \geq \sum_{j=1}^{n} \underbrace{\chi_Q(q_j)}_{=1} (x_j - x_{j-1}) = \underbrace{\sum_{j=1}^{n} (x_j - x_{j-1})}_{\text{telescope}} = 1 - 0 = 1$$

as $1 \geq \chi_Q$

Characteristic or indicator function of Q

Hence $\underline{\int_0^1} \chi_Q = 0 < 1 = \overline{\int_0^1} \chi_Q$, so χ_Q is not integralable on $[0,1]$

Theorem (Cauchy Criterion for Integralability)

Let $a < b$ in R, $f: [a,b] \Rightarrow R$ be bounded, Then TFAE (the following are equivalent 相同)

(i) f is integralable on $[a,b]$,

(ii) given $\varepsilon > 0$, there exist a Partition P_ε of $[a,b]$ s.t. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

(iii) given $\varepsilon > 0$, there exist a Partition P_ε of $[a,b]$ so for every refinement P of P_ε, $U(f,P) - L(f,P) < \varepsilon$

Proof: (i) $\Rightarrow$ (ii) we assume that $\sup\{L(f,P) : P \text{ Partition of } [a,b]\} = \underline{\int_a^b} f = \overline{\int_a^b} f = \inf\{U(f,P) : P \text{ Partition of } [a,b]\}$

Let $\varepsilon > 0$, by first equality above, there is a Partition P_1 of $[a,b]$ s.t.

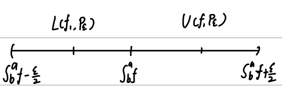
$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f, P_1) \leq \underline{\int_a^b} f$$

and by third equality above, there is a Partion P_2 s.t.

$$\overline{\int_a^b} f \leq U(f, P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P_1 \cup P_2$, a refinement of P_1 and of P_2, then

Since $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$, we find $\int_a^b f - \frac{\varepsilon}{2} < L(f, P_1) \overset{\text{lemma}}{\leq} L(f, P_\varepsilon) \leq U(f, P_\varepsilon) < \int_a^b f + \frac{\varepsilon}{2}$



$$\Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

(ii) $\Rightarrow$ (iii) we use the lemma, if $P_\varepsilon \subseteq P$ we have

$$L(f, P_\varepsilon) \leq L(f, P) \leq U(f, P) \leq U(f, P_\varepsilon)$$

Hence $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \Rightarrow U(f, P) - L(f, P) < \varepsilon$

(iii) $\Rightarrow$ (ii) $P_\varepsilon \subseteq P_\varepsilon$, i.e. refines itself

(ii) $\Rightarrow$ (i) Given $\varepsilon > 0$, there is P_ε, a partition of $[a, b]$, so $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

we have $L(f, P_\varepsilon) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P_\varepsilon) \Rightarrow \left| \overline{\int_a^b} f - \underline{\int_a^b} f \right| < \varepsilon$, this occurs for any $\varepsilon > 0$ so $\underline{\int_a^b} f = \overline{\int_a^b} f$

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Goal today: $f: [a,b] \to \mathbb{R}$, Continuous $\Rightarrow$ integrable

Aside: Uniform Continunity $\quad I = [a,b], (a,b), (a,b], [a,b)$

Recall: $f: I \to \mathbb{R}$ is Continuous if $\forall x \in I$, for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(x')| < \varepsilon$ if $|x - x'| < \delta$

Note: This defn is "local" first we choose x, ε, then δ

Defn: $f: I \to \mathbb{R}$ is uniformly Continuous if for $\forall \varepsilon > 0$, $\exists \delta > 0$
$|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ for $x, x' \in I$

Note: In uniform Continunity, δ is chosen globally for ε, i.e. not depend on x

Proposition (Sequential test of uniform Continunity)
let $f: I \to \mathbb{R}$, then f is unifonly continuous $\Rightarrow$ for any sequences $x_n, x_n' \in I$ with $\lim_{n\to\infty} |x - x_n'| = 0$ we have $\lim_{n\to\infty} |f(x) - f(x')| = 0$ [Fact $\Leftarrow$ also true]

Proof. given $\varepsilon > 0$, let δ be as in defn of uniform continuity. Since $\lim_{n\to\infty} |x_n - x_n'| = 0$, $\exists N_0 \in \mathbb{N}$ so for $n \geq N_0$ we have $|x_n - x_n'| < \delta$, but then for $n \geq N_0$, we also have that $|f(x_n) - f(x_n')| < \varepsilon$ I.e. $\lim_{n\to\infty} |f(x_n) - f(x_n')| = 0$

Ex. (1) $f: (0,1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Notice that f is Continuous

let $x_n = \frac{1}{n}$, $x_n' = \frac{1}{2n}$, $|x_n - x_n'| = \frac{1}{2n} \to 0$

$|f(x_n) - f(x_n')| = |n - 2n| = n$, Hence, not uniform Continuous

(2) $g: (0,1] \to \mathbb{R}$ $\quad g(x) = \sin(\frac{1}{x})$, g is Continuous

$x_n = \frac{1}{\pi n}$, $x_n' = \frac{2}{(2n+1)\pi}$ $\quad |x_n - x_n'| = \frac{1}{\pi n (2n+1)} \to 0$

$|g(x_n) - g(x_n')| = |\sin(n\pi) - \sin(\frac{2n+1}{2}\pi)| = 1$, it's not Uniform Continuous (ex. $\varepsilon = 1$)

Theorem: let $f:[a,b]\to\mathbb{R}$ be continuous, then f is uniformly continuous

Proof: let us suppose that f is continuous but not uniformly continuous

Hence $\exists\, \varepsilon>0$, s.t. $\forall\, \delta>0$, there are $x, x'\in[a,b]$ so $|f(x)-f(x')|\geq\varepsilon$ while $|x-x'|<\delta$

Let us consider $\delta=\frac{1}{n}$ so there are $x_n, x_n'\in[a,b]$ s.t. $|f(x_n)-f(x_n')|\geq\varepsilon$ (*) while $|x_n-x_n'|<\frac{1}{n}$

By Bolzano–Weierstrass, there is a subsequence x_{n_k} of x_n s.t. $x=\lim\limits_{k\to\infty}x_{n_k}$ exist in $[a,b]$

Then notice that $|x-x_{n_k}'|\leq|x-x_{n_k}|+|x_{n_k}-x_{n_k}'|<|x-x_{n_k}|+\frac{1}{n_k}$

Hence, by squeeze principle $\lim\limits_{k\to\infty}x_{n_k}'=x$, since f is continuous, we have that $\lim\limits_{n\to\infty}f(x_{n_k})=f(x)=\lim\limits_{k\to\infty}f(x_{n_k}')\Rightarrow\lim\limits_{k\to\infty}|f(x_{n_k})-f(x_{n_k}')|=0$

This contradices that each $|f(x_{n_k})-f(x_{n_k}')|\geq\varepsilon$ in (*)

Thus, by contradiction argument, f must be uniformly continuous

Theorem (Continuous on closed bounded interval $\Rightarrow$ integrable): Let $f:[a,b]\to\mathbb{R}$ be continuous, then f is integrable

Proof: let $\varepsilon>0$, then, by uniform continuity (last theorem) of f, there is $\delta>0$ s.t. $|f(x)-f(x')|<\frac{\varepsilon}{b-a}$ whenever $|x-x'|<\delta$ for $x, x'\in[a,b]$

Thus, we let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be any partition with length $\ell(P)=\max\limits_{j=1,\dots,n}(x_j-x_{j-1})<\delta$

[E.g.: $P_n=\{a<a+\frac{b-a}{n}<a+2\cdot\frac{b-a}{n}<\cdots<a+(n-1)\frac{b-a}{n}<b\}$, then $\lim\limits_{n\to\infty}\ell(P_n)=0$]

Now, by Extreme Value Theorem we have $x_j^{*}\in[x_{j-1},x_j]$ s.t. $f(x_j^{*})=\inf\{f(x): x\in[x_{j-1},x_j]\}=m_j$, $x_j^{**}\in[x_{j-1},x_j]$ s.t. $f(x_j^{**})=\sup\{f(x): x\in[x_{j-1},x_j]\}=M_j$

Then, $L(f,P)=\sum\limits_{j=1}^{n}m_j(x_j-x_{j-1})=\sum\limits_{j=1}^{n}f(x_j^{*})(x_j-x_{j-1})$, and $U(f,P)=\sum\limits_{j=1}^{n}M_j\cdot(x_j-x_{j-1})=\sum\limits_{j=1}^{n}f(x_j^{**})\cdot(x_j-x_{j-1})$

$$U(f,P)-L(f,P)=\sum_{j=1}^{n}\left(f(x_j^{**})-f(x_j^{*})\right)\cdot(x_j-x_{j-1})$$

$$\leq\sum_{j=1}^{n}\underbrace{|f(x_j^{**})-f(x_j^{*})|}_{<\frac{\varepsilon}{b-a}\text{ as }|x_j^{**}-x_j^{*}|<\delta}\cdot(x_j-x_{j-1})<\sum_{j=1}^{n}\frac{\varepsilon}{b-a}(x_j-x_{j-1})=\frac{\varepsilon}{b-a}\cdot(b-a)=\varepsilon$$

Hence, by the Cauchy Criterion of Integrabity.

Corollary: If $f:[a,b]\to\mathbb{R}$ is continuous then $\int_a^b f=\lim\limits_{n\to\infty}\sum\limits_{j=1}^{n}f(a+j\cdot\frac{b-a}{n})\cdot\frac{b-a}{n}$

Proof: We have $a+j\cdot\frac{b-a}{n}\in[a+(j-1)\frac{b-a}{n}, a+j\cdot\frac{(b-a)}{n}]_{j=1,\dots,n}$,

so $m_j\leq f(a+j\cdot\frac{(b-a)}{n})\leq M_j$ and thus $L(f,P_n)\leq\sum\limits_{j=1}^{n}f(a+j\cdot\frac{b-a}{n})\cdot\frac{b-a}{n}\leq U(f,P_n)$, where $P_n=\{a<a+\frac{b-a}{n}<a+2\frac{b-a}{n}<\cdots<b\}$

also we have $L(f,P_n)\leq\int_a^b f\leq U(f,P_n)$

The proof of theorm shows that $\lim\limits_{n\to\infty}(U(f,P_n)-L(f,P_n))=0$ as $\lim\limits_{n\to\infty}\ell(P_n)=\lim\limits_{n\to\infty}\frac{b-a}{n}=0$

and have Cauchy criterim is satisfied hence $\int_a^b f$ exists and is $\lim\limits_{n\to\infty}L(f,P_n)$, $\lim\limits_{n\to\infty}U(f,P_n)$ by squeeze theorm.

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Last time: Thm: $f:[a,b]\to R$ Continuous $\Rightarrow$ integrable (*)

Cor: $\int_a^b f = \lim_{n\to\infty}\sum_{j=1}^{n} f(a+j\cdot\frac{b-a}{n})\,\frac{b-a}{n}$

(*) Used uniform Continuity of f, gain for any $\varepsilon>0$, $\exists\,\delta>0$ s.t.

$U(f,P)-L(f,P)<\varepsilon$ whenever $\ell(P)<\delta$

Ex.: We will let $a>0$ and and Compute $\int_0^a x^p\,dx$ for $p=0,1,2$

(i) $p=0$, $x^p=1$, $P=\{0=x_0<x_1=a\}$

$L(1,P)=a=U(1,P)$ [P' refines P, then $L(1,P)\le L(1,P')\le U(1,P')\le U(1,P)$ and $L(1,P')\le U(1,P)=a$, $a=L(1,P)\le U(1,P')$]

It follows that $\int_0^a 1\,dx = a$

(ii) From last Corollary $\int_0^a x\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}(j\cdot\frac{a}{n})\frac{a}{n} = \lim_{n\to\infty}\frac{a^2}{n^2}\sum_{j=1}^{n} j = \lim_{n\to\infty}\frac{a^2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{a^2}{2}$

(iii) We need a formula for $\sum_{j=1}^{n} j^2$

$\int_0^a x^2\,dx = \lim_{n\to\infty}\sum_{j=1}^{n}(j\cdot\frac{a}{n})^2\cdot\frac{a}{n} = \lim_{n\to\infty}\frac{a^3}{n^3}\sum_{j=1}^{n} j^2$

$= \lim_{n\to\infty}\frac{a^3}{3n^3}\left[(n+1)^3-1-n-\frac{n(n+1)}{2}\right] = \frac{a^3}{3}$

Trick $(n+1)^3-1 = \sum_{j=1}^{n}\left[(j+1)^3-j^3\right]$ (telescope)

$= \sum_{j=1}^{n}\left[\sum_{k=0}^{3}\binom{3}{k} j^k - j^3\right]$ (binomial Thm)

$= \sum_{j=1}^{n}\sum_{k=0}^{2}\binom{3}{k} j^k$

$= \sum_{k=0}^{2}\binom{3}{k}\sum_{j=1}^{n} j^k = \overbrace{\sum_{j=1}^{n} j^0}^{n} + 3\overbrace{\sum_{j=1}^{n} j}^{\frac{n(n+1)}{2}} + 3\sum_{j=1}^{n} j^2$

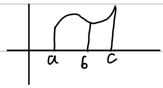
$\Rightarrow \sum_{j=1}^{n} j^2 = \frac{1}{3}\left[(n+1)^3-1-n-\frac{n(n+1)}{2}\right]$

Basic Properties of Integrals (additivity over integrals, Riemann Sums, linearity, order Properties)

Proposition: (Additivity over integrals)

Let $a<b<c$ in $\mathbb{R}$ and $f:[a,c]\to\mathbb{R}$ satisfies that f is integrable on each of $[a,b]$, $[b,c]$

Then f is integrable on $[a,c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$



Proof Given $\varepsilon>0$, the Cauchy criterion integrability

- a Partition P_1 of $[a,b]$ s.t. $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2}$
- a Partition P_2 of $[b,c]$ s.t. $U(f,P_2)-L(f,P_2)<\frac{\varepsilon}{2}$

lower sum on $[a,c]$ Lower sum on $[a,b]$ Lower sum on $[b,c]$

Let P be any refinement of $P_1\cup P_2$, then $L(f,P)\geq L(f,P_1\cup P_2) = L(f,P_1)+L(f,P_2)$

$$U(f,P)\leq U(f,P_1\cup P_2) = U(f,P_1)+U(f,P_2)$$

Then $U(f,P)-L(f,P)\leq U(f,P_1)-L(f,P_1)+U(f,P_2)-L(f,P_2) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$, So f is integrable on $[a,c]$

Let P as above, be written $P=\{a=x_0<\cdots<x_m=b<\cdots<x_n=c\}$ and we let $Q_1=\{a=x_0<\cdots<x_m=b\}$, $Q_2=\{b=x_m<\cdots<x_n=c\}$

We have $L(f,Q_1)\leq\int_a^b f\leq U(f,Q_1)$. $L(f,Q_2)\leq\int_b^c f\leq U(f,Q_2)$

Then, as in (*), we have $L(f,P)=L(f,Q_1)+L(f,Q_2)\leq\int_a^b f+\int_b^c f\leq U(f,Q_1)+U(f,Q_2)=U(f,P)$

Since f is integrable on $[a,c]$ we have $\int_a^c f=\sup\{L(f,P): P \text{ is Partn of } [a,c]\}\leq\int_a^b f+\int_b^c f\leq\inf\{U(f,P): P \text{ is Partn of } [a,c]\}$

$\Longrightarrow \int_a^c f = \int_a^b f+\int_b^c f$

Riemann Sum Let $f:[a,b]\to\mathbb{R}$, $P=\{a=x_0<\cdots<x_n=b\}$.

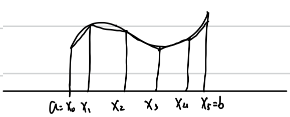
A **Riemann Sum** is any sum of the form : $S(f,P)=\sum_{j=1}^n f(t_j)(x_j-x_{j-1})$, where $t_j\in[x_{j-1},x_j]$ for $j=1,\cdots,n$

Some "canonical" choices: **Left Sum**: $S_l(f,P)=\sum_{j=1}^n f(x_{j-1})\cdot(x_j-x_{j-1})$

Right Sum: $S_R(f,P)=\sum_{j=1}^n f(x_j)\cdot(x_j-x_{j-1})$

mid Point $S_m(f,P)=\sum_{j=1}^n f(\frac{x_{j-1}+x_j}{2})(x_j-x_{j-1})$

An "almost" Rieman Sum, **trapezoid Sum** $T(f,P)=\frac{1}{2}[S_l(f)+S_r(f)]=\sum_{j=1}^n\frac{f(x_{j-1})+f(x_j)}{2}\cdot(x_j-x_{j-1}) = \frac{1}{2}\cdot f(a)\cdot(x_1-a)+\sum_{j=2}^{n-1}f(x_j)(x_j-x_{j-1})+\frac{1}{2}f(b)\cdot(b-x_{n-1})$



Theorem If $f : [a,b] \to R$, then TFAE (都等价)

(i) f is integrable

(ii) there is a number I_f s.t. given any $\varepsilon > 0$, $\exists P_\varepsilon$ of $[a,b]$

s.t. • for any refinement P of P_ε and

• any Riemann Sum $S(f,P)$

we have $|S(f,P) - I_f| < \varepsilon$, further more $I_f = \int_a^b f$

Given $t_j \in [x_{i-1}, x_i]$ and $f, g : [a,b] \to R$ we have for $\alpha, \beta \in R$ $S(\alpha f + \beta g, P) = \alpha \cdot S(f,P) + \beta \cdot S(g,P)$

Remark: If $f : [a,b] \to R$ is continuous, P is a partition of $[a,b]$, then each of $L(f,P)$, $U(f,P)$ are Riemann Sum

Proof: See the proof of integrability of continuous

1.15

Theorem: Let $f:[a,b]$, then TFAE

(i) f is integrable on $[a,b]$ and

(ii) there is a number. I_f so given $\varepsilon>0$, there is a Partition P_ε s.t.

- for any refinere P of P_ε and any Riemann Sum $S(f,P)$

We have $|S(f,P)-I_f|<\varepsilon$, further more $I_f=\int_a^b f$

Proof: (i) $\Rightarrow$ (ii) Given $\varepsilon>0$, the Cauchy Criterion Provides P_ε so for any refinement P of P_ε

$$U(f,P)-L(f,P)<\varepsilon \quad ①$$

Write $P=\{a=x_0<x_1<\cdots<x_n=b\}$, and let $j=1,\cdots,n$ $t_j\in[x_{j-1},x_j]$

We define that $m_j=\inf\{f(x): x\in[x_{j-1},x_j]\}\leq f(t_j)\leq \sup\{f(x): x\in[x_{j-1},x_j]\}=M_j$

and hence $\sum_{j=1}^n m_j(x_j-x_{j-1})\leq \sum_{j=1}^n f(t_j)(x_j-x_{j-1})\leq \sum_{j=1}^n M_j(x_j-x_{j-1})$

$$L(f,P)\leq S(f,P)\leq U(f,P) \quad ②$$

Also $L(f,P)\leq \int_a^b f\leq U(f,P)$ ③

Then ①.②.③ $\Rightarrow$ $|S(f,P)-\int_a^b f|<\varepsilon$, In Particular, take $I_f=\int_a^b f$

(ii) $\Rightarrow$ (i) We let for $\varepsilon>0$, given, $P_{\frac{\varepsilon}{4}}$ be a Partition So $|S(f,P_{\frac{\varepsilon}{4}})-I_f|<\frac{\varepsilon}{4}$ for P is a refinement of $P_{\frac{\varepsilon}{4}}$, $S(f,P)$ is a Riemann Sum

We fix such $P=\{a=x_0<x_1<\cdots<x_n=b\}$

For $j=1,\cdots,n$ Let m_j, M_j be as ~~above~~ (below) we then find for each j

$x_j^*, x_j^{**}\in[x_{j-1},x_j]$. s.t. $f(x_j^*)<m_j+\frac{\varepsilon}{4(b-a)}$ and $M_j-\frac{\varepsilon}{4(b-a)}<f(x_j^{**})$

We then Consider Riemann Sums

$$S^*(f,P)=\sum_{j=1}^n f(x_j^*)\cdot(x_j-x_{j-1}),\quad S^{**}(f,P)=\sum_{j=1}^n f(x_j^{**})\cdot(x_j-x_{j-1})$$

Notice that $S^*(f,P)-L(f,P)=\sum_{j=1}^n \underbrace{[f(x_j^*)-m_j]}_{<\frac{\varepsilon}{4(b-a)}}(x_j-x_{j-1})<\sum_{j=1}^n\frac{\varepsilon}{4(b-a)}\cdot[x_j-x_{j-1}]=\frac{\varepsilon}{4(b-a)}\cdot(b-a)=\frac{\varepsilon}{4}$

and likewise $U(f,P)-S^{**}(f,P)<\frac{\varepsilon}{4}$

Thus $U(f,P)-L(f,P)=U(f,P)-S^{**}(f,P)+S^{**}(f,P)-I_f+I_f-S^*(f,P)+S^*(f,P)-L(f,P)$

$$<\frac{\varepsilon}{4}+|S^{**}(f,P)-I_f|+|I_f-S^*(f,P)|+\frac{\varepsilon}{4}<\varepsilon$$

use assumption on P

Hence, by Cauchy Criterion shows f is integrable. $\square$

Proposition: Let $f, g\ [a,b] \to \mathbb{R}$ each be integrable and $\alpha, \beta \in \mathbb{R}$

Then $\alpha f + \beta g : [a,b] \to \mathbb{R}$ is integrable, and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$

Proof: Let $\varepsilon > 0$, then find partitions of $[a,b]$

- P_1 s.t. for any refinement P of P_1, and any Riemann sum $S(f,P)$: $\left| S(f,P) - \int_a^b f \right| < \frac{\varepsilon}{2|\alpha|+1}$
- P_2 s.t. for any refinement 包含 Q of P_2 and any Riemann sum $S(g,P)$: $\left| S(g,Q) - \int_a^b f \right| < \frac{\varepsilon}{2|\beta|+1}$

Let $P = P_1 \cup P_2$, which refines each of P_1 and P_2, Write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and choose $t_j \in [x_{j-1}, x_j]$ for each j

Then, $S(\alpha f + \beta g, P) = \sum_{j=1}^{n} (\alpha f(t_j) + \beta g(t_j)) \cdot (x_j - x_{j-1}) = \alpha \sum_{j=1}^{n} f(t_j) \cdot (x_j - x_{j-1}) + \beta \sum_{j=1}^{n} g(t_j) \cdot (x_j - x_{j-1})$

We then have $\left| \underbrace{S(\alpha f + \beta g, P)}_{\alpha S(f,P) + \beta S(g,P)} - \left[\alpha \int_a^b f + \beta \int_a^b g \right] \right| \leq |\alpha| \cdot \left| S(f,P) - \int_a^b f \right| + |\beta| \cdot \left| S(g,P) - \int_a^b g \right|$

$$< |\alpha| \cdot \frac{\varepsilon}{2|\alpha|+1} + |\beta| \frac{\varepsilon}{2|\beta|+1}$$

$$< \varepsilon$$

<u>Proposition</u> (order Properties of integrals)

Let $f, g : [a,b] \to R$ each be integrable Then

(i) $f \geqslant 0 \Rightarrow \int_a^b f \geqslant 0$

(ii) $f \geqslant g \Rightarrow \int_a^b f \geqslant \int_a^b g$

(iii) $|f| : [a,b] \to R$ is integrable with $|\int_a^b f| \leq \int_a^b |f|$

(iv) $f \vee g$, $f \wedge g : [a,b] \to R$: ($f \vee g = \max\{f(x), g(x)\}$, $f \wedge g = \min\{f(x), g(x)\}$) are each integrable

<u>Proof</u> (ii) $f-g$ is integrable with $f-g \geqslant 0$ so $\int_a^b f - \int_a^b g = \int_a^b (f-g) \geqslant 0$ by (i), $\int_a^b f - \int_a^b g \geqslant 0 \Rightarrow \int_a^b f \geqslant \int_a^b g$

(iii) let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and for each $j = 1, \cdots, n$

$M_j' - m_j' = \sup\{|f(x)| : x \in [x_{j-1}, x_j]\} - \inf\{|f(x')| : x' \in [x_{j-1}, x_j]\}$

$= \sup\{|f(x)| : x \in [x_{j-1}, x_j]\} + \sup\{-|f(x')| : x' \in [x_{j-1}, x_j]\}$

$= \sup\{|f(x)| - |f(x')| : x, x' \in [x_{j-1}, x_j]\}$

$\leq \sup\{|f(x) - f(x')| : x, x' \in [x_{j-1}, x_j]\}$

$= \sup\{f(x) - f(x') : x, x' \in [x_{j-1}, x_j]\}$ (Symmetry)

$= M_j - m_j$, with $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$, $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

Then $U(|f|, P) - L(|f|, P) = \sum_{j=1}^{n} (M_j' - m_j') \cdot (x_{j-1} - x_j) \leqslant \sum_{j=1}^{n} (M_j - m_j) \cdot (x_{j-1} - x_j) = U(f, P) - L(f, P)$

Thus f satisfies Cauchy Criterion $\Rightarrow |f|$ Satisfies Cauchy Criterion

Then, for any Riemann sums we have

$|S(f, P)| = |\sum_{j=1}^{n} f(t_j) \cdot (x_j - x_{j-1})| \leqslant \sum_{j=1}^{n} |f(t_j)| \cdot (x_j - x_{j-1}) = S(|f|, P)$

It follows (how?) (exercise): that $|\int_a^b f| \leq \int_a^b |f|$

(iv) $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

1.17 Elementary ($\neq$ easy) theory of integration

lower, upper sum, Riemann Sums

$\underbrace{\hspace{3cm}}$

See close $\Rightarrow$ defn integrability

Cauchy Criterion

Cauchy Criterion

- Continuous $\Rightarrow$ integrable (uniform continuous)
- (A1) monotone $\Rightarrow$ integrable

} uniform Partitions i.e. $\ell(P) < \delta$ E.g. right uniform Sums

- additivity over integrals (A1) $[a,b] \Rightarrow$ integrable on $[a_1,b_1] \subseteq [a,b]$
- Riemann Sum. $L(f,P) \leq S(f,P) \leq U(f,P)$
- linearity
- order Properties

Towards Fundamental Theorems of Calculus

Proposition: Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$, Define $F:(a,b] \to \mathbb{R}$, $F(x) = \int_a^x f = \int_a^t f(t)\,dt$ we may call this "integrals accumulation function"

Then (i) F is continuous on $(a,b]$

(ii) $\lim_{x \to a^+} F(x) = 0$

Hence we define $F(a) = 0 = \int_a^a f$. Thus, $F:[a,b] \to \mathbb{R}$ and it's continuous on $[a,b]$

Proof: (i) A1, Q5(c) assumes that f is integrable on each $[a,x]$, $x \in [a,b]$

So $F(x) = \int_a^x f$ make sense. Now, let $a < x < x' \leq b$, and

we compute
$$F(x') - F(x) = \int_a^{x'} - \int_a^x f$$
$$= \int_a^x f + \int_x^{x'} f - \int_a^x f$$
$$= \int_x^{x'} f$$

Since f is integrable, it is bounded, i.e. $\sup_{x \in [a,b]} |f(x)| = M < \infty$, Thus $|f| \leq M$ on $[a,b]$. Hence, by Properties

$$|F(x') - F(x)| = \left|\int_x^{x'} f\right| \leq \int_x^{x'} |f| \leq \int_x^{x'} \underbrace{M}_{\text{Constant}} = M(x'-x) = M|x'-x|$$

Thus, given $\varepsilon > 0$, Let $\delta = \frac{\varepsilon}{M+1}$ and we have $|x'-x| < \delta \Rightarrow |F(x') - F(x)| \leq M \cdot \delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$

Hence F is (uniformly) continuous on $[a,b]$

(ii) We use M as above

$$\left|\int_a^x f - 0\right| = \left|\int_a^x f\right| \leq \int_a^x |f| \leq \int_a^x M = M \cdot (x-a)$$ Proceed as above $\square$

Theorem (Mean Value for Integrals) or (Average value for integrals)

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then there exists $c \in [a,b]$ s.t. $\int_a^b f = f(c)\cdot(b-a)$

Proof: We use two importance facts about continuous function

Extreme Value Theorem $\exists\, x^*, x^{**} \in [a,b]$ s.t. $f(x^*) = m = \min\{f(x) : x \in [a,b]\}$ and $f(x^{**}) = M = \max\{f(x) : x \in [a,b]\}$

Then $m \le f \le M$ on $[a,b]$ so order properties provide $m\cdot(b-a) = \int_a^b m \le \int_a^b f \le \int_a^b M = M\cdot(b-a)$

So $f(x^*) = m \le \frac{1}{b-a}\int_a^b f \le M = f(x^{**})$

Intermediate Value Theorem: Since $f(x^*) \le \frac{1}{b-a}\int_a^b f \le f(x^{**})$, there is c between x^*, x^{**} and Hence $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$ $\square$

f integrable $\Rightarrow F(x) = \int_a^x f$ defines a continuous function

f continuous $\Rightarrow F$ differentiable (BELOW)

Fundamental Theorem of Calculus I

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then satisfies that • F is differentiable on $[a,b]$ with $F' = f$ on $[a,b]$

Proof: Let $x \in [a,b]$ we wish to examine the quotient $\frac{F(x+h)-F(x)}{h}$ when $x+h \in [a,b]$

$h>0$: $\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f = \frac{1}{h}\cdot f(c_h)\cdot(x+h-x)$, by M.V.T, where $c_h \in [x, x+h]$

just like continuity proposition $= f(c_h)$

$h<0$: $\frac{F(x+h)-F(x)}{h} = \frac{F(x)-F(x+h)}{-h} = -\frac{1}{h}\int_{x+h}^{x} f = -\frac{1}{h}\cdot f(c_h)\cdot(x-(x+h)) = f(c_h)$ where $c_h \in [x+h, x]$

Hence $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c_h) \underset{f \text{ is continuous}}{=} f(\lim_{h\to 0} c_h) \underset{\text{by squeeze theorem}}{=} f(x)$

Thus $F'(x)$ exists, and equals $f(x)$, for $x \in [a,b]$ $\square$

Remark: Notice that we really found

- left derivative at $x=b$
- right derivative at $x=a$

1.20

This week and after : Tutorial in RCH 305

Notation : Let $-\infty \le a \le b \le \infty$ in R_1, $f:(a,b) \to R$ be continuous. Fix $c \in (a,b)$ and define

$$F:(a,b) \to R \quad , \quad F(x) = \begin{cases} \int_c^x f & , \text{ if } x \geq c \\ -\int_x^c f & , \text{ if } x < c \end{cases}$$

We know, from the Fundamental Theorem of C. that $F'(x) = f(x)$ for $x > c$

Let us complete $F'(x)$ for $x < c$. Let $c' \in (a,c)$ for $x \in (c',c)$

We have $\int_{c'}^c f = \int_{c'}^x f + \int_x^c f \Rightarrow F(x) = -\int_x^c f = \int_{c'}^x f - \int_{c'}^c f$

$$\Rightarrow F'(x) = \frac{d}{dx}\left[\int_{c'}^x f - \int_{c'}^c f\right] = f(x)$$

It will be convenient, here after, to let $\int_c^x f = -\int_x^c f$ if $x < c$

and we have F.T. of C.I : $\frac{d}{dx}\int_c^x f = f(x)$, for $x \in (a,b)$

Logarithm and Exponential a rigourous approach

Define for $x \in (0,\infty)$, $L(x) = \int_1^x \frac{1}{t} dt$, we shall use only integral and differentiation rules to gain theory of L

F.T. of C.I $\Rightarrow$ $L'(x) = \frac{1}{x}$, also $L(1) = 0$

Proposition : If $a, b > 0$, then $L(ab) = L(a) + L(b)$

Proof: Let $F(x) = L(a \cdot x)$, then chain rules provides $F'(x) = \frac{1}{ax} \cdot \frac{d}{dx} ax = \frac{1}{x} = L'(x)$

Hence $F' - L' = 0 \Rightarrow F - L = C$ (constant), by Mean Value Theorem

$$\Rightarrow F = L + C \; *$$

Then, $L(a) = F(1) = L(1) + C \underset{\uparrow \text{defn of } f}{=} C$, also $L(ab) = F(b) \underset{\uparrow (*)}{=} L(b) + L(a)$ □

Proposition: For $a > 0$, $q \in Q$ $L(a^q) = q\, L(a)$ (Convention $a^0 = 1$)

Proof: First, if $n \in N$, $L(a^n) = L(a) + L(a^{n-1}) = \cdots\cdots = \underbrace{L(a) + L(a) + \cdots + L(a)}_{n} = n \cdot L(a)$

Second $L(a) = L((a^{\frac{1}{n}})^n) \Rightarrow L(a^{\frac{1}{n}}) = \frac{1}{n} \cdot L(a)$ (1)

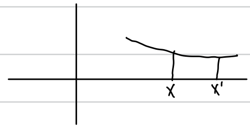
Third $0 = L(1) = L(a \cdot a^{-1}) = L(a) + L(a^{-1}) \Rightarrow L(a^{-1}) = -L(a)$ (2)

Then (1) and (2) $\Rightarrow L(a^m) = m \cdot L(a)$, for $m \in Z$. For $q = \frac{m}{n}$, $m \in Z$, $n \in N$

We combine (1), (2), (3) to get $L(a^q) = m\, L(a^{\frac{1}{n}}) = \frac{m}{n} L(a)$ $(q = \frac{m}{n})$

Proposition: (i) L is increasing: $0<x<x'$ then $L(x) < L(x')$

(ii) $\lim_{x\to 0^+} L(x) = -\infty$, $\lim_{x\to\infty} L(x) = \infty$



Proof (i) $L(x') - L(x) = \int_x^{x'} \frac{1}{t}\,dt \geqslant \int_x^{x'} \frac{1}{x'}\,dt = \frac{1}{x'}(x'-x) > 0$

[Alternatively: $L'(x) = \frac{1}{x} > 0$, M.V.T. $\Rightarrow L$ is strictly increasing]

(ii) To see that $\lim_{x\to\infty} L(x) = \infty$, it suffices to find $(a_n)_{n=1}^{\infty} \subset (0,\infty)$ s.t. $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} L(a_n) = \infty$ [exercise]

Consider $(2^n)_{n=1}^{\infty}$, and we have $\lim_{n\to\infty} L(2^n) = \lim_{n\to\infty} n\underbrace{L(2)}_{>L(1)=0} = \infty$. Likewise, $\lim_{n\to\infty} 2^{-n} = 0$ and $\lim_{n\to\infty} L(2^{-n}) = \lim_{n\to\infty} (-n)\cdot \underbrace{L(2)}_{>0} = -\infty$ $\square$

Corollary: $L: (0,\infty) \to R$ is one to one and onto

Proof: Increasing $\Rightarrow$ one to one. Since $\lim_{x\to\infty} L(x) = -\infty$, $\lim_{x\to\infty} L(x) = \infty$, and I.V.T provides that L is onto

Define: $E: R \to (0,\infty)$ to be L^{-1} (inverse function)

Hence $E(L(x)) = x$, $x \in (0,\infty)$ and $L(E(y)) = y$ if $y \in R$

Derivative of E: if $y\in R$, $L(E(y)) = y$ ($E = L^{-1}$ is differentiable) $\xRightarrow{\text{chain rule}}$ $\frac{1}{E(y)}\cdot E'(y) = 1$ $\Rightarrow E(y) = E'(y)$

Rules for E: Let $c, d \in R$ (i) $E(c+d) = E(c)\cdot E(d)$ (ii) $E(-c) = \frac{1}{E(c)}$ (iii) $E(0) = 1$ (iv) $E(qc) = (E(c))^q$, $q \in Q$

Pf: (i) let $c = L(a)$, $d = L(b)$ (L is onto) $E(c+d) = E(L(a)+L(b)) = E(L(ab)) = ab = E(c)\cdot E(d)$

(iii) $L(1) = 0$, so $E(0) = 1$

(ii) use (i) and (iii)

(iv) $E(qc) = E(q\cdot L(a)) = E(L(a^q)) = a^q = (E(c))^q$

What's $E(1)$? we note that $\lim_{h\to 0} \frac{L(1+h)-L(1)}{h} = \frac{L(1+h)}{h} = L'(1) = \frac{1}{1} = 1$

Hence $1 = \lim_{n\to\infty} \frac{L(1+\frac{1}{n}) - L(1)}{\frac{1}{n}} = \lim_{n\to\infty} n\cdot L(1+\tfrac{1}{n}) = \lim_{n\to\infty} L((1+\tfrac{1}{n})^n)$

Since, E is continuous. $E(1) = E(\lim_{n\to\infty} L((1+\frac{1}{n})^n)) = \lim_{n\to\infty} E[L((1+\frac{1}{n})^n)] = \lim_{n\to\infty} (1+\frac{1}{n})^n = e$

From rule (iv), $E(q) = e^q$ for $q \in Q$, if $x \in R$, write $x = \lim_{n\to\infty} q_n$, each $q_n \in Q$, and we define $e^x = E(x) = \lim_{n\to\infty} E(q_n) = \lim_{n\to\infty} e^{q_n}$

Define: for $a>0$, we have $a = E(L(a)) = e^{L(a)}$, and we let $a^x = E(L(a)x) = e^{L(a)\cdot x}$

Exercise with chain rule (i) $\frac{d}{dx}(a^x) = L(a)\cdot a^x$ (ii) $L(a^x) = L(a)x = x\cdot L(a)$ (iii) $P \in R$, $x>0$. $x^P = e^{P\cdot L(x)}$

$\frac{d}{dx} x^P = P\cdot x^{P-1}$

We call L: the logarithm function, $L = Log = ln = log_e$, E: the exponential function, $E(x) = e^x$

1.22

Earlier: F.T. of C.I: f continuous $\Rightarrow \frac{d}{dx}\int_a^x f = f(x)$ i.e. there exists F s.t. $F' = f$

<u>Fundamental Theorem of Calculus II</u>: Let $f, F : [a,b] \to \mathbb{R}$ satisfy that
- f is integrable
- F is continuous on $[a,b]$
- F is differentiable on (a,b), with $F' = f$ on (a,b)

Then, $F(b) - F(a) = \int_a^b f$

<u>Proof</u>: Let $\varepsilon > 0$. Find a partition P_ε on $[a,b]$, so
- for every refinement P of P_ε
- for every Riemann sum $S(f,P)$, we have

$$\left| S(f,P) - \int_a^b f \right| < \varepsilon$$

Take P as above, write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

Now, let us consider F on each $[x_{j-1}, x_j]$
- F is continuous on $[x_{j-1}, x_j]$
- F is differentiable on (x_{j-1}, x_j) [can use closed interval, except for $j = 0, n$]

Thus, Mean Value Theorem tell us:
- there exists $c_j \in (x_{j-1}, x_j) \subset [x_{j-1}, x_j]$

s.t. $F(x_j) - F(x_{j-1}) = F'(c_j) \cdot (x_j - x_{j-1})$ (*)

Now, we consider $F(b) - F(a) = \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})]$ (telescope)

$= \sum_{j=1}^{n} f(c_j) \cdot (x_j - x_{j-1})$ (by *)

$= S(f,P)$ (a Riemann sum)

Hence, $\left| F(b) - F(a) - \int_a^b f \right| = \left| S(f,P) - \int_a^b f \right| < \varepsilon$ (thanks to choice of P ☺)

Since $\varepsilon > 0$ is arbitrary, we see desired result □

Remarks: (i) Suppose $F, G : [a,b] \to$, both satisfy $F' = f = G'$, for integrable f. Then $(F-G)' = F' - G' = f - f = 0 \overset{M.V.T}{\Longrightarrow} F - G = C$ - constant

Hence $F(x) = G(x) + C$ for any x in $[a,b]$

(ii) If $f : [a,b] \to \mathbb{R}$ is <u>continuous</u>, then f is integrable (theorem from earlier) and $F(x) = \int_a^x f$ defines an antiderivative

<u>Moral</u>: f continuous $\Rightarrow$ an antiderivative exists

Notation: If f is continuous (on some intervals), and F is an antiderivative of f. i.e. $F' = f$ (on interval of said intervals)

Write $\int f(x)\,dx = F(x) + C$

Antiderivatives we use everyday: ① $P \neq -1$, $\int x^P dx = \frac{x^{P+1}}{P+1} + C$. $P = -1$, $\int \frac{1}{x} dx = \log|x| + C = \log(k \cdot |x|)$ where $k = e^C$

② $\int e^x dx = e^x + C$

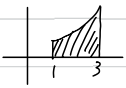
Accepting Theory of trigonometric functions we have : ① $\int \cos dx = \sin x + C$ ③ $\int \sin dx = -\cos x + C$

② $\int \sec^2 x\,dx = \tan x + C$ ④ $\int \csc^2 x\,dx = -\cot x + C$

⑤ $\int \frac{1}{x^2+1} dx = \arctan x + C$ — $(-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow R$ one to one onto

⑥ $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ — $[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ one to one onto

Eg: $\int_1^3 x^3 dx = \frac{1}{4} x^4 \Big|_1^3 = \frac{1}{4}[3^4 - 1^4] = \frac{1}{4}[81 - 1] = 20$



Theorem (Change of Variables / Substitution / "Reverse Chain Rule")

Suppose:
- $g[a,b] \to R$ is differentiable with g' continuous
- f is defined on $g([a,b])$ with $f \circ g : [a,b] \to R$ continuous

Then $\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$

Change Endpoints!!!

Proof: Let F be any antiderivative of f [$g([a,b]) = [c,d]$. Let $F(x) = \int_c^x f$ F.T. of C. I]

Let $H: [a,b] \to R$ be given by $H(x) = F(g(x))$, then chain rule provides $H'(x) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$

and F.T. of C. II provides: $H(b) - H(a) = \int_a^b f(g(x))\, g'(x)\, dx$

but F.T. of C, again provides $\int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a)) = H(b) - H(a)$

Antiderivative form

$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du \Big|_{u = g(x)}$

Ex: (i) $\int xe^{-x^2}dx = -\frac{1}{2}\int e^{-x^2}(-2x)\,dx = -\frac{1}{2}\int e^{u}du = -\frac{1}{2}e^{u}+C = -\frac{1}{2}e^{-x^2}+C$

$\because$ try again : $u=-x^2 \Rightarrow du=-2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

(ii) $\int_1^3 x(x^2+4)^{91}dx$, let $u = x^2+4 \Rightarrow du = 2x\,dx$

$= \frac{1}{2}\int_5^{13} u^{91}du = \frac{1}{2}\cdot\frac{u^{92}}{92}\Big|_5^{13} = \frac{1}{184}\cdot[13^{92}-5^{92}]$

(iii) $\int \cos^m x\cdot\sin^n x\,dx$ with $n=2k+1$ (odd)

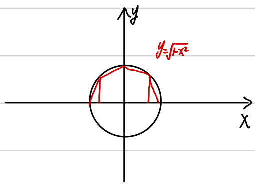
$= \int \cos^m x\cdot\sin^{2n}x\cdot\sin x\,dx$

$= \int \cos^m x\cdot(1-\cos^2 x)^k\cdot\sin x\,dx$ let $u=\cos x$, $du=-\sin x\,dx$

$= -\int u^m(1-u^2)^k\,du\,\Big|_{u=\cos x}$

Tutorial 1.22

Def'n $\pi = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx$



Using Trapezoidal Sums:

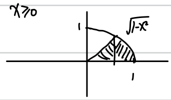
$\pi \geqslant 2T\left(\sqrt{1-\bullet^2}, \{-1<-\frac{4}{5}<0<\frac{4}{5}<1\}\right)$

$= 2\left[1+\frac{14}{25}\right] > 3$

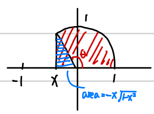
Let for $-1\leq x\leq 1$. $\operatorname{Arcos}(x) = x\sqrt{1-x^2} + 2\int_x^1\sqrt{1-u^2}\,du$

Then $\frac{1}{2}\operatorname{Arcos}(x)$ is the area is the area of :

$x\geqslant 0$



$x<0$



Note : $\frac{1}{2}\operatorname{Arcos}x$ is proportional to the angle θ. Hence it's reasonable to measure $\theta = \operatorname{Arcos}(x)$

$\operatorname{Arcos}(-1) = \pi$

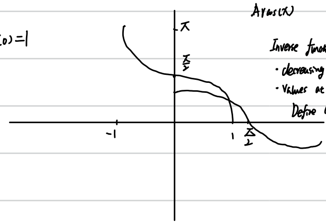
$\operatorname{Arcos}(0) = 2\int_0^1\sqrt{1-u^2}\,du \overset{\text{symmetry}}{=} \int_{-1}^{1}\sqrt{1-u^2} = \frac{\pi}{2}$

$\operatorname{Arcos}(1) = 0$

Derivatives: $\mathrm{Arcos}'(x) = \underbrace{\sqrt{1-x^2} + x\cdot\frac{1}{2}\frac{1}{\sqrt{1-x^2}}\cdot(-2x)}_{\text{Product by chain rules}} + \underbrace{-2\sqrt{1-x^2}}_{\int_x^1 = -\int_1^x}$ F.T. of C.I

$$= -\frac{x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2}\,\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

Hence, $\mathrm{Arcos}'(x) < 0 \overset{M.V.T}{\Longrightarrow}$ decreasing. $\lim_{x\to -1^+} \mathrm{Arcos}'(x) = -\infty = \lim_{x\to 1^-} \mathrm{Arcos}'(x)$ $\quad \mathrm{Arcos}'(0) = 1$



Inverse function: $[-1,1] \to [0,\pi]$
- decreasing ⟹ one to one
- Values at $x = \pm 1$, I.V.T ⟹ on to

Define $\cos = \mathrm{Arcos}^{-1}\ [0,\pi] \to [-1,1]$

$\mathrm{Arcos}'' = -\frac{x^2}{(1-x^2)^{\frac{3}{2}}}$: $\mathrm{Arcos}''(x) > 0$ if $x < 0$ ⟹ Cave up

$\mathrm{Arcos}''(x) < 0$ if $x > 0$ ⟹ Cave down

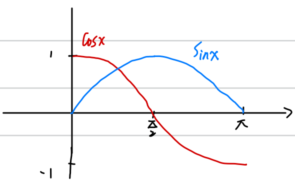
Define $\sin\theta = \sqrt{1-\cos^2\theta}$. Hence $[0,\pi] \to [0,1]$ with $\sin(0) = 0$, $\sin(\frac{\pi}{2}) = 1$, $\sin\pi = 0$

Derivatives of Sin, Cos

$\mathrm{Arcos}(\cos\theta) = \theta \overset{\text{chain rule}}{\Longrightarrow} -\frac{1}{\sqrt{1-\cos^2\theta}}\cos'\theta = 1 \Rightarrow \cos'\theta = -\sin\theta$

$\sin'\theta = \frac{d}{d\theta}\sqrt{1-\cos^2\theta} = \frac{1}{2}(1-\cos^2\theta)^{-\frac{1}{2}}(-2\cos\theta\cdot\cos'\theta) = \cos'\theta$

Hence $\sin'(0) = 1$, $\sin'(\frac{\pi}{2}) = 0$, $\sin'(\pi) = -1$ and $\sin''(\theta) = -\sin\theta < 0$, if $0 < \theta < \pi$ ⟹ Concave down



Extension to R

(I) we define $\sin, \cos\ [-\pi,\pi] \to [-1,1]$

Cos is even: $\cos(-\theta) = \cos\theta$, $\cos\theta = \cos\theta$ if $\theta \geq 0$

Sin is odd: $\sin(-\theta) = -\sin\theta$, $\sin\theta = \sin\theta$ if $\theta \geq 0$

(II) we define $\cos, \sin\ R \to [-1,1]$, $\cos(\theta + 2\pi n) = \cos\theta$, $\sin(\theta+2\pi n) = \sin\theta$, $\theta \in [-\pi,\pi]$ $n \in \mathbb{Z}$

Angle Sum If $s,t \in \mathbb{R}$: $\cos(s+t) = \cos s \cdot \cos t - \sin s \cdot \sin t$

Lemma: Let $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable, with $\cdot\ f(0) = f'(0)$ $\cdot\ f''+f=0$. Then $f=0$

Proof: Let $g = (f')^2 + f$ Then $g(0)=0$ and $g' = 2f' \cdot f'' + 2ff' = 2f'[f''+f] = 0 \Rightarrow g$ constant Hence $g=0$ Then $0 \le f^2 \le g$ □

Proof of D.A. for cos: Let $a,b,s \in \mathbb{R}$ be fixed Define $f: \mathbb{R} \to \mathbb{R}$

$f(t) = \cos(s+t) - a\cos t + b\sin t$ Then $f'(t) = -\sin(s+t) + a\sin t + b\cos t$

$f''(t) = -\cos(s+t) + a\cos t - b\sin t$

$\Rightarrow f''+f=0$

Now, we wish to choose a,b to satisfy : $f(0)=0$, hence $0 = f(0) = \cos s - a \Rightarrow a = \cos s$

$f'(0)=0$ hence $0 = f'(0) = -\sin s + b \Rightarrow b = \sin s$.

With these choices of a,b, the lemma tells us that $f(t)=0$,

Hence $0 = \cos(s+t) - [\cos s\cos t - \sin s\sin t]$

Double angle formula Since $\cos^2 t + \sin^2 t = 1$, the angle sum formula gives $\cos 2t = \cos^2 t - \sin^2 t = \begin{cases} 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1}{2}[1-\cos 2t] \\ 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1}{2}[1+\cos 2t] \end{cases}$

Angle Sum formula for sin $\sin(s+t) = \cos s\sin t + \sin s\cos t$

Proof Fix s in $\mathbb{R}$, for t consider $\cos(s+t) = \cos\cos t - \sin s\sin t$

and take derivative both sides □

Double angle : $\sin 2t = 2\sin t\cos t$

1.24

Fun with trig (and, the like)

Angle Sum identities : $s, t \in \mathbb{R}$

$\cos(\frac{\pi}{12})$
$\Downarrow$
$\cos(\frac{\pi}{3} - \frac{\pi}{4})$

$$\cos(s+t) = \cos s \cdot \cos t - \sin t \cdot \sin t \implies \begin{cases} \cos^2 x = \frac{1}{2}(1+\cos 2x) \\ \sin^2 x = \frac{1}{2}(1-\cos 2x) \end{cases}$$

$$\sin(s+t) = \sin s \cos t + \cos s \cdot \sin t \implies \sin\cos x = \frac{1}{2}\sin 2x$$

E.g. (i)
$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$ ← usually, stop here
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C$$

(ii)
$$\int \cos^4 x\,dx = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$$
$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$
$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}[1+\cos 4x]\right) dx$$
$$= \frac{1}{4}\left(x + \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x\right) + C$$

(iii) $\int \sin x \cos^4 x\,dx$, $u = \cos x$, $du = -\sin x\,dx$
$$= -\int u^4\,du \Big|_{u=\cos x}$$
$$= -\frac{\cos^5 x}{5} + C$$

(iv)
$$\int \sin^2 x \cdot \cos^4 x\,dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos^2 x\,dx$$
$$= \int \frac{1}{4}\sin^2 2x \cdot \frac{1}{2}[1+\cos 2x]\,dx$$
$$= \frac{1}{8}\int (\underbrace{\sin^2 2x}_{\frac{1}{2}[1-\cos 4x]} + \underbrace{\sin^2 2x \cdot \cos 2x}_{u=\sin 2x})\,dx \quad \cdots\cdots$$

Change of Variable (Antiderivative) $\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du \Big|_{u=g(x)}$, f continuous , g' continuous

Inverse form: Suppose we try $x = g(u)$ ("nice" g)
$$\int f(x)\,dx = \int f(g(u))\cdot g'(u)\,du \ \Big|_{x=g(u)}$$ hopefully: we can resolve formula in u, back in terms of x

Trig Sustitution

Forms	Substitution	main identity	"dx"
a^2-x^2	$x = a\sin\theta$	$a^2-x^2 = a^2\cos^2\theta$	$dx = a\cdot\cos\theta\, d\theta$
x^2+a^2	$x = a\cdot\tan\theta$	$x^2+a^2 = a^2\cdot\sec^2\theta$ $\tan^2\theta+1 = \sec^2\theta$	$dx = a\cdot\sec^2\theta\, d\theta$

Ex: (i) $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}}$, $x = 3\cos\theta$, $dx = 3\cos\theta d\theta$

$-3<x<3 \Rightarrow -1<\sin\theta<1 \Rightarrow -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$$= \int \frac{3\cos\theta}{(3\cos^2\theta)^{\frac{3}{2}}}\, d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\cos^3\theta}\, d\theta = \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta + C$$

$$= \frac{1}{9}\cdot\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C$$

$$= \frac{1}{9}\cdot\frac{\frac{1}{3}x}{\sqrt{1-(\frac{1}{3}x)^2}} + C$$

$$= \frac{1}{9}\cdot\frac{x}{\sqrt{9-x^2}} + C$$

(ii) $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$ $x+1 = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$

$$= \int \frac{2\sec^2\theta}{2^2\sec^2\theta}\, d\theta$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\cdot \mathrm{Arctan}\left(\frac{x+1}{2}\right) + C$$

(iii) $\int \sqrt{1-x^2}\, dx$ $x = \sin\theta$, $dx = \cos\theta d\theta$ ($\cos\theta>0$, since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$)

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int [1+\cos 2\theta]\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{2}\left[\mathrm{Arcsin}(x) + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\mathrm{Arcsin}(x) + x\cdot\sqrt{1-x^2} + C$$

$$\Rightarrow \mathrm{Arcsin}(x) = 2\int\sqrt{1-x^2}\, dx - x\sqrt{1-x^2} + C'$$

$\left[\mathrm{Arcsin}(x) = \frac{\pi}{2} - \mathrm{Arccos}(x)\right]$

(iv) $\int \frac{dx}{\sqrt{x^2+1}} dx$, $x = \tan\theta$, $d\theta = \sec^2\theta d\theta$.

$x\in\mathbb{R}$, $-\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow \cos\theta>0$

i.e $\sec\theta>0$

$\sqrt{\sec^2\theta} = \sec\theta$

$$= \int \frac{\sec\theta\, d\theta}{\sec\theta}$$

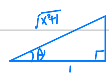
$$= \int \sec\theta\, d\theta$$

(dirty trick!) $= \int \sec\theta\cdot\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$

$$= \int \frac{\sec^2\theta + \tan\theta\sec\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \log|\sec\theta + \underbrace{\tan\theta}_{x}| + C$$

$$= \log(\sqrt{x^2+1} + x) + C$$



$\tan\theta = \frac{x}{1}$

$\sec\theta = \frac{\sqrt{x^2+1}}{1}$

(iv) $\int \frac{1}{\sqrt{x^2+1}}\,dx$

$= \int \frac{\cosh t}{\cosh t}\,dt$

$= \int dt$

$= t + C$

$= \log(x+\sqrt{x^2+1}) + C$

, $\cosh t = \frac{e^t+e^{-t}}{2}$, $\sinh t = \frac{e^t-e^{-t}}{2} = \cosh' t$

$\cosh^2 t - \sinh^2 t = 1$ (hyperbolic, identity)

$x = \sinh t$, $dx = \cosh t\,dt$

$x = \frac{e^t-e^{-t}}{2} \Rightarrow 2x = e^t - e^{-t} \Rightarrow 0 = e^{2t} - 2xe^t - 1 \Rightarrow e^t = \frac{2x \pm \sqrt{(2x)^2+4}}{2} \Rightarrow e^t = x+\sqrt{x^2+1}$, $e^t > 0$

$\Downarrow$

$t = \log e^t = \log(x+\sqrt{x^2+1})$

1.27

Method of Partical fractions

Warm up: $\int \sec\theta \, d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} d\theta$

$$= \int \frac{\sin^2\theta}{\cos\theta} d\theta + \int \cos\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cos\theta \, d\theta + \int \cos\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{1-\sin^2\theta} \cos\theta \, d\theta + \int \cos\theta \, d\theta . \quad \text{Let } u = \sin\theta$$

$$= \int \frac{u^2}{1-u^2} du + \int \cos\theta \, d\theta$$

$$= \int \frac{u^2+1-1}{1-u^2} du + \int \cos\theta \, d\theta$$

$$= \int \frac{1}{1-u^2} - 1 + \int \cos\theta \, d\theta$$

$$= \int \frac{1}{2} \cdot \frac{1}{1+u} + \frac{1}{2} \cdot \frac{1}{1-u} - 1 + \int \cos\theta \, d\theta$$

$$= \frac{1}{2}\left(\log|1+u| + \log|1-u|\right) - u + \sin\theta + C$$

$$= \frac{1}{2}\left(\log|\sin\theta+1| + \log|1-\sin\theta|\right) + C$$

Theorem (i) Let $q \neq 0$ be a Polynomial with $\mathbb{R}$-coefficencs Then we may write

$$q(x) = a(x-r_1)^{m_1} \cdot \cdots (x-r_m)^{m_m} \cdot (x^2+b_1x+c_1)^{n_1} \cdot \cdots\cdots (x^2+b_Nx+c_N)^{n_N}$$

Where $a \neq 0$, $r_1, r_2, \ldots, r_m$ are distince $\mathbb{R}$-roots of q and $b_1, \ldots b_n, c_1, \ldots c_n \in \mathbb{R}$ $\quad b_j^2 - 4c_j < 0$ for $j = 1, \ldots, n$

$= \deg q$

also $m_1, \ldots, m_n, n_1, \ldots, n_N \in \mathbb{N}$

(ii) Let P be $\mathbb{R}$-Polynomial with $\deg P < \deg q$, there are unique numbers $A_1, \ldots, B_n, \ldots C_n, \ldots$

So $\displaystyle \frac{P(x)}{q(x)} = \sum_{j=1}^{M} \sum_{k=1}^{m_j} \frac{A_{jn}}{(x-r_j)^n} + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{B_{jk}x + C_{jk}}{(x^2+b_jx+c_j)^k}$

Eg: $\frac{x^2+4x+3}{(x-1)^2(x^2+3x+4)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{Bx+C}{x^2+3x+4}$

$x^2+4x+3 = A_1(x-1)(x^2+3x+4) + A_2(x^2+3x+4) + (Bx+C)(x-1)^2$ (*)

Method ①:
- Work out RHS
- Match coefficients
- Solve 4x4 system linear equations

Method ②: Take advantage of the root $x=1$, Sub-in $x=1$ to (*)

$8 = 1+4+3 = A_1 0 + A_2(1+3+4) + (B\cdot 1+C)\cdot 0 = 8A_2 \Rightarrow A_1 = 1$

Put back into (*), and set $A_1 = A$

$x-1 = A_1[x^3+2x^2+x-4] + (Bx+C)\cdot(x^2-2x+1)$

Match coeffs
x^3: $0 = A+B$
x^2: $0 = 2A-2B+C$
x: $1 = A+B-2C$
x^0: $-1 = -4A+C$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ -4 & 0 & 1 & -1 \end{array}\right] \to$ $\frac{1}{8} = A$, $-\frac{1}{8} = B$, $-\frac{1}{2} = C$ ← Consistent!

$\frac{x^2+4x+3}{(x-1)^2(x^2+3x+4)} = \frac{1}{8}\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{8}\cdot\frac{x+4}{(x+\frac{3}{2})^2+\frac{5}{4}}$

Algorithm for integrating a rational function

Find $\int \frac{P(x)}{q(x)}\,dx$, P, q are Polynomials $q \neq 0$

(1) If $\deg P \geq \deg q$, Perform Polynomial divisions

$$q(x)\overline{\smash{)}\,\begin{array}{l} S(x) \\ P(x) \\ \hline r(x) \end{array}} \leftarrow \text{remainder} \qquad \Rightarrow S(x) + \frac{r(x)}{q(x)}$$

$\deg r < \deg q$

(2) Factor $q(x)$ [Warning : either obvious or really really hard!]

(3) We reduce $\frac{r(x)}{q(x)}$ by Partial fractions

(4) We must now integrate terms like $\int \frac{A}{(x-r)^m}\,dx$, $m \in \mathbb{N}$. $\int \frac{Bx+C}{(x^2+bx+c)^n}\,dx$ ☺ $b^2-4c<0$, $n \in \mathbb{N}$

To deal with ☺ We consider $x^2+bx+c = (x+\frac{b}{2})^2 + c - \frac{b^2}{4}$

$n=1 \Rightarrow$ easy , done before use log, Arctan

$n \geq 2$ $\int \frac{Bx+C}{[(x+\frac{b}{2})^2+D^2]^n}\,dx$, $x+\frac{b}{2} = D\tan\theta$ $dx = D\sec^2\theta\, d\theta$

$$= \int \frac{B(\tan\theta - \frac{b}{2})+C}{D^{2n}\sec^{2n}\theta} \cdot D\sec^2\theta\, d\theta$$

$$= \frac{B}{D^{2n-1}} \int \cos^{2n-1}\theta \sin\theta\, d\theta \quad , \; u = \cos\theta$$

Eventually, we want to deal with terms like $\int \cos^{2n}\theta\, d\theta$

Partial fractions reduction of a rational function.

Theorem. *Let $q \neq 1$ be a polynomial with $\mathbb{R}$-coefficients.*

(i) *We have factorization*

$$q(x) = a(x-r_1)^{m_1} \dots (x-r_M)^{m_M}(x^2+b_1x+c_1)^{n_1} \dots (x^2+b_Nx+c_n)^{n_n}$$

where $a \in \mathbb{R} \setminus \{0\}$, $r_1, \dots, r_M$ denote the real roots of q with respective multiplicities $m_1, \dots, m_M$, and $b_1, c_1, \dots, b_N, c_N \in \mathbb{R}$ with $b_j^2 - 4c_j < 0$ for $j = 1, \dots, N$

(ii) *If p is a polynomial with $\mathbb{R}$-coefficients and*

$$\deg p < \deg q$$

then there is a unique sequence $A_{1,1}, \dots, B_{N,m_N}, C_{N,m_N}$ of real numbers for which

$$\frac{p(x)}{q(x)} = \frac{1}{a}\left[\sum_{j=1}^{M}\sum_{k=1}^{m_j}\frac{A_{j,k}}{(x-r_j)^k} + \sum_{j=1}^{N}\sum_{k=1}^{n_j}\frac{B_{j,k}x + C_{j,k}}{(x^2+b_jx+c_j)^k}\right] \qquad (*)$$

Remark: Observe that $m_1 + \dots + m_M + 2[n_1 + \dots + n_N] = \deg q$.

You will be unlikely, in practice, to have to use this horrible formula $(*)$ in settings with $\deg q > 5$. Hence consider sample $\deg q = 5$ cases, where notation simplifies (e.g. we drop double subscripts where it shall not lead to confusion):

$$\frac{ap(x)}{(x-r_1)\dots(x-r_5)} = \sum_{j=1}^{5}\frac{A_j}{x-r_j}$$

$$\frac{ap(x)}{(x-r_1)\dots(x-r_3)(x-r_4)^2} = \sum_{j=1}^{4}\frac{A_j}{x-r_j} + \frac{B}{(x-r_4)^2} \qquad [\text{here } B = A_{4,2}]$$

$$\frac{ap(x)}{(x-r_1)\dots(x-r_3)(x^2+bx+c)} = \sum_{j=1}^{3}\frac{A_j}{x-r_j} + \frac{Bx+C}{x^2+bx+c}$$

$$\frac{ap(x)}{(x-r)(x^2+bx+c)^2} = \frac{A}{x-r} + \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2}.$$

Sketch proof of Theorem. **(i)** The Fundamental Theorem of Algebra gives a set of $\deg q$ $\mathbb{C}$-roots of q, including multiplicity. The non-$\mathbb{R}$ roots come in conjugate pairs, and give rise to the polynomials $x^2 + b_j x + c_j$, as suggested.

(ii) Let us consider the RHS (right hand side) of $(*)$, multiplied by $aq(x)$. What we and up with is a $\mathbb{R}$-linear span of polynomials from the set

$$F_q = \left\{ \frac{q(x)}{(x-r_j)^k} : j = 1, \dots, M, k = 1, \dots, m_j \right\} \cup \\ \left\{ \frac{q(x)}{(x^2 + b_j x + c_c)^k}, \frac{xq(x)}{(x^2 + b_j x + c_c)^k} : j = 1, \dots, N, k = 1, \dots, n_j \right\}.$$

Notice that this set has $m_1 + \cdots + m_M + 2[n_1 + \cdots + n_N] = \deg q$ elements. Furthermore,

$$\deg q = \dim V_q, \text{ where } V_q = \{p : p \text{ is a polynomial with } \deg p < \deg q\}.$$

We shall liberally use facts from linear algebra. We will be done once we establish that

$$F_q \text{ is a basis for } V_q. \tag{$**$}$$

The method is by (a horrible) induction.

We first establish some **base cases**. [Understanding these tells us a large part of why this Theorem is true.]

Case $\deg q = 0$. Then $q(x) = a$ (constant) and $V_q = \{0\}$, so there is nothing to prove.

Case $\deg q = 1$. Then $q(x) = a(x-r)$ and V_q is the space of constant polynomials, clearly spanned by $\frac{q(x)}{a(x-r)} = 1$ (constant).

Cases $\deg q = 2$. Here $V_q = \{Ax + B : A, B \in \mathbb{R}\}$. We have

$$q(x) = \begin{cases} a(x-r_1)(x-r_2) & r_1 \neq r_2 & \Rightarrow F_q = \{x - r_1, x - r_2\} \\ a(x-r)^2 & & \Rightarrow F_q = \{x - r, 1\} \\ a(x_2 + bx + c) & b^2 - 4c < 0 & \Rightarrow F_q = \{1, x\}. \end{cases}$$

In each case above, F_q is a basis for V_q.

Induction steps. [Now the real work begins.] We assume that $(**)$ holds for any q with $\deg q \leq n$, and fix such q.

Linear factor. Let $r \in \mathbb{R}$. By assumption $(**)$ we see that

- $(x-r)F_q = \{(x-r)f(x) : f \in F_q\}$ is a basis for $(x-r)V_q = \{(x-r)p(x) : p \in V_q\}$ (consider why this is a subspace of polynomials),
- if $q(r) \neq 0$ then q is linearly independent of $(x-r)V_q$ (think about this), and
- if $q(r) = 0$ then $r = r_j$ and $\frac{q(x)}{(x-r_j)^{m_j}}$ is linearly independent of $(x-r)V_q$.

Combining the facts, we see that

$$F_{(x-r)q} = \begin{cases} \{q\} \cup [(x-r)F_q] & \text{if } q(r) \neq 0 \\ \left\{\frac{q(x)}{(x-r_j)^{m_j}}\right\} \cup [(x-r)F_q] & \text{if } r = r_j \end{cases}$$

is a linearly independent set in $V_{(x-r)q}$ of size $\deg[(x-r)q] = \deg q + 1$, and hence a basis.

Quadratic factor. Let $b, c \in \mathbb{R}$ with $b^2 - 4c < 0$. By assumption $(**)$ we see that

- $(x^2+bx+c)F_q = \{(x^2+bx+c)f(x) : f \in F_q\}$ is a basis for $(x^2+bx+c)V_q = \{(x^2+bx+c)p(x) : p \in V_q\}$,
- if $b, c \neq b_j, c_j$ for any $j = 1, \ldots, M$ then q and xq are linearly independent of each other and of$(x^2+bx+c)V_q$ (think harder about this), and
- if $b, c = b_j, c_j$ for some j then $\frac{q(x)}{(x^2+b_jx+c_j)^{n_j}}$ and $\frac{xq(x)}{(x^2+b_jx+c_j)^{n_j}}$ are is linearly independent of each other and of $(x^2+bx+c)V_q$.

Combining the facts, we see that

$$F_{(x^2+bx+c)q} = \begin{cases} \{q, xq\} \cup [(x^2+bx+c)F_q] & \text{if } b, c \neq b_j, c_j \\ \left\{\frac{q(x)}{(x^2+b_jx+c_j)^{n_j}}, \frac{xq(x)}{(x^2+b_jx+c_j)^{n_j}}\right\} \cup [(x^2+bx+c)F_q] & \text{if } b, c = b_j, c_j \end{cases}$$

is a linearly independent set in $V_{(x^2+bx+c)q}$ of size $\deg[(x^2+bx+c)q] = \deg q + 2$, and hence a basis.

Hence we have all cases where $\deg q = n + 1$, and all cases where q admits no roots with $\deg q = n+2$ (hence n is even). Thus, by induction $(**)$ holds generally. $\square$

Written by Nico Spronk, for use by students of Math 148 at University of Waterloo.

An error estimate for trapezoidal sums

Theorem. *Let $f : [a,b] \to \mathbb{R}$ stisfy that*

- *f is twice differentiable on $[a,b]$, and*
- *$K = \sup_{x\in[a,b]} |f''(x)| < \infty$.*

Then, for any parititon $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we have that the trapezoid sums satisfy

$$\left| T(f,P) - \int_a^b f \right| \leq \frac{K}{12} \sum_{j=1}^{n} (x_j - x_{j-1})^3.$$

Remark. Notice that the error estimate is dominated by $\frac{K}{12} n \ell(P)^3$ where $\ell(P) = \max_{j=1,\dots,n}(x_j - x_{j-1})$. Hence we expect this to be small if $\ell(P)$ is small.

Proof. (I) (Big trapezoid estimate.) Let us first suppose that $P = \{a < b\}$ is a most trivial partition and we wish to consider the difference

$$T(f,P) - \int_a^b f = \frac{[f(a) + f(b)](b-a)}{2} - \int_a^b f.$$

We let the accumlated difference between a and $a+x$:

$$g : [0, b-a] \to \mathbb{R}, \quad g(x) = \frac{[f(a) + f(a+x)]x}{2} - \int_a^{a+x} f.$$

Let us consider derivatives, using product rule, chain rule and F.T. of C. I:

$$\begin{aligned} g'(x) &= \frac{f'(a+x)x}{2} + \frac{f(a) + f(a+x)}{2} - f(a+x) \\ &= \frac{f(a)}{2} + \frac{f'(a+x)x}{2} - \frac{f(a+x)}{2} \\ g''(x) &= \frac{f''(a+x)x}{2} + \frac{f'(a+x)}{2} - \frac{f'(a+x)}{2} = \frac{f''(a+x)x}{2}. \end{aligned}$$

Notice that

$$g(0) = 0 \quad \text{and} \quad g'(0) = 0.$$

Since $x \geq 0$ and $a + x \in [a, b]$ our assumptions on f'' provide that

$$|g''(x)| = \frac{|f''(a+x)|}{2} x < \frac{K}{2} x \quad \Rightarrow \quad -\frac{K}{2} x \leq g''(x) \leq \frac{K}{2} x.$$

We thus have

$$\int_0^x \left[-\frac{K}{2} u\right] du \leq \int_0^x g''(u)\, du \leq \int_0^x \frac{K}{2} u\, du \qquad \text{(order properties)}$$

$$\Longrightarrow \quad -\frac{K}{4} x^2 \leq g'(x) - g'(0) = g'(x) \leq \frac{K}{4} x^2 \qquad \text{(F.T. of C. II)}$$

We apply exactly the same reasoning (order properties, then F.T. of C. II) to the last estimate to see that

$$-\frac{K}{12} x^3 \leq g(x) \leq \frac{K}{12} x^3.$$

We then substitute $x = b - a$ to get

$$-\frac{K}{12}(b-a)^3 \leq g(b-a) = \frac{[f(a) + f(b)](b-a)}{2} - \int_a^b f \leq \frac{K}{12}(b-a)^3$$

which is the same as saying that

$$\left| \frac{[f(a) + f(b)](b-a)}{2} - \int_a^b f \right| \leq \frac{K}{12}(b-a)^3. \qquad (\heartsuit)$$

(II) Now let us refine this vulgar big trapezoid estimate into smaller trapezoids. For $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, as given, we apply $(\heartsuit)$ to each interval $[x_{j-1}, x_j]$:

$$\begin{aligned}
\left| T(f,P) - \int_a^b f \right| &= \left| \sum_{j=1}^n \frac{f(x_{j-1}) + f(x_j)}{2}(x_j - x_{j-1}) - \int_a^b f \right| \\
&= \left| \sum_{j=1}^n \frac{f(x_{j-1}) + f(x_j)}{2}(x_j - x_{j-1}) - \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f \right| && \begin{pmatrix}\text{additivity} \\ \text{over intervals}\end{pmatrix} \\
&\leq \sum_{j=1}^n \left| \frac{[f(x_{j-1}) + f(x_j)](x_j - x_{j-1})}{2} - \int_{x_{j-1}}^{x_j} f \right| && \text{(triangle ineq.)} \\
&\leq \sum_{j=1}^n \frac{K}{12}(x_j - x_{j-1})^3 = \frac{K}{12} \sum_{j=1}^n (x_j - x_{j-1})^3 && \text{by } (\heartsuit) \quad \square.
\end{aligned}$$

Example. Let us try to use geometric partitions to estimate $\log 2 = \int_1^2 \frac{dx}{x}$.

We let $f(x) = \frac{1}{x}$ and $P_n = \{1 < 2^{\frac{1}{n}} < 2^{\frac{2}{n}} < \cdots < 2\}$.

Let us compute the trapezoidal sum:

$$\begin{aligned} T(f, P_n) &= \sum_{j=1}^{n} \frac{2^{-\frac{j-1}{n}} + 2^{-\frac{j}{n}}}{2} (2^{\frac{j}{n}} - 2^{\frac{j-1}{n}}) \\ &= \sum_{j=1}^{n} \frac{1 + 2^{-\frac{1}{n}}}{2} (2^{\frac{1}{n}} - 1) = \frac{n}{2} (2^{\frac{1}{n}} - 2^{-\frac{1}{n}}). \end{aligned}$$

Now we compute the error estimate of the last theorem. We have $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$ so

$$K = \sup_{x \in [1,2]} |f''(x)| = 2.$$

We have error estimate

$$\begin{aligned} |T(f, P_n) - \log 2| &\leq \frac{K}{12} \sum_{j=1}^{n} (2^{\frac{j}{n}} - 2^{\frac{j-1}{n}})^3 \\ &= \frac{1}{6} \sum_{j=1}^{n} 2^{3\frac{j-1}{n}} (2^{\frac{1}{n}} - 1)^3 = \frac{(2^{\frac{1}{n}} - 1)^3}{6} \sum_{j=1}^{n} 2^{\frac{3}{n}(j-1)} \\ &= \frac{(2^{\frac{1}{n}} - 1)^3}{6} \cdot \frac{2^3 - 1}{2^{\frac{3}{n}} - 1} = \frac{7(2^{\frac{1}{n}} - 1)^3}{6(8^{\frac{1}{n}} - 1)} \end{aligned}$$

Hence we have

$$\left| \frac{n}{2} (2^{\frac{1}{n}} - 2^{-\frac{1}{n}}) - \log 2 \right| \leq \frac{7(2^{\frac{1}{n}} - 1)^3}{6(8^{\frac{1}{n}} - 1)} < \frac{7}{6} (2^{\frac{1}{n}} - 1)^2.$$

Notice that

$$\frac{7}{6} (2^{\frac{1}{n}} - 1)^2 < 10^{-6} \quad \Leftrightarrow \quad 2 < \left(\frac{\sqrt{7/6}}{10^3} + 1 \right)^n.$$

Let us estimate by first noticing that $1 < \sqrt{7/6}$ so it suffices to find n for which $2 < (\frac{1001}{1000})^n = (1.001)^n$. A bit of a computational experimentation shows that

$$n = 695$$

suffices. [We really do not wish to compute $\log_2$ of anything, since this is the same computational complexity as computing $\log 2$, and defeats the purpose of devising numerical error estimates.]

Remark. If we look at the proof of the Theorem we can refine the estimate to

$$\left|T(f,P) - \int_a^b f\right| \leq \frac{1}{12}\sum_{j=1}^n K_j(x_j - x_{j-1})^3 \quad \text{where each} \quad K_j = \sup_{x\in[x_{j-1},x_j]} |f''(x)|.$$

This has the disadvantage of often making the estimates more computationally complex, and improves only when we expect $|f''(x)|$ to be unusually large on a few small subintervals of $[a,b]$.

Remark. If, in the theorem above, we only assume that f is differentiable with

$$L = \max_{x\in[a,b]} |f'(x)| < \infty$$

[but not necessarily twice differentiable], we gain a much weaker estimate. Indeed consider the following difference of a midpoint sum with $P = \{a < b\}$:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f &= \left[f\left(\frac{a+b}{2}\right) - f(c)\right](b-a) && \begin{pmatrix}\text{M./A.V.T. for}\\ \text{Integrals}\end{pmatrix} \\ &= f'(d)\left(\frac{a+b}{2} - c\right)(b-a) && \begin{pmatrix}\text{M.V.T. for}\\ \text{Derivatives}\end{pmatrix} \end{aligned}$$

where c in $[a,b]$ arises from the Mean/Average Value Theorem for Integrals, and d between c and the midpoint $\frac{a+b}{2}$ arises from Mean Value Theorem for Derivatives, in particular $|\frac{a+b}{2} - c| < \frac{1}{2}(b-a)$. Hence we get estimate

$$\left|f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f\right| \leq \frac{L}{2}(b-a)^2.$$

As in (II) of the proof of the theorem we gain for $P = \{a = x_0 < \cdots < x_n = b\}$ midpoint estimate

$$\left|S_m(f,P) - \int_a^b f\right| \leq \frac{L}{2}\sum_{j=1}^n (x_j - x_j - 1)^2.$$

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Solving a system of linear equations

I wish to show a notionally convenient way of solving linear equations. I shall proceed by example.

Consider equations

$$\begin{array}{lrcrcrcr}
(1) & A & + & B & & & = & 0 \\
(2) & 2A & - & 2B & + & C & = & 0 \\
(3) & A & + & B & - & 2C & = & 1 \\
(4) & -4A & & & + & C & = & -1
\end{array}$$

Her is the typical strategy for solving:

$$\left\{\begin{array}{rcrcrcr}
A & + & B & & & = & 0 \\
2A & - & 2B & + & C & = & 0 \\
A & + & B & - & 2C & + & 1 \\
-4A & & & + & C & + & -1
\end{array}\right.$$

$$\begin{array}{c}(3)-(1)\\ (2)-2\times(1)\\ \longrightarrow\\ \\ \end{array}
\left\{\begin{array}{rcrcrcr}
A & + & B & & & = & 0 \\
 & - & 4B & + & C & = & 0 \\
 & & & - & 2C & = & 1 \\
-4A & & & + & C & = & -1
\end{array}\right.$$

$$\begin{array}{c}(2)\div(-4)\\ (3)\div(-2)\\ \longrightarrow\\ \\ \end{array}
\left\{\begin{array}{rcrcrcr}
A & + & B & & & = & 0 \\
 & & B & - & C/4 & = & 0 \\
 & & & & C & = & -1/2 \\
-4A & & & + & C & = & -1
\end{array}\right.$$

$$\begin{array}{c}(2)+(3)\div 4\\ \text{then}\\ (1)-(2)\\ \longrightarrow\\ \\ \end{array}
\left\{\begin{array}{rcrcrcr}
A & & & & & = & 1/8 \\
 & & B & & & = & -1/8 \\
 & & & & C & = & -1/2 \\
-4A & & & + & C & = & -1
\end{array}\right.$$

Hence we conclude that

$$A = \frac{1}{8}, \quad B = -\frac{1}{8} \quad \text{and} \quad C = -\frac{1}{2}$$

The last equation is consistent, as $-4 \cdot \frac{1}{8} + (-\frac{1}{2}) = -1$, and serves as an error check. This was exhusting. TURN TO NEXT PAGE.

Let us consider, again, the same system of equations:

$$\begin{array}{rcrcrcr} A & + & B & & & = & 0 \\ 2A & - & 2B & + & C & = & 0 \\ A & + & B & - & 2C & = & 1 \\ -4A & & & + & C & = & -1 \end{array}$$

We consider a *matrix* (really just an array of numbers) whose, rows reflect the equations, whose first 3 columns are the coefficients of the unknowns, and whose the last column is the values of the equations. The last separated from the "coefficient matrix".

$$\begin{array}{cc} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} & \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ -4 & 0 & 1 & -1 \end{array}\right] \\ & \begin{array}{cccc} A & B & C & \text{value} \end{array} \end{array}$$

The operations on the equations, last page, are now simply row operations on the matrix, where we treat each row as a vector:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ -4 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_3 - R_1 \\ R_2 - 2\cdot R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ -4 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\substack{-\frac{1}{4}R_2 \\ -\frac{1}{2}R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ -4 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_2 + \frac{1}{4}R_3 \\ \text{then} \\ R_1 - R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{2} \\ -4 & 0 & 1 & -1 \end{array}\right]$$

The process in reverse, interprets the last matix as equations

$$R_1:\ A = \frac{1}{8}, \quad R_2:\ B = -\frac{1}{8} \quad \text{and} \quad R_3:\ C = -\frac{1}{2}$$

where the last equation $R_3 : -4A + C = -1$ serves as an error check.

Written by Nico Spronk, for use by students of Math 148 at University of Waterloo.

1.29

Theorem: (Integration by Parts / "reverse Product")

Let $f, g, F : [a,b] \to \mathbb{R}$ satisfy
- f is integrable on $[a,b]$
- $F' = f$ on $[a,b]$
- g' is integrable on $[a,b]$

Then, $\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,dx$

Proof: Product rule : $\frac{d}{dx}[F(x)g(x)] = F'(x)g(x) + F(x)g'(x)$

$$= f(x)g(x) + F(x)g'(x)$$

F.T. of C. II $\quad F(b)g(b) - F(a)g(a) = \int_a^b [f(x)g(x) + F(x)g'(x)]\,dx$

$$\Longrightarrow F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,dx = \int_a^b f(x)g(x)\,dx$$

Antiderivative form: $\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$. $F(x) = \int f(x)\,dx$ (choose $C=0$)

$$\int f'g = fg - \int fg'$$

Ex. (i) $\int \mathrm{Arctan}(x)\,dx$

$= \int 1 \cdot \mathrm{Arctan}(x)\,dx$

$= x\,\mathrm{Arctan}(x) - \int x \cdot \frac{1}{1+x^2}\,dx$

$= x\,\mathrm{Arctan}(x) - \frac{1}{2}\log(1+x^2) + C$

(ii) $\int \underset{g}{x^2}\,\underset{f}{e^x}\,dx = x^2e^x - \int 2x\,e^x\,dx$

$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

iii) $I_n(x) = \int \cos^{2n} dx \quad , \; n \geq 1$

$= \int \cos x \cdot \cos^{2n-1} x \, dx$

$= \sin x \cos^{2n-1} x - \int \sin (2n-1) \cdot \cos^{2n-2} \cdot (-\sin x) dx$

$= \sin x \cos^{2n-1} x + (2n-1) \int (1-\cos^2 x) \cos^{2n-2} x \, dx$

$= \sin x \cos^{2n-1} x + (2n-1) \left[\int \cos^{2n-2} x \, dx - \int \cos^{2n} x \, dx \right]$

$= \sin x \cos^{2n-1} x + (2n-1) \left[I_{n-2}(x) - I_n(x) \right]$

$\Rightarrow 2n I_n(x) = \sin x \cos^{2n-1} x + (2n-1) I_{n-1}(x)$

$I_n(x) = \frac{1}{2n} \sin x \cdot \cos x^{2n-1} + \frac{2n-1}{2n} I_{n-1}(x)$ (Reduction formula)

Specific examples: $n=0$, $I_0(x) = \int \cos^0 x \, dx = \int 1 \, dx = x + C$

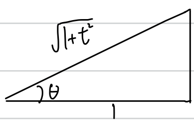
Hence $\int \cos^2 x \, dx = I_1(x) = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C_1$

(Double angle): $\int \cos^2 x \, dx = \frac{1}{2} \int (1+\cos 2x) dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + C = \frac{1}{2} x + \frac{1}{2} \sin x \cos x + C$

$n=4$, $\int \cos^4 x \, dx = I_2(x) = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \left[\frac{1}{2} x + \frac{1}{2} \sin x \cos x \right] + C_2$

$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$

(iii) $\int \frac{dt}{(t^2+1)^3}$, $t = \tan\theta$, $dt = \sec^2\theta \, d\theta$



$\tan\theta = t$

$= \int \frac{\sec^2\theta}{(\sec\theta)^3} d\theta = \int \cos^4\theta \, d\theta = \frac{1}{4} \sin\theta \cos^3\theta + \frac{3}{8} \sin\theta \cos\theta + \frac{3}{8} \theta + C$

$= \frac{1}{4} \frac{t}{(1+t^2)^2} + \frac{3}{8} \frac{t}{1+t^2} + \frac{3}{8} \mathrm{Arctan}(t) + C$

$\cos\theta = \frac{1}{\sqrt{1+t^2}}$

$\sin\theta = \frac{t}{\sqrt{1+t^2}}$

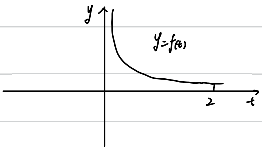
Improper Integrals

Recall: Integration involves upper and lower sums and have requires :
- bounded functions and
- bounded integrals

Definition Let $a<b$ and $f:(a,b]\to\mathbb{R}$ satisfies that: f is integrable on $[x,b]$ for each $x\in(a,b]$

Then we define the (improper) integral by $\int_a^b f = \lim_{x\to a^+}\int_x^b f$ provided the limit exists

Ex. (i) $f(t)=\frac{1}{\sqrt{t}}$ on $(0,2]$



integrable on $[x,2]$, $0<x<2$

Compute $\displaystyle\int_x^1 \frac{dt}{\sqrt{t}} = \int_x^1 t^{-\frac{1}{2}}dt = 2t^{\frac{1}{2}}\Big|_x^2 = 2\sqrt{2}-2\sqrt{x}$

Then, $\displaystyle\int_0^2 \frac{dt}{\sqrt{t}} = \lim_{x\to0^+}\int_x^2\frac{dt}{\sqrt{t}} = \lim_{x\to0^+}[2\sqrt{2}-2\sqrt{x}] = 2\sqrt{2}$

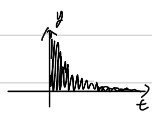
(ii) $g(t)=\frac{1}{t^2}$ on $(0,2]$ $\quad$ g is continuous, so integrable on each $[x,2]$ $\quad 0<x<2$

$\displaystyle\int_x^2\frac{dt}{t^2} = -\frac{1}{t}\Big|_x^2 = \frac{1}{x}-\frac{1}{2}$

So $\displaystyle\lim_{x\to0^+}\int_x^2\frac{dt}{t^2} = \lim_{x\to0^+}[\tfrac{1}{x}-\tfrac{1}{2}] = \infty.$

We write "$\int_0^2\frac{dt}{t^2}=\infty$" or $\int_0^2\frac{dt}{t^2}$ D.N.E.

(iii) $h(t)=\dfrac{|\sin(\frac{1}{t})|}{\sqrt{t}}$, $t\in(0,2]$ $\quad$ h is continuous on $[x,2]$ $\quad 0<x<2$



How can we show if this is improperly integrable?

Comparison method $\quad 0\le|\sin(\frac{1}{t})|\le1 \Rightarrow 0\le\frac{|\sin(\frac{1}{t})|}{\sqrt{t}}\le\frac{1}{\sqrt{t}} \Rightarrow 0\le\int_x^2\frac{|\sin(\frac{1}{t})|}{\sqrt{t}}dt\le\int_x^2\frac{dt}{\sqrt{t}} = 2\sqrt{2}-2\sqrt{x}\le2\sqrt{2}$

$H(x)=\int_x^1\frac{|\sin(\frac{1}{t})|}{\sqrt{t}}$ is non-increasing

if $0<x'<x<2$, $H(x')-H(x)=\int_{x'}^2 h-\int_x^2 h = \int_{x'}^x h+\int_x^2 h-\int_x^2 h = \int_{x'}^x h\ge0$, So $H(x')\ge H(x)$

[$H'(x)=-h(x)\le0$ by F.T. of C.I $\quad$ MVT $\Rightarrow H$ is non increasing]

$H(x)$ is bounded on $(0,2]$ and monotonic $\quad$ MCT!

So $\displaystyle\lim_{x\to0^+}H(x)=\int_0^2\frac{|\sin(\frac{1}{t})|}{\sqrt{t}}dt$ exists

1.31

Last time: $\int_a^b f = \lim_{x\to a^+} \underbrace{\int_x^b f}_{F(x)}$, f integrable on each $[x,b]$ for $a<x<b$. Typically f unbounded on $(a,b]$ improper integral

Hence, we wish to test $\lim_{x\to a} F(x)$

Facts from Math 147

(I) $\lim_{x\to a} F(x) = L \iff$ for every sequence $(a_n)_{n=1}^\infty$ s.t. $\lim_{n\to\infty} a_n = a$ provides that $\lim_{n\to\infty} F(a_n) = L$

a_n is "Cauchy"

(II) Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$, then $\lim_{n\to\infty} a_n$ exists $\iff \forall \varepsilon>0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $|a_m - a_n| < \varepsilon$ whenever $m,n \geq n_\varepsilon$ (Cauchy Criterion [Deep fact - B-W T])

Theorem (Cauchy Criterion for limit function)

Let $F:(a,b]\to\mathbb{R}$, then $\lim_{x\to a^+} F(x)$ exists $\iff \forall\varepsilon>0,\ \exists\delta>0$ s.t. $|F(u)-F(v)|<\varepsilon$ whenever $|u-a|<\delta$, $|v-a|<\delta$ for $u,v\in[a,b]$

Proof: $\Rightarrow$ Let $L = \lim_{x\to a^+} F(x)$. then given $\varepsilon>0$, $\exists\delta>0$ s.t. $|F(u)-L|<\frac{\varepsilon}{2}$ whenever $|u-a|<\delta$ and $u\in[a,b]$

Hence if $u,v\in[a,b]$. $|u-a|<\delta$. $|v-a|<\delta$ then $|F(u)-F(v)| \leq |F(u)-L| + |L-F(v)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

$\Leftarrow$ We will verify the Fact (I) , Hence $(a_n)_{n=1}^\infty \subset (a,b]$ be any sequence s.t. $\lim_{n\to\infty} a_n = a$.

We wish to see that $(F(a_n))_{n=1}^\infty$ is Cauchy , Hence by the fact (II), is convergent

Let $\varepsilon>0$ be given, find $\delta>0$ as in $(*)$

$\lim_{n\to\infty} a_n = a \Rightarrow \exists n_\delta\in\mathbb{N}$ s.t. $|a-a_n|<\delta$ whenever $n\geq n_\delta$

Hence if $m,n\geq n_\delta$ we have $\begin{cases}|a-a_m|<\delta\\|a-a_n|<\delta\end{cases} \Rightarrow |a_m-a_n|<\delta$ (both a_n, a_m are to the right of a)

Thus, $(*)$ Provides that $|F(a_n)-F(a_m)|<\varepsilon$

Summary we have that $n_\varepsilon = n_\delta$ s.t. $|F(a_n)-F(a_m)|<\varepsilon$ whenever $n,m\geq n_\varepsilon$

Last time: $\int_0^2 \frac{|\sin(\frac{1}{t})|}{\sqrt{t}}dt = \lim_{x\to 0^+} \underbrace{\int_x^2 \frac{|\sin(\frac{1}{t})|}{\sqrt{t}}dt}_{H(x)}$. H is monotonic and bounded $\Rightarrow \lim_{x\to 0^+} H(x)$ exists

Ex. Consider $\int_0^1 \frac{\sin\frac{1}{t}}{\sqrt{t}}dt = \lim_{x\to 0^+} \int_x^1 \frac{|\sin(\frac{1}{t})|}{\sqrt{t}}dt$, $-1 \le \sin(\frac{1}{t}) \le 1 \Rightarrow -\frac{1}{\sqrt{t}} \le \frac{\sin(\frac{1}{t})}{\sqrt{t}} \le \frac{1}{\sqrt{t}} \xRightarrow{\text{order Properties}} -\int_x^1 \frac{dt}{\sqrt{t}} \le \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt \le \int_x^1 \frac{dt}{\sqrt{t}}$

Now we consider $0<u<v<1$, again order properties $-\int_u^v \frac{dt}{\sqrt{t}} \le \int_u^v \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt \le \int_u^v \frac{dt}{\sqrt{t}} \Rightarrow -2(\sqrt{v}-\sqrt{u}) \le \underbrace{F(v)-F(u)}_{\text{adding of integrals (check)}} \le 2(\sqrt{v}-\sqrt{u})$

Then, $|F(v)-F(u)| \le 2(\sqrt{v}-\underbrace{\sqrt{u}}_{\ge 0}) \le 2\sqrt{v}$ if $\delta = \frac{\varepsilon^2}{4}$, and if $0<u<v<\delta$

So $|F(v)-F(u)| < 2\sqrt{\delta} = \varepsilon$, Hence $\lim_{x\to 0^+} F(x) = \int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt$ exists!

Other types of improper integral

$\int_a^b$, f is integrable on each $[a,x]$ $a<x<b$, but unbounded

$\int_{-1}^1 \frac{1}{\sqrt{|t|}}dt = \int_{-1}^0 \frac{dt}{\sqrt{t}} + \int_0^1 \frac{dt}{\sqrt{t}}$ (sum of two improper integrals)

Def'n: Let $a \in \mathbb{R}$, $f:[a,\infty) \to \mathbb{R}$ satisfy that: • f is integrable on each $[a,x]$, $a<x$

Let the improper integral given by $\int_a^\infty f = \lim_{x\to\infty} \int_a^x f$, if limit exists

Ex. (1) $\int_1^\infty \frac{dt}{t^2} = \lim_{x\to\infty} \int_1^x \frac{dt}{t^2} = \lim_{x\to\infty} \left[-\frac{1}{t}\Big|_1^x\right] = \lim_{x\to\infty} (1-\frac{1}{x}) = 1$

(2) $\int_0^\infty x^2 e^{-x}dx = \lim_{x\to\infty} \int_0^x t^2 e^{-t}dt$

$\int_0^x t^2 e^{-t}dt = -t^2e^{-t}\Big|_0^x + \int 2t\cdot e^{-t}dt$

$= -x^2e^{-x} + 2\left[[-te^{-t}]_0^x + \int_0^x e^{-t}dt\right]$

$= -x^2e^{-x} - xe^{-x} + 2(1-e^{-x})$

$= -x^2e^{-x} - 2xe^{-x} + 2(1-e^{-x}) \xrightarrow[x\to\infty]{} 2$

(ii) $\int_0^\infty e^{-t^2}dt$ Very important for statistics

$x>0$, $\int_0^x e^{-t^2}dt$, $F'(x) = e^{-x^2} > 0 \Rightarrow F$ is increasing (M.V.T) F increasing and bounded $\Rightarrow \lim_{x\to\infty} F(x) = \int_0^\infty t\cdot e^{-t^2}dt$ exists

$t \ge 1$, $0 \le e^{-t^2} \le te^{-t^2}$

$x>1$ $\int_0^x e^{-t^2}dt = \underbrace{\int_0^1 e^{-t^2}dt}_{\text{finite}} + \int_1^x e^{-t^2}dt \le \int_0^1 e^{-t^2}dt + \int_1^x t\cdot e^{-t^2}dt$

$= \int_0^1 e^{-t^2}dt - \frac{1}{2}e^{-t^2}\Big|_1^x$

$= \int_0^1 e^{-t^2}dt - \frac{1}{2}[e^{-x^2} - \frac{1}{e}] \le \int_0^1 e^{-t^2}dt + \frac{1}{2e} \Rightarrow F$ bounded

2.3

Examples of Improper integrals

(iv) $\int_{\frac{1}{\pi}}^{\infty} \sin(\frac{1}{t})\,dt$

Attempt #1 if $0 \le \theta \le \pi$ then $0 \le \sin\theta \le \theta$

$g(\theta) = \theta - \sin\theta$

$g'(\theta) = 1 - \cos\theta \ge 0 \Rightarrow g$ is non-decreasing (M.V.T)

$\Rightarrow 0 \le \sin\frac{1}{t} \le \frac{1}{t}$, $x > \frac{1}{\pi}$

$0 \le \int_{\frac{1}{\pi}}^{x} \sin(\frac{1}{t})dt \le \int_{\frac{1}{\pi}}^{x} \frac{1}{t}\,dt = \log x - \log\frac{1}{\pi}$ Conclusion: not enough information

Attempt #2 $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta - \sin 0}{\theta - 0} = \sin'(0) = \cos 0 = 1$. Hence there is $N > 0$ s.t. $\frac{\sin\frac{1}{t}}{\frac{1}{t}} > \frac{1}{2}$ number smaller than 1

i.e. $t > N \Rightarrow \sin\frac{1}{t} \ge \frac{1}{2t}$

We suppose $N \ge \frac{1}{\pi}$, $x > N$ and we have $\int_{\frac{1}{\pi}}^{x} \sin\frac{1}{t}\,dt = \underbrace{\int_{\frac{1}{\pi}}^{N} \sin(\frac{1}{t})dt}_{\text{finite}} + \int_{N}^{x} \sin(\frac{1}{t})dt \ge \int_{\frac{1}{\pi}}^{N} \sin\frac{1}{t}\,dt + \int_{N}^{x} \frac{1}{2t}\,dt = \int_{\frac{1}{\pi}}^{N} \sin\frac{1}{t}\,dt + \frac{1}{2}[\log x - \log N]$ N fixed, so this is a fixed number

Conclusion: $\int_{\frac{1}{\pi}}^{\infty} \sin(\frac{1}{t})dt$ diverges

Notes on Comparison Test Consider improper integral $\int_a^b f$ Either, f is unbounded at a or at b just one of a, b is $\infty/-\infty$

$\int_{-\infty}^{\infty} t\,dt = \lim_{x\to\infty} \int_{-x}^{x} t\,dt = \lim_{x\to\infty} 0$!!! $\quad \underbrace{\int_{-\infty}^{0} t\,dt}_{\text{diverge}} + \underbrace{\int_{0}^{\infty} t\,dt}_{\text{diverges}}$

Case #1 $f \ge 0$ on $[a,b)$: • If we can find $0 \le f \le g$ on (a,b) and $\int_a^b g$ Converges $\Rightarrow \int_a^b f$ Converges (use MCT. boundness)

• If we can find $0 \le g \le f$ on (a,b) and $\int_a^b g$ diverges $\Rightarrow \int_a^b f$ diverges $[\int f \ge \int g \to \infty]$

Case #2 : f not necessarily non-negative on (a,b) : • If we can find $g, h \ge 0$ with $-g \le f \le h$, and both $\int_a^b g$, $\int_a^b h$ Converge $\Rightarrow f$ Converges

Theorem (Cauchy Criterion for limit at ∞)

If $F:[a,\infty] \to \mathbb{R}$, then $\lim_{x\to\infty} F(x)$ exists $\iff$ given $\varepsilon>0$ there $\exists N>a$ s.t. $|F(u)-F(v)|<\varepsilon$ whenever $u,v>N$

Sketch Proof: ($\Leftarrow$) Let $(a_n)_{n=1}^{\infty} \subset [a,\infty)$ with $\lim_{n\to\infty} a_n = \infty$, then there is $n_0 \in \mathbb{N}$, so $m,n \geq n_0$ $|F(u)-F(v)|<\varepsilon$ — Verify!

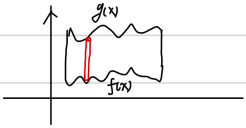
Hence $(F(a_n))_{n=1}^{\infty}$ is Cauchy, hence admits limit L

Check that $\forall (b_n)_{n=1}^{\infty} \subset [a,\infty)$, $\lim_{n\to\infty} b_n \Rightarrow \lim_{n\to\infty} F(b_n) = L$

Check that this implies that $\lim_{x\to\infty} F(x) = L$ $\square$

Applications:

Area: Let $f,g:[a,b]\to\mathbb{R}$ be integrable with $f\leq g$, let $S=\{(x,y): a\leq x\leq b, y \text{ lies between } f(x) \text{ and } g(x), a\leq x\leq b\}$

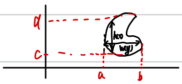


Partition $P=\{a=x_0<x_1<\cdots<x_n=b\}$ $\quad s_j, t_j \in [x_{j-1}, x_j]$ (Say by $\ell(P)<\delta$, A2 Q1)

$$\text{Area}(S) \approx \sum_{j=1}^{n} \underbrace{(g(t_j)-f(t_j))}_{\text{height}} \underbrace{(x_j-x_{j-1})}_{\text{width}} = \sum_{j=1}^{n} g(t_j)(x_j-x_{j-1}) - \sum_{j=1}^{n} f(s_j)(x_j-x_{j-1}) \approx \int_a^b g - \int_a^b f = \int_a^b (g-f)$$

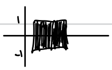
Hence, we define $\text{area}(S) = \int_a^b [g(x)-f(x)]dx$

If S is a "nice" region



$\text{Area}(S) = \int_a^b h_S(x)\,dx = \int_c^d w_S(y)\,dy$

Warning example $S=\{(x,y): 0\leq y\leq 1 \text{ if } x\in\mathbb{Q}, -1\leq y\leq \text{ if } x\in\mathbb{R}\setminus\mathbb{Q}\}$



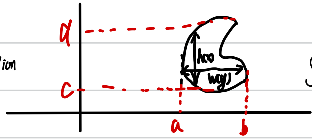
Notice "height" $h_S(x)=1$ by we should not image that $\text{area}(S) = \int_0^1 h_S(x)\,dx = 1$!!!

Let $f=-\chi_{ir}$, $g=\chi_{\mathbb{Q}}$, really we use $\underbrace{t_j \in \mathbb{Q}, \ s_j \in \mathbb{R}\setminus\mathbb{Q}}_{(*)=2}$

$\left.\begin{array}{l} t_i \text{ irrational} \\ s \text{ rational} \end{array}\right\} \Rightarrow (*) = 0$

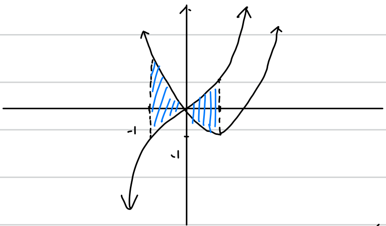
2.5

S is a "nice" region



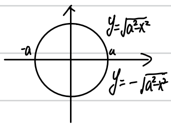
$S \subset \mathbb{R}^2$ $\quad \text{Area}(S) = \int_a^b h_S(x)\,dx = \int_c^d w_S(y)\,dy$

Ex. (i) $S = \{(x,y) : y \text{ between } y = x^3,\ y = x^2 - 2x,\ -1 \le x \le 1\}$



$$\text{Area}(S) = \int_{-1}^{1} |x^3 - (x^2-2x)|\,dx$$
$$= \int_{-1}^{0} x^2-2x-x^3\,dx + \int_0^1 x^3-x^2+2x\,dx$$

(ii) Circle of radius $a > 0$ $\quad x^2+y^2=a^2$



$$\text{Area}(C) = \int_{-a}^{a} (\sqrt{a^2-x^2} - (-\sqrt{a^2-x^2}))\,dx$$
$$= 2\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$$

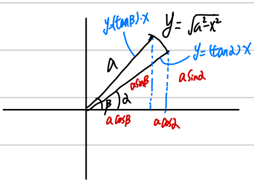
method #1 : $x = au$, $dx = a\,du$

$$= 2\int_{-1}^{1} \sqrt{a^2-(au)^2}\,du$$
$$= a^2 \cdot \underbrace{2\int_{-1}^{1}\sqrt{1-u^2}\,du}_{=\pi \text{ by defn}} = a^2\pi$$

method #2 $\quad x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\text{Area}(S) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2-(a\sin\theta)^2}\cdot a\cos\theta\,d\theta$$
$$= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$
$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1+\cos 2\theta]\,d\theta$$
$$= a^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= a^2\pi + 0 = a^2\pi$$

(iii) Let W be a circular wedge $a > 0$, $0 \le \alpha \le \beta \le \frac{\pi}{2}$



$$\text{Area}(W) = \int_0^{a\cos\beta} (\tan\beta - \tan\alpha)x\,dx + \int_{a\cos\beta}^{a\cos\alpha} [\sqrt{a^2-x^2} - (\tan\alpha)x]\,dx$$
$$= \int^{a\cos\beta} \tan\beta\, x\,dx - \int_0^{a\cos\alpha} \tan\alpha \cdot x\,dx + \int_{a\cos\beta}^{a\cos\alpha} \sqrt{a^2-x^2}\,dx$$
, Let $x = a\cos\theta$, $dx = -a\sin\theta\,d\theta$
$$= \frac{a^2}{2}\sin\beta\cos\beta - \frac{a^2}{2}\sin\alpha\cos\alpha - a^2\int_\beta^\alpha \sin^2\theta\,d\theta$$
$$= \frac{a^2}{2}\sin\beta\cos\beta - \frac{a^2}{2}\sin\alpha\cos\alpha + a^2\int_\alpha^\beta \sin^2\theta\,d\theta$$
$$= \frac{a^2}{2}\sin\beta\cos\beta - \frac{a^2}{2}\sin\alpha\cos\alpha + \frac{a^2}{2}\int_\alpha^\beta (1-\cos 2\theta)\,d\theta$$
$$= \frac{a^2}{2}(\underbrace{\sin\beta\cos\beta}_{\frac{1}{2}\sin 2\beta} - \underbrace{\sin\alpha\cos\alpha}_{\frac{1}{2}\sin 2\alpha}) + \frac{a^2}{2}\left[(\beta-\alpha) - \tfrac{1}{2}(\sin 2\beta - \sin 2\alpha)\right]$$
$$= a^2\cdot\frac{\beta-\alpha}{2} = a^2\pi\underbrace{\frac{(\beta-\alpha)}{2\pi}}_{\text{Proportion of full angle}}$$

$$\text{Area}(W) = \frac{a^2}{2}(\beta-\alpha)$$

Average Value : $a=\{a_1,a_2,\dots,a_n\}\subset\mathbb{R}$

$a_{ave}=\frac{a_1+\dots+a_n}{n}$

Let $f:[a,b]\to\mathbb{R}$ be an integrable function, we wish to figure out "integral average value" f_{ave}

Uniform Partition: $\{a<a+\frac{b-a}{n}<a+2\frac{b-a}{n}<\dots<b\}=P_n$, $t_j\in[a+(j-1)\frac{b-a}{n},\ a+j\frac{b-a}{n}]$ $j=1,2,\dots,n$

Expect: $f_{ave}\approx\frac{\sum_{j=1}^{n}f(t_j)}{n}=\frac{1}{b-a}\sum_{j=1}^{n}f(t_j)\frac{b-a}{n}=\frac{1}{b-a}S(f,P_n)$

A2, Q1 $\Rightarrow \lim_{n\to\infty}\frac{1}{b-a}S(f,P_n)=\frac{1}{b-a}\int_a^b f$

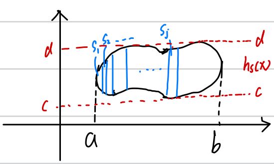
Define $f_{ave}=\frac{1}{b-a}\int_a^b f$

Weighted average $a=\{a_1,\dots,a_n\}\subset\mathbb{R}$: weights $\underbrace{w_1,\dots,w_n}_{w}>0$

$a_{w\,ave}=\frac{a_1w_1+\dots+a_nw_n}{w_1+\dots+w_n}$, We have $f:[a,b]\to\mathbb{R}$, $P=\{a=x_0<x_1<\dots<x_n=b\}$ Sample Points $t_j\in[x_{j-1},x_j]$

$f_{w\,ave}=\frac{f(t_1)(x_1-x_0)+\dots+f(t_n)(x_n-x_{n-1})}{(x_1-x_0)+\dots+(x_n-x_{n-1})}=\frac{\sum_{j=1}^{n}f(t_j)(x_j-x_{j-1})}{b-a}=\frac{1}{b-a}S(f,P)$. Riemann Sum

Centroid S "nice" region



x-center $\overline{x}_S$, $P=\{a=x_0<\dots<x_n=b\}$, tags : $t_j\in[x_{j-1},x_j]$, $j=1,\dots,n$

$S_j=\{(x,y)\in S:\ x_{j-1}\le x\le x_j\}$. $\overline{x}_S\approx\frac{\sum_{j=1}^{n}t_j\cdot area(S_j)}{\sum_{j=1}^{n}area(S_j)}$ ← Area(S)

Suppose h_S is continuous , $area(S_j)=\int_{x_{j-1}}^{x_j}h_S(u)\,du=h_S(c_j)\cdot(x_j-x_{j-1})$, $c_j\in[x_{j-1},x_j]$

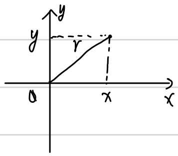
Mean / Average Value Theorem for Integrals (F.T.of.C.I)

We have $\overline{x}_S\approx\frac{\sum_{j=1}^{n}c_j h_S(c_j)\cdot(c_j-c_{j-1})}{area(S)}=\frac{1}{area(S)}S(H_S,P)$, $H_S(x)=x\,h_S(x)$

$\overline{x}_S=\frac{1}{area(S)}\cdot\int_a^b x\cdot h_S(x)\,dx$

$\overline{y}_S=\frac{1}{area(S)}\int_d^c y\,w_S(y)\,dy$

Polar Coordinates : Euclidean Coordinates (x,y) in $\mathbb{R}^2$



$r = \sqrt{x^2+y^2}$ $\quad x = r\cos\theta,\ y = r\sin\theta$

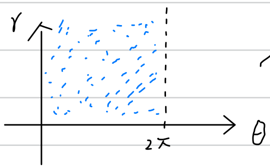
Find θ: $x>0$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\qquad \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \operatorname{Arctan}(\frac{y}{x}) = \theta$

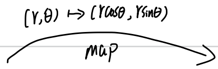
$x<0$: check $\theta = \operatorname{Arctan}(\frac{y}{x}) + \pi$

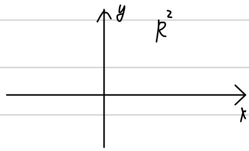
$x=0$ $\quad y>0,\ \theta = \frac{\pi}{2},\ \ y<0,\ \theta = -\frac{\pi}{2}, \frac{3\pi}{2}$

Given $r>0$, $0 \le \theta < 2\pi$

$(x,y) = (r\cos\theta, r\sin\theta)$ is a unique point in $\mathbb{R}^2 \setminus \{0,0\}$







$r=0,\ 0\le\theta<2\pi \xrightarrow[\text{to}]{\text{map}} \{(0,0)\}$

Polar equation(s) (i) Circle $x^2+y^2=a^2$ $(a>0)$

$x = r\cos\theta$, $y = r\sin\theta$

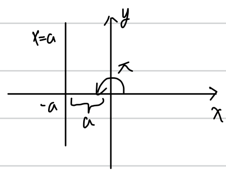
$a^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2(\sin^2\theta + \cos^2\theta) = r^2 \Rightarrow a = \pm r$, but $a=r$ survives as $r>0$

So Circle in Polar: $r = a$

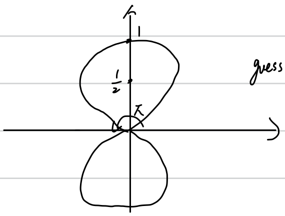
(ii) Vertical line : $x=a$, $a>0$, $r = \cos\theta\, a \Rightarrow r = \frac{a}{\cos\theta} = a\cdot\sec\theta$

$a<0$ $\quad r = \frac{a}{\cos\theta}$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$x=0$, $\underbrace{\theta = \frac{\pi}{2},\ \theta = \frac{3\pi}{2}}_{2 \text{ rays}}$



(iii) $r = |\sin\theta|$, $0 \le \theta \le 2\pi$



guess: Circle $r = \frac{1}{2}$, $(0, \frac{1}{2})$

$\frac{1}{4} = x^2 + (y-\frac{1}{2})^2$ try $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$

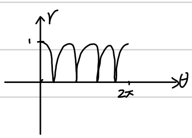
$x(\theta)^2 + (y(\theta)-\frac{1}{2})^2 = x(\theta)^2 + y(\theta)^2 - y(\theta) + \frac{1}{4} = (\sin\theta\cos\theta)^2 + (\sin\theta\sin\theta)^2 - \sin^2\theta + \frac{1}{4}$

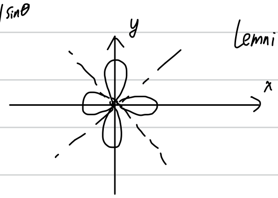
$= \sin^2\theta(\cos^2\theta + \sin^2\theta) - \sin^2\theta + \frac{1}{4}$

$= \sin^2\theta - \sin^2\theta + \frac{1}{4} = \frac{1}{4}$ ☺

If $\pi \le \theta < 2\pi$ $|\sin\theta| = -\sin\theta$ Check that $x(\theta) = -\sin\theta\cos\theta$ $y(\theta) = -\sin\theta\sin\theta$ Satisfy $x(\theta)^2 + (y(\theta)-\frac{1}{2})^2 = \frac{1}{4}$

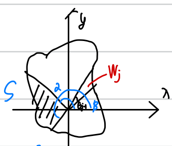
(iv) $r(\theta) = |\cos 2\theta|$, $x(\theta) = |\cos 2\theta| \cdot \cos\theta$, $y(\theta) = |\cos 2\theta| \sin\theta$





Lemniscate

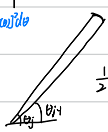
Area Let $r: [0, 2\pi] \to [0, \infty)$ be a continuous function, $r(0) = r(2\pi)$



Area$(S) = \frac{1}{2}\int_\alpha^\beta [r(\theta)]^2 d\theta$

$P = \{0 = \theta_0 < \cdots < \theta_n = 2\pi\}$

Area$(R) = \sum_{j=1}^{n}$ area W_j , $m_j = \min_{\theta \in [\theta_{j-1}, \theta_j)} r(\theta)$ $M_j = \max_{\theta \in [\theta_{j-1}, \theta_j)} r(\theta)$



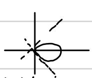
$\frac{1}{2} m_j^2 (\theta_j - \theta_{j-1}) \le \text{area}(W_j) \le \frac{1}{2} M_j (\theta_j - \theta_{j-1})$

area of the large circular wedge radius M_j

$L(\frac{1}{2}r^2, P) = \sum_{j=1}^{n} \frac{1}{2} m_j^2 (\theta_j - \theta_{j-1}) \le \text{Area}(R) \le \sum_{j=1}^{n} \frac{1}{2} M_j^2 (\theta_j - \theta_{j-1}) = U(\frac{1}{2}r^2, P)$

$\frac{1}{2}r^2$ continuous $\to$ integrable Conclusion : Area$(R) = \frac{1}{2}\int_0^{2\pi} [r(\theta)]^2 d\theta$

Ex. lemniscate , one lobe $r(\theta) = \cos(2\theta)$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



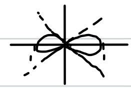
Area$(L) = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos(4\theta)) d\theta$

$= \frac{1}{4}[\theta + \frac{1}{4}\sin(4\theta)]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}$

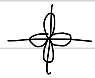
2.7

Note (on terminology)

i) lemniscate $r(\theta) = 2\sqrt{\cos(2\theta)}$, Domain : $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and $\frac{3\pi}{4} \le \theta \le \frac{7\pi}{4}$

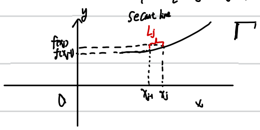


ii) "flower" $r(\theta) = |\cos(2n\theta)|$ $n \in \mathbb{N}$ $\quad n=1$



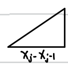
Arc length $f : [a,b] \to \mathbb{R}$ Continuous

$\Gamma = \{(x,y) : y = f(x),\ a \le x \le b\}$



$P = \{a = x_0 < \cdots < x_n = b\}$

$\text{Length}(L_j) = \sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2}$



$$\text{Length}(\Gamma) \approx \sum_{j=1}^{n} \text{Length}(L_j) = \sum_{j=1}^{n} \sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2}$$

Add assumptions : • f' exists on $[a,b]$ and is Continuous on $[a,b]$: MVT $\Rightarrow$ $f(x_j) - f(x_{j-1}) = f'(c_j)\cdot(x_j - x_{j-1})$ $\quad c_j \in (x_{j-1}, x_j)$

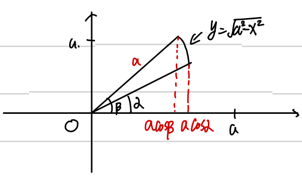
$$\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2} \Rightarrow \sqrt{(x_j - x_{j-1})^2 + [f'(c_j)(x_j - x_{j-1})]^2} \Rightarrow \sqrt{1 + [f'(c_j)]^2}\cdot(x_j - x_{j-1})$$

Then, $\text{Length}(\Gamma) \approx \sqrt{1 + [f'(c_j)]^2}\cdot(x_j - x_{j-1}) = S(\underbrace{\sqrt{1+(f')^2}}_{\text{Continuous function}}, P)$ Riemann Sum

Defn If f' exists and is Continuous on $[a,b]$, $\Gamma = \{(x,y) : y = f(x),\ a \le x \le b\}$ (graph)

$$\text{length}(\Gamma) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Ex. (i) $0 \le \alpha \le \beta \le \pi$, $a > 0$



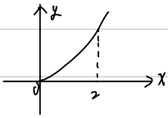
$$\text{Length}(\Gamma) = \int_{a\cos\beta}^{a\cos\alpha} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx$$

$$= \int_{a\cos\beta}^{a\cos\alpha} \frac{a}{\sqrt{a^2 - x^2}}\, dx \quad , \ x = a\cdot\cos\theta \ , \ dx = a\sin\theta\, d\theta \ , \ \sin\theta \ge 0$$

$$= \int_{\beta}^{\alpha} \frac{a}{\sqrt{a^2 - a^2\cos^2\theta}}\cdot(-a\sin\theta\, d\theta)$$

$$= \int_{\alpha}^{\beta} a\, d\theta = a\cdot(\beta - \alpha)$$

(ii) $\Gamma = \{(x,y) : y=x^2,\ 0\le x\le 2\}$



$\frac{dy}{dx} = 2x$

Length$(\Gamma) = \int_0^2 \sqrt{1+(2x)^2}\,dx$, $2x = \sinh t = \frac{e^t-e^{-t}}{2}$, $dx = \frac{1}{2}\cosh t\,dt$, $\cosh t = \frac{e^x+e^{-x}}{2}$, $t = \log(2x+\sqrt{(2x)^2+1})$ inverse of sinh

$= \int_0^{\log(4+\sqrt{17})} \underbrace{\sqrt{1+(\sinh t)^2}}_{\cosh t}\cdot\frac{1}{2}\cosh t\,dt$

$= \frac{1}{2}\int_0^{\log(4+\sqrt{17})} \cosh^2 t\,dt$, $\cosh^2 t = \frac{1}{2}[\cosh(2t)+1]$

$= \frac{1}{4}\int_0^{\log(4+\sqrt{17})} (\cosh(2t)+1)\,dt$

$= \frac{1}{4}\left[\frac{1}{2}\sinh(2t)+t\right]\Big|_0^{\log(4+\sqrt{17})}$, $\sinh^2 t = \frac{1}{2}\sinh t\cdot\underbrace{\cos t}_{\sqrt{1+\sinh^2 t}}$

= evaluate

<u>Method #2</u> $2x = \tan t$, $dx = \frac{1}{2}\sec^2 t\,dt$

Length$(\Gamma) = \int_0^{\arctan 4} \sqrt{1+\tan^2 t}\cdot\frac{1}{2}\sec^2 t\,dt$

$= \frac{1}{2}\int_0^{\arctan(4)} \sec^3 t\,dt$

$\int \sec^3 t\,dt = \int \sec^2 t\sec t\,dt = \tan t\sec t - \int \underbrace{\tan t\tan t}_{\sec^2 t-1}\sec t\,dt$

$= \tan t\sec t - \int(\sec^3 t - \sec t)\,dt$

$\Longrightarrow 2\int \sec^3 t\,dt = \tan t\sec t + \int \sec t\,dt = \tan t\sec t + \log|\sec t+\tan t| + C$ $\sec(\arctan(4)) = \sqrt{1+\tan^2(\arctan 4)} = \sqrt{1+16} = \sqrt{17}$

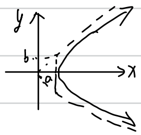
$\Longrightarrow$ Evaluate $\frac{1}{2}\int_0^{\arctan(4)} \sec^3 t\,dt$

Parameterization: We regard $x, y : [a,b] \to \mathbb{R}$ (coordinates are each functions)

$$\Gamma = \{(x(t), y(t)) : t \in [a,b]\}$$

Examples: A Polar Curves: (i) $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$

(ii) hyperbolic coordinates $a, b > 0$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



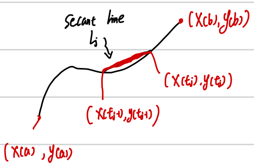
right lobe $x(t) = a\cosh t\, dt$

$y(t) = b\sinh t$

We wish to compute/define Length(Γ)

Assumptions: • $x'(t)$, $y'(t)$ always exist on $[a,b]$, $x', y' : [a,b] \to \mathbb{R}$ are each continuous

$$P = \{a = x_0 < \cdots < x_n = b\}$$



$$\text{Length} \approx \sum_{j=1}^{n} \text{Length}(L_i)$$

$$= \sum_{j=1}^{n} \sqrt{(x(t_j) - x(t_{j-1}))^2 + [y(t_j) - y(t_{j-1})]^2}$$

$$= \sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2} \cdot (t_j - t_{j-1})$$

$$\approx S\left(\sqrt{(x')^2 + (y')^2}, P\right)$$

MVT $\Rightarrow$ $x(t_j) - x(t_{j-1}) = x'(c_j)(t_j - t_{j-1})$, $c_j \in (t_{j-1}, t_j)$

$y(t_j) - y(t_{j-1}) = y'(c_j)(t_j - t_{j-1})$, $c_j^* \in (t_{j-1}, t_j)$

$$\text{Length}(\Gamma) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

ARCLENGTH FOR PARAMETERIZED CURVES

We let $x, y : [a, b] \to \mathbb{R}$ each be functions. We suppose that

- x', y' each exist on $[a, b]$, and define continuous functions there.

Examples: (i) Our main class of examples use *polar coordinates*:

$$r : [\alpha, \beta] \to [0, \infty),\ x(\theta) = r(\theta)\cos\theta,\ y(\theta) = r(\theta)\sin\theta \text{ for } \alpha \le \theta \le \beta$$

where we assume that r' exists and is continuous on $[\alpha, \beta]$.

(ii) *Hyperbolic coordinates*: if $a, b > 0$, then

$$x, y : \mathbb{R} \to \mathbb{R},\ x(t) = a\cosh t,\ y(y) = b\sinh t$$

parameterize the right half of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Derivation of arclength formula. We consider

$$\Gamma = \{(x(t), y(t)) : a \le t \le b\}.$$

We partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, and let for $j = 1, \dots, n$, L_j denote the secant line connecting $(x(t_{j-1}), y(t_{j-i}))$ to $(x(t_j), y(t_j))$. Pythagoreas' Theorem provides that

$$\text{length}(L_j) = \sqrt{[x(t_j) - x(t_{j-1})]^2 + [y(t_j) - y(t_{j-1})]^2}.$$

If we invoke the M.V.T. from differential calculus, we find $c_j, c_j^* \in (t_{j-1}, t_j)$ for which

$$x(t_j) - x(t_{j-1}) = x'(c_j)(t_j - t_{j-1}) \text{ and } y(t_j) - y(t_{j-1}) = y'(c_j)(t_j - t_{j-1}).$$

and hence

$$\text{length}(L_j) = \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2}(t_j - t_{j-1}).$$

Now any reasonable notion of arclength should satisfy

$$\begin{aligned} \text{length}(\Gamma) &\approx \sum_{j=1}^{n} \text{length}(L_j) \\ &= \sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2}(t_j - t_{j-1}) \qquad (*) \end{aligned}$$

The looks almost like a Riemann sum $S(\sqrt{(x')^2+(y')^2}, P)$ except that we do not know if $c_j = c_j^*$.

Uniform continuity comes to the rescue. We let $F(s,t) = \sqrt{[x'(s)]^2 + [y'(t)]^2}$.

Theorem. *Given $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|F(s,t) - F(s,s)| < \varepsilon \text{ whenever } |s-t| < \delta \text{ for } s,t \in [a,b].$$

Proof. We suppose not. Then we can find an $\varepsilon > 0$ which allows for each n in $\mathbb{N}$ us to find s_n and t_n in $[a,b]$ so

$$|F(s_n,t_n) - F(s_n,s_n)| \geq \varepsilon \text{ while } |s_n - t_n| < \frac{1}{n}. \qquad (\dagger)$$

However, Bolzano-Weierstrauss provides us with a subsequence $(s_{n_k})_{k=1}^{\infty}$ for which $s_0 = \lim_{k\to\infty} s_{n_k}$ exists. Then

$$|t_{n_k} - s_0| \leq |t_{n_k} - s_{n_k}| + |s_{n_k} - s_0| < \frac{1}{n_k} + |s_{n_k} - s_0|$$

and the Squeeze Principle implies that $\lim_{k\to\infty} t_{n_k} = s_0$ too. But, since x', y' are assumed to be continuous functions of t we have

$$\begin{aligned}\lim_{k\to\infty} F(s_{n_k}, t_{n_k}) &= \lim_{k\to\infty} \sqrt{[x'(s_{n_k})]^2 + [y'(t_{n_k})]^2} \\ &= \sqrt{[x'(s_0)]^2 + [y'(s_0)]^2} = F(s_0, s_0)\end{aligned}$$

while $(\dagger)$ tells us that $|F(s_{n_k}, t_{n_k}) - F(s_0,s_0)| \geq \varepsilon$. This contradiction tells us that this result, as stated, must hold. $\square$

With the last theorem in hand, given $\varepsilon > 0$, let us choose $\delta > 0$ so

$$|F(s,t) - F(s,s)| < \frac{\varepsilon}{b-a} \text{ whenever } |s-t| < \delta \text{ for } s,t \in [a,b].$$

We now assume that $\ell(P) < \delta$, and we compare the formula $(*)$ above to the Riemann sum

$$S(\sqrt{(x')^2+(y')^2}, P) = \sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j)]^2}(t_j - t_{j-1}).$$

We use the triangle inequlity to get

$$\begin{aligned}
&\left|\sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2}(t_j - t_{j-1}) - S(\sqrt{(x')^2 + (y')^2}, P)\right| \\
&\quad \leq \sum_{j=1}^{n} \left|\sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2} - \sqrt{[x'(c_j)]^2 + [y'(c_j)]^2}\right| (t_j - t_{j-1}) \\
&\quad = \sum_{j=1}^{n} |F(c_j, c_j^*) - F(c_j, c_j)|(t_j - t_{j-1}) \\
&\quad \leq \sum_{j=1}^{n} \frac{\varepsilon}{b-a}(t_j - t_{j-1}) = \varepsilon
\end{aligned}$$

We combine the estimate at $(*)$ with the estimate above to see that

$$\text{length}(\Gamma) \approx S(\sqrt{(x')^2 + (y')^2}, P), \text{ provided that } \ell(P) \text{ is small.}$$

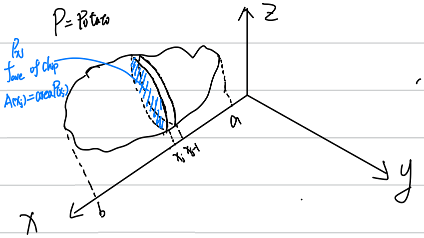
Thus, taking $\ell(P) \to 0$ we define

$$\boxed{\text{length}(\Gamma) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.}$$

2.10

Volume $S \subset R^3$ "nice" region, typically by define surfaces with defineable cross-sections

P = Potato



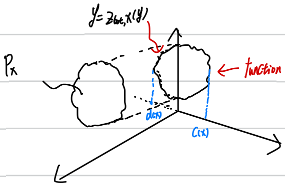
Partition $\{a = x_0 < x_1 < \cdots < x_n = b\} = Q$

$\text{Volume}(P) = \sum_{j=1}^{n} \text{Volume}(P_j)$

$\approx \sum_{j=1}^{n} \underbrace{A(x_j)}_{\text{area}} \underbrace{(x_j - x_{j-1})}_{\text{width of chip}}$ Riemann Sum

We define $\text{Volume}(P) = \int_a^b A(x)dx$

P_x

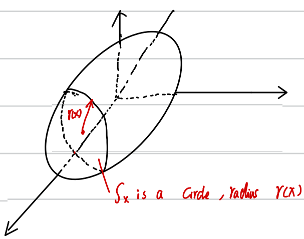


$Z_{top,x}(y) = y$

$$A(x) = \int_{c(x)}^{d(x)} \left[Z_{top,x}(y) - Z_{bot,x}(y) \right] dy$$

Hard Part: Figure out $Z_{top,x}$, $Z_{bot,x}$, $c(x)$, $d(x)$

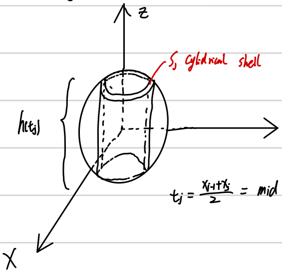
Circular Symmetry: Circular symmetry above x-axis, cross sections are circles



$A(x) = \pi \cdot (r(x))^2$, $\text{Volume}(S) = \pi \int_a^b [r(x)]^2 dx$

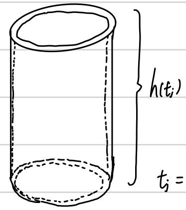
Method of discs

Method of Cylindrical shells. Suppose that $R \subset \mathbb{R}^3$ is circularly symmetric about z-axis



$t_j = \frac{x_{j-1}+x_j}{2}$ = mid point

$P = \{0 = x_0 < \cdots < x_n = b\}$



$t_j = \frac{x_{j-1}+x_j}{2}$

Volume(S_j) = Volume (cylinder, height $h(t_j)$, radius x_j)

— Volume (cylinder, height $h(t_j)$, Radius x_{j-1})

$= \underbrace{\pi x_j^2}_{A(t_j)} \cdot \underbrace{h(t_j)}_{\text{height}} - \pi \cdot x_{j-1}^2 \cdot h(t_j)$

$= \pi (x_j^2 - x_{j-1}^2) h(t_j)$

$= 2\pi \left(\frac{x_j + x_{j-1}}{2}\right) \cdot (x_j - x_{j-1}) \cdot h(t_j)$

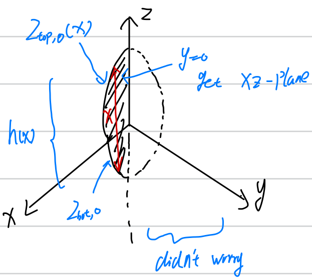
$= 2\pi t_j \cdot h(t_j)\, (x_j - x_{j-1})$

$$\text{Volume}(R) \underset{\text{CA}}{\overset{\text{by}}{=}} \sum_{j=1}^{n} \text{Volume}(S_j)$$

$$= \sum_{j=1}^{n} 2\pi t_j \cdot h(t_j)\,(x_j - x_{j-1})$$

$$= 2\pi \int_{mid}(H, P) \quad H(x) = x \cdot h(x)$$

$$\text{Volume}(R) = 2\pi \int_0^b x\, h(x)\, dx$$

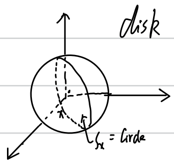


$$h(x) = z_{top,0}(x) - z_{bot,0}(x)$$

$$\text{Volume}(R) = 2\pi \int_0^b x \cdot [z_{top,0}(x) - z_{bot,0}(x)]\, dx$$

Example. S-sphere, $r = a > 0$ $\quad x^2 + y^2 + z^2 = a$

disk

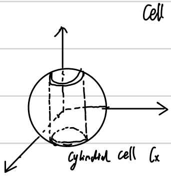


Compute Volume(S) : Disks: Fix x, for the moment $-a \le x \le a$. Project S_x into y-z Plane

Set $y=0$, $x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 \Rightarrow r(x) = \sqrt{a^2 - x^2}$

$$\text{Volume}(S) = \pi \int_{-a}^{a} (\sqrt{a^2 - x^2})^2 dx = \frac{4}{3}\pi a^3$$

Cell



$h(x) = \sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2}) = 2\sqrt{a^2 - x^2}$

$\text{Volume}(S) = 2\pi \int_0^a x \cdot 2\sqrt{a^2 - x^2}\, dx$

$= 4\pi \int_0^a x \cdot \sqrt{a^2 - x^2}\, dx = \frac{4}{3}\pi a^3$

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Midterm ends at A3

Hint for A3 Q8

S = spherical shell, inner radius u, outer radius v, $u<v$

- Show that $\text{Volume}(S) = \frac{4}{3}\pi\,\underbrace{(v^2+uv+u^2)}_{\text{show that this equals } 3t^2 \text{ for some } u \le t \le v}(v-u)$

- $\text{mass}(S) \approx \rho(t)\,\text{Volume}(S)$ if S is thin, ρ is density function

(Can be done rigoriously, but a decent heuristic will suffice)

Application of antiderivatives

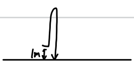
Constant gravity : Near surface of the earth, force due to gravity, hence acceleration of an object in "free-fall" is constant $g \approx 9.8\,m/s^2$

Problem: A ball is thrown straight up in the air, released 1m from the ground at a speed 15m/s

Determine : (a) how hight will the ball fly?

(b) At what time, after release, will it get there?

(c) when will it hit the ground?



$S(t)$ = displacement (hight from ground), $t=0$, $S(0)=1m$

$V(t) = S'(t)$ = Velocity . $V'(t) = a(t) = -g$ (negative as ball is falling down)

Thus, $V(t) = -gt + V_0$, $V_0 = V(0) = 15m/s$

$S(t) = -\frac{g}{2}t^2 + V_0 t + S_0$, $S_0 = S(0) = 1m$

(a).(b) We know that S attains its maximum $\Rightarrow S' = 0 = V$, the time t_{max}, at which happens satisfies

$0 = V(t_{max}) = -g t_{max} + V_0 \Rightarrow t_{max} = \frac{V_0}{g} \approx \frac{15m/s}{9.8m/s^2} \approx 1.5s$

$S(\frac{V_0}{g}) = -\frac{g}{2}(\frac{V_0}{g})^2 + V_0(\frac{V_0}{g}) + S_0 = \frac{V_0^2}{2g} + S_0 \approx \frac{(15m/s)^2}{2\cdot 9.8m/s^2} + 1m \approx \cdots$

Can't move back in t

(c) We want so $S(t)=0$, i.e. $0 = -\frac{g}{2}t^2 + V_0 t + S_0 \Rightarrow t = \frac{-V_0 \pm \sqrt{V_0^2+2gS_0}}{-g} = \frac{V_0}{g} + \sqrt{(\frac{V_0}{g})^2 + \frac{2S_0}{g}}$

Differential Equations

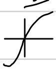
1st order D.E., standard form $\underbrace{y' = f(x,y)\ , \quad \underbrace{y(x_0) = y_0}_{\text{initial value}}}_{\text{Initial Value Problem}}$

I.V.P

Caratheodory Existence Theorem $f(x,y)$ is continuous near $(x_0, y_0) \Rightarrow$ Solution to I.V.P exists

Picard-Lindelöf Theorem "nice" assumption of 2nd variable of f near $(x_0, y_0) \Rightarrow$ unique soln to I.V.P exists

Ex: $y' = x \cdot y^{\frac{1}{3}}\ , \quad y(0) = x_0$



Solution #1 : $y(x) = 0$ i.e $y_0 = 0$

Solution #2 Assume $y(0) \neq 0$, hence $y(x) \neq 0$ in neiburhood of x $(c+y_1)$

$$\frac{y'}{y^{\frac{1}{3}}} = x \Rightarrow \frac{3}{2} y^{\frac{2}{3}} = \frac{1}{2}x^2 + C \Rightarrow y^{\frac{2}{3}} = \frac{1}{3}x^2 + C' \Rightarrow y = \left(\frac{1}{3}x^2 + C'\right)^{\frac{3}{2}} \Rightarrow y_0 = y(0) = \left(\frac{1}{3}0^2 + C'\right)^{\frac{3}{2}} = C'^{\frac{3}{2}} \Rightarrow C' = y_0^{\frac{2}{3}}$$

Separation of Variables (Happiest time)

I.V.P. $y' = p(x)q(y)\ , \quad y(x_0) = y_0$.

$q(y) \neq 0$ near y_0 . $\dfrac{y'}{q(y)} = p(x) \Rightarrow \underbrace{\displaystyle\int_{y_0}^{y(x)} \frac{dy}{q(y)} \ \overset{\text{Change of Variables}}{=\!=} \int_{x_0}^{x} \frac{y'(s)}{q(y(s))}\, ds}_{\text{"Solve" to learn } y = y(x)} = \displaystyle\int_{x_0}^{x} p(s)\, ds$

Eg (i) $x + y \cdot y' = 0 \Rightarrow y \cdot y' = -x$, I.V.P $(0, y_0)$

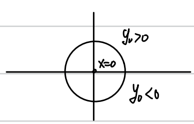
$$\Rightarrow \underbrace{\int_{y_0}^{y(x)} y\, dy}_{\frac{1}{2}(y^2(x) - y_0^2)} = \int_0^x (-s)\, ds = -\frac{1}{2}x^2$$

$\frac{1}{2}(y^2(x) - y_0^2) \Rightarrow y(x) = \pm\sqrt{y_0^2 - x^2}$

$y_0 = 0$, domain$(y) = \{0\}$

$y_0 > 0 \quad y_0 = y(0) = \sqrt{y_0^2}$, so $y(x) = \sqrt{y_0^2 - x^2}$

$y_0 < 0$, $y_0 = y(0) = -\sqrt{y_0^2}$. So $y(x) = -\sqrt{y_0^2 - x^2}$



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Recall I.V.P with separation of variables

$$y' = P(x)q(y), \quad y(x_0) = y_0$$

$$\Rightarrow \frac{y'}{q(y)} = P(x) \qquad [q(y_0) \neq 0]$$

$$\Rightarrow \int_{y_0}^{y(x)} \frac{dy}{q(y)} = \int_{x_0}^{x} \frac{y'(s)}{q(y(s))} ds = \int_{x_0}^{x} P(s) ds$$

Changes of variables

Antiderivative form: $P(x)$, $P' = p$

$$Q(y) = \int \frac{dy}{q(y)} = P(x) + C$$

(ii) Radioactive decay: The mass x of an unstable elements, decays at a rate in time proportional to remaining mass

$$\underbrace{x'(t)}_{\text{rate of decay in time}} = \underbrace{-k}_{\text{negative proportionality as mass decreases}} \cdot x(t)$$

$$\frac{x'}{x} = -k \Rightarrow \log x = -kt + C$$

$x > 0$ as mass > 0

$$\Rightarrow x(t) = Ce^{-kt}, \quad C = e^c > 0 \text{ as mass } > 0$$

I.V.P. $x(0) = x_0$ initial mass $\Rightarrow x_0 = x(0) = C$

What about k? New piece of data: halflife, t_h at which half of x_0 remains

$$\frac{1}{2}x_0 = x_0 e^{-kt_h} \Rightarrow -\log 2 = -kt_h \Rightarrow k = \frac{\log 2}{t_h}, \quad x(t) = x_0 e^{-\frac{\log 2}{t_h} \cdot t}$$

(iii) An object falls from a stand still to the earth from hight H (H is "large" $H < R$ radius of Earth)

As the object falls, it experiences wind residence proportional velocity

Displacement: $s(0) = H$, $v = s'$ — velocity

acceleration proportional to force $\quad v' = \underbrace{-g}_{\text{force due to gravity}} \underbrace{-kv}_{k>0,\text{ in opposition of gravity}} < 0$

$$\Rightarrow \frac{v'}{kv+g} = 1 \Rightarrow \frac{1}{k}\log|kv+g| = -t + C, \quad g + kv = Ce^{-kt}, \quad C = e^c$$

$$\Rightarrow g = Ce^0 = C, \text{ thus } g + kv = ge^{-kt}, \quad kv = g(e^{-kt} - 1), \quad v = \frac{g}{k}(e^{-kt} - 1)$$

$$\lim_{t \to \infty} v(t) = -\frac{g}{k} = \text{"terminal velocity"}$$

Parachute: k large — low magnitude of terminal velocity — approach it very quickly

Ball bearing: k small — hight magnitude of term- velocity — approach slower

Variable replacement : Eg

$y' = f(x,y)$, $f(tx,ty) = f(x,y)$ (homogeneous degree 0)

If we are away from $x=0$, set $z = \frac{y}{x}$, $y = zx$

$\frac{d}{dx}(zx) = z'x + z = y' = f(x,y) = f(1, \frac{y}{x}) = f(1,z) \Rightarrow z'x = f(1,z) - z \Rightarrow \frac{z'}{f(1,z)-z} = \frac{1}{x}$

Eg: $y' = \frac{x-y}{x+y}$, $z = \frac{y}{x}$ (away from $x=0$)

$\Rightarrow \frac{z'}{\frac{1-z}{1+z} - z} = \frac{1}{x}$

$\hookrightarrow \frac{(1+z)z'}{1-z-(z+z^2)} = -\frac{(1+z)z'}{z^2+2z-1}$

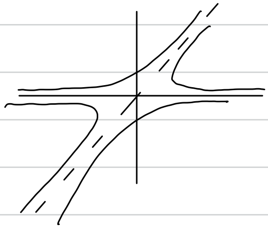
$\Rightarrow -\frac{1}{2}\log|z^2+2z-1| = \log|x| + C \Rightarrow -\log|z^2+2z-1| = \log x^2 + C'$

Qualitative analysis: $C'' = \log|x^2 \cdot (z^2+2z-1)|$

$k = \pm x^2 \cdot (z^2+2z-1)$

$k = \pm (y^2 - 2yx - x^2)$

$k = \pm ((y-x)^2 - 2x^2)$ - hyperbolas



2.24

First order linear D.E.

$y' = p(x)y + q(x)$ $\quad$ p, q continuous on same domain

Facts: Any I.V.P with such a D.E. i.e. $y(x_0) = y_0$ always admits a unique sol'n, assuming that p, q are continuous in neighborhood of x_0 [Picard-Lindelöf Thm, PM 351, AM331]

How to solve?

Homogeneous Case: $y' = p(x)y$, i.e. $q(x) = 0$ $\Rightarrow$ $\frac{y'}{y} = p(x)$ $\Rightarrow$ $\log|y| = P(x) + C$ $\quad$ $P(x) = \int p(x)dx$

$\Rightarrow$ $y = k \cdot e^{P(x)}$, $k = e^c > 0$

Non-homogeneous Case: Let $P(x) = \int p(x)dx$, as above

$\underbrace{y' = p(x)y}_{\text{"p- term"}} + q(x)$ $\quad$ "Trick" $(e^{-P(x)}y)' = e^{-P(x)}y' + e^{-P(x)}(-p(x))\cdot y$

$= e^{-P(x)} \cdot (y' - p(x)y)$

$= e^{-P(x)} \cdot q(x)$

$\Rightarrow e^{-P(x)} y = \int e^{-P(x)} \cdot q(x) dx \Rightarrow y = e^{P(x)} \int e^{-P(x)} q(x) dx$ $\quad$ do not forget constant

$e^{-P(x)} = e^{-\int p(x)dx}$ $\quad$ "integrating factor"

Eg Solve $xy' - 3y = x^6$

$y' = \frac{3}{x}y + x^5$, $p(x) = \frac{3}{x}$, $P(x) = \int \frac{3}{x}dx = 3\log|x| = \log|x^3|$ $\quad$ Did not worry about C

$e^{-P(x)} = \frac{1}{|x^3|}$, $e^{P(x)} = |x^3|$

$$y = |x|^3 \cdot \int \frac{x^5}{|x|^3} dx = \begin{cases} x^3 \cdot (\frac{1}{3}x^3 + C), & x > 0 \\ -x^3 \cdot (\int \frac{x^5}{-x^3} dx), & x < 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{3}x^6 + Cx^3, & x > 0 \\ \frac{1}{3}x^6 - Cx^3, & x < 0 \end{cases}$$

Note: Equation does not allow $x = 0$ in domain, we have for either $x > 0$ or $x < 0$

$y(x) = \frac{1}{3}x^6 + Cx^3$ (sign absorbed into C)

Linear 2nd order linear D.E.

$$y''+P(x)y'+q(x)y=r(x)$$

Facts (i) If P, q, r are continuous on an open interval, then a "general solution" exists: $\varphi_1 y_1+\varphi_2 y_2$. y_1, y_2 are linearly independent. φ_1, φ_2 are differentiable function or constant

(ii) I.V.P $y(x_0)=y_0$, $y'(x_0)=V_0$

Method to solve:

(I) Homogeneous Case: $y''+P(x)y'+q(x)y=0$

- Can be very difficult to compute sol'n, unless P, q are constant (A4)
- A general solution always exists: of form $C_1y_1+C_2y_2$, y_1, y_2 linearly independent sol'ns

[In I.V.P situations, use initial data to learn C_1, C_2] $\begin{cases} y_1 \neq Cy_2 \\ Cy_1 \neq y_2 \end{cases}$, $C \in \mathbb{R}$

(II) (Variation of Parameters – L. Euler)

Idea, replace C_1, C_2 from homogeneous case with functions

We assume: we have φ_1, φ_2 diff'ble with continuous φ_1', φ_2' and we consider (f) $y=\varphi_1y_1+\varphi_2y_2$ (y_1, y_2 are linearly indep. solutions to homogeneous case)

And $\varphi_1'y_1+\varphi_2'y_2=0$ (*)

Let us consider for y in (f) $y'=(\varphi_1y_1+\varphi_2y_2)'=\underbrace{\cancel{\varphi_1'y_1+\varphi_2'y_2}}_{\text{by }(*)}+\underline{\varphi_1y_1'+\varphi_2y_2'}$

$$y''=(\varphi_1y_1'+\varphi_2y_2')'=\varphi_1'y_1'+\varphi_2'y_2'+\varphi_1y_1''+\varphi_2y_2''$$

and then
$$\begin{aligned} y''+Py'+qy &= \varphi_1'y_1'+\varphi_2'y_2'+\varphi_1y_1''+\varphi_2y_2''+P(\varphi_1y_1'+\varphi_2y_2')+q(\varphi_1y_1+\varphi_2y_2) \\ &= \varphi_1'y_1'+\varphi_2'y_2'+\varphi_1\underbrace{(y_1''+Py_1'+qy_1)}_{=0}+\varphi_2\underbrace{(y_2''+Py_2'+qy_2)}_{=0} \\ &= \varphi_1'y_1'+\varphi_2'y_2' \quad (**) \end{aligned}$$

If we wish to solve (♡) Hence we have

"Linear equations" $\begin{cases} \varphi_1'y_1'+\varphi_2'y_2'=r & \text{by (♡) and (**)} \\ \varphi_1'y_1+\varphi_2'y_2=0 & \text{by assumption (*)} \end{cases}$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \xRightarrow{\text{Cramer's rule}} \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \frac{1}{y_1y_2'-y_1'y_2} \cdot \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$W=y_1\cdot y_2'-y_1'y_2 \Rightarrow \begin{cases} \varphi_1' = -\frac{y_2 r}{W} \\ \varphi_2' = \frac{y_1 r}{W} \end{cases} \Rightarrow \begin{cases} \varphi_1(x) = -\int \frac{y_2(x)r(x)}{W(x)}\,dx \\ \varphi_2 = \int \frac{y_1(x)r(x)}{W(x)}\,dx \end{cases}$$

do not forget constant ☺

General solution: $y(x)=\varphi_1(x)\cdot y_1(x)+\varphi_2(x)y_2(x)$

get y_1, y_2 from homogeneous case.

2.26

Taylor's Theorem: Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$ be $(n+1)$ times differentiables with f^{n+1} continuous.
Then for $a \in I$, we have $f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n}_{P_n(x)} + \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$

$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-n)^{n+1}$ – Lagrange remainder theorem $\quad C = \frac{f(x) - P_n(x)}{(x-a)^{n+1}}$

Sketch proof: Let $C \in \mathbb{R}$ satisfy $f(x) - P_n(x) = C \cdot (x-a)^{n+1}$, fix x, for t between a and x

$\varphi(t) = f(x) - \left[f(t) + f'(t)(x-t) + \cdots + \frac{f^n(t)}{n!}(x-t)^n + C \cdot (x-t)^{n+1}\right]$

$\varphi(x) = 0 = \varphi(a)$, Rolle's Theorem $\Rightarrow \varphi'(c) = 0$ for some c between x and a

Calculate: $\varphi'(t) = -\frac{f^{n+1}(t)}{n!}(x-t)^n + (n+1) \cdot C \cdot (x-t)^n$, solve to get $C = \frac{f^{n+1}(c)}{(n+1)!}$ $\square$

another version

Taylor's Theorem: Let $I \subseteq \mathbb{R}$ be an open interval, $f: I \to \mathbb{R}$ be $(n+1)$ times differentiables with f^{n+1} continuous.
Then for $a \in I$, we have $f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\int_a^x \frac{f^{n+1}(t)}{n!}(t-a)^n \, dt}_{R_n(x),\ \text{Cauchy form remainder for } x \in I}$

Proof We have $f(x) = f(a) + \int_a^x f'(t)\,dt$, F.T. of C.

$= f(a) + \int_a^x f'(t)(x-t)^0 \, dt$

$= f(a) - f'(t)(x-t)\Big|_{a=t}^{x=t} + \int_a^x f''(t)(x-t)\,dt$ $\quad$ Integration by Part

$= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt$

Inductive step: $\int_a^x f^m(t)(x-t)^{m-1}\,dt = -\frac{1}{m} f^m(t)(x-t)\Big|_{t=a}^{t=x} + \frac{1}{m}\int_a^x f^{m+1}(t) \cdot (x-t)^m\,dt$

$= \frac{1}{m} f^m(a)(x-a)^m + \frac{1}{m}\int_a^x f^{m+1}(t)(t-a)^m\,dt$

$(*) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x f^3(t)(x-t)^2\,dt$

$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\left[\frac{1}{3} f^3(a)(x-a)^3 + \frac{1}{3}\int_a^x f^4(a)(x-a)^3\,dt\right]$

$\vdots$

$= \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{n+1}(t)(x-t)^n\,dt$ $\square$

Remark: We assumed f^{n+1} is continuous the M/A.V.T for integrals provides $C = C_x$ between a and x s.t.

$\frac{1}{n!}\int_a^x f^{n+1}(t)(x-t)^n\,dt = \frac{f^{n+1}(c)}{n!}(x-c)^n \cdot \underbrace{(x-a)}_{\text{length of interval}}$

Compare : Lagrange form $\qquad$ 2^{nd} Cauchy form

$$\frac{f^{n+1}(c_x)}{(n+1)!}(x-a)^{n+1} = \underset{\underset{f(x)-P_n(x)}{\parallel}}{R_n(x)} = \frac{f^{n+1}(c_x^*)}{n!}(x-c_x^*)(x-a)$$

$c_x \neq c_x^*$ in general

Remark: Given $f: I \to R$, a as above, $P_n(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k$, we have that P_n is the unique polynomial with $\deg P_n \leq n$ s.t. $\lim_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^{n+1}} = 0$

Sketch Proof : Suppose Q is a polynomial, $\deg Q \leq n$ with $\lim_{x\to a} \frac{f(x)-Q(x)}{(x-a)^{n+1}} = 0$.

then $Q(x) + [f(x)-Q(x)] = f(x) = P_n(x) + R_n(x)$

$\Rightarrow Q(x) - P_n(x) = R_n(x) - [f(x)-Q(x)]$

$x \neq a$, $\frac{Q(x)-P_n(x)}{(x-a)^n} = \frac{R_n(x)}{(x-a)^n} - \frac{f(x)-Q(x)}{(x-a)^n}$

$$= \frac{\frac{f^n(c_x)}{n!}(x-c_x)(x-a)}{(x-a)^n} - \frac{f(x)-Q(x)}{(x-a)^n}$$

$$\left[\frac{f^n(c_x)}{n!} \cdot \underbrace{\frac{(x-c_x)^n}{(x-a)^n}}_{|\cdot| \leq 1} \cdot (x-a)\right] \xrightarrow[\substack{x\to a \\ c_x \to a}]{} 0 - 0 = 0$$

E.g. $e^x = \sum_{k=0}^{n} \frac{1}{k!}x^k + \frac{e^c}{(k+1)!}x^{n+1}$, $f(x) = e^x$, $f'(x) = e^x$, centered at $a=0$

$$e^{-x^2} = \sum_{k=0}^{n} \frac{1}{k!}(-x^2)^k + \frac{e^c}{(n+1)!}(-x^2)^{n+1}$$

$$= \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{k!}x^{2k}}_{\text{degree } 2n} + \frac{e^c(-1)^{n+1}}{(n+1)!}x^{2n+2}$$

Conclusion: $\frac{e^{-x^2} - \sum_{k=0}^{n}\frac{(-1)^k}{k!}x^{2k}}{x^{2n+1}} = \frac{\frac{(-1)^{n+1}e^{c_x}}{(n+1)!}x^{2n+2}}{x^{2n+1}} \xrightarrow[x\to 0]{} 0$. We know that $P_{2n}(x) = \sum_{k=0}^{n}\frac{(-1)^k}{k}x^{2k}$, for $f(t) = e^{-t^2}$ around $a=0$

We can learn $f^k(p)$ just from this polynomial for $k=0,\cdots,n$

2.28

Recall : f $(n+1)$-times diffarentiable in an interval containing $f(x) = \sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^k + R_n(x)$

Lagrange . $\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} = R_n(x) = \frac{f^{n+1}(c^*)}{n!}(x-c^*)^n(x-a)$ 2nd Cauchy form , c, c^* between a and x

Example : (Integral functions) $E(x) = \int_0^x e^{-t^2}dt$. Wish to estimate $E(1)$ with a polynomial in 1

...... $E(x)$ .... in x, $x \in [0,1]$

$$a=0 ,\ e^x = \left(\sum_{k=0}^{n} \frac{x^k}{k!}\right) + \frac{e^c}{(n+1)!}\cdot x^{n+1}$$

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!} + \frac{(-1)^{n+1}e^c}{(n+1)!}\cdot t^{2n+2}$$

$$E(x) = \int_0^x e^{-t^2}dt = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \underbrace{\int_0^x t^{2k}dt}_{\frac{x^{2k+1}}{2k+1}} + \frac{(-1)^{n+1}}{(n+1)!}\int_0^x e^c t^{2n+2}dt$$

c between 0 and t^2

$$\left|E(x) - \sum_{k=0}^{n}\frac{(-1)^k \cdot x^{2k+1}}{(2k+1)\cdot k!}\right| = \left|\frac{(-1)^{n+1}}{(n+1)!}\int_0^x e^c \cdot t^{2n+2}dt\right| \quad x\in[0,1]$$

$$\le \frac{1}{(n+1)!}\cdot\int_0^x \left|e^c\cdot t^{2n+2}\right|dt$$

$$\le \frac{e^1}{(n+1)!}\int_0^x t^{2n+2}dt ,\ 0<e^c\le 1 ,\ c\in[-1,0]$$

$$= \frac{x^{2n+3}}{(n+1)!\cdot(2n+3)} \le \frac{1}{(n+1)!\cdot(2n+3)} \quad \text{as } x\in[0,1]$$

Uniform estimate : Estimate holds for any $x\in[0,1]$

Rate of decay of estimate

Ratio of estimates

$$\begin{array}{l} n^{th}: \\ \\ n+1^{th}: \end{array} \left[\frac{\frac{1}{(n+1)!\,(2n+3)}}{\frac{1}{(n+1+1)!\cdot(2(n+1)+3)}}\right]^{-1} \longrightarrow \underbrace{\frac{2n+3}{2n+5}}_{<1}\cdot\frac{1}{n+2} = \frac{e_{n+1}}{e_n}$$

$\boxed{e_n : \text{error in } n \ .\ e_n \sim r^n \ (0<r<1) ,\ \frac{e_{n+1}}{e_n} = r \ (\text{fixed})}$

Series:

Def'n: Let $(a_k)_{k=1}^{\infty} \subset \mathbb{R}$ be a sequence. We define the series $\underbrace{\sum_{k=1}^{\infty}}_{\text{Series: "improper-sum"}} a_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n}}_{\text{Sums}} a_k$, Provided the limits exists

$$\int_0^{\infty} f(t)\,dt = \lim_{x\to\infty} \underbrace{\int_0^{x} f(t)\,dt}_{\text{integral}}$$

Terminology: we say $\sum_{k=1}^{\infty} a_k$ converges, Provide $\left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty}$ Converges

Eg (Essential example - geometric series)

Let $a \in \mathbb{R}$, when does $\sum_{k=0}^{\infty} a^k$ converge?

Let $S_n = \sum_{k=0}^{n} a^k = 1 + a + \cdots + a^n$ $\Rightarrow$ $S_n = \begin{cases} \frac{1-a^{n+1}}{1-a} & \text{if } a \neq 1 \\ n+1 & \text{if } a = 1 \end{cases}$

$S_n(1-a) = 1 - a^{n+1}$

Fact: $\lim_{n\to\infty} a^{n+1} = \begin{cases} 0, & |a|<1 \\ \text{DNE} & |a|\geq 1,\ a\neq 1 \\ 1, & a_n = 1 \end{cases}$ $\Bigg|$ $a=-1$, $a^{n+1} \equiv (-1)^{n+1}$ oscillates

Hence: $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ if $|a|<1$

Example (Sometimes, we get Lucky)

Consider $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, $S_n = \sum_{k=0}^{n} \frac{1}{k(k+1)} = \sum_{k=0}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \frac{1}{n+1}$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right] = 1 \implies \text{series Converges}$$

3.2

Eg. $a \in \mathbb{R}$ $\sum_{k=0}^{\infty} a^k \begin{cases} \frac{1}{1-a}, & |a|<1 \\ D.N.E & |a| \geq 1 \end{cases}$

Eg. $0.3333\cdots = \frac{3}{10} + \frac{3}{10^2} + \cdots\cdots = \sum_{k=1}^{\infty} \frac{3}{10^k}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{10^k} = \frac{3}{10} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^{k-1}}$

$= \frac{3}{10} \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{10^k}$

$= \frac{3}{10} \cdot \frac{1}{1-\frac{1}{10}}$ (by Geo series $a = \frac{1}{10}$)

$= \frac{3}{9} = \frac{1}{3}$

Fundamental Question of Series : Given $\sum_{k=1}^{\infty} a_k$. does it converge?

Proposition: (nth term test — weakest necessity result)

$\sum_{k=1}^{\infty} a_k$ converges $\Longrightarrow \lim_{k\to\infty} a_k = 0$

Proof Let $S_n = \sum_{k=1}^{n} a_k$, then $a_n = S_n - S_{n-1}$, we assume $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$ exists

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$ exists, hence by differences of limit sequences

$0 = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} [S_n - S_{n-1}] = \lim_{n\to\infty} a_n$ $\square$

Eg: $|a| \geq 1 \Rightarrow \sum_{k=0}^{\infty} a^k$ D.N.E Indeed $\lim_{k\to\infty} a^k \neq 0$ (or does exist)

Theorem (Cauchy Criterion for Series Convergence)

$\sum_{k=1}^{\infty} a_k$ converges $\Longleftrightarrow$ given $\varepsilon > 0$, there exist n_ε in $\mathbb{N}$ s.t $\left|\sum_{k=m}^{n} a_n\right| < \varepsilon$ whenever $n > m \geq n_\varepsilon$

Proof Let $S_n = \sum_{k=1}^{n} a_k$

Then $\sum_{k=1}^{\infty} a_k$ converges $\Longleftrightarrow \lim_{n\to\infty} S_n$ exists

$\Longleftrightarrow \begin{cases} \text{given } \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \\ \text{s.t. } |S_n - S_m| < \varepsilon \text{ whenever } n > m \geq n_\varepsilon \end{cases}$

Note that $S_n - S_{m-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m-1} a_k = \sum_{k=m}^{n} a_k$ $\blacksquare$

E.g. Consider $\sum_{k=1}^{\infty} \frac{1}{k}$ (harmonic series)

[$\lim_{k\to\infty} \frac{1}{k} = 0$, yet we do not have enough information for n^{th} term test]

$\sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$, Hence for $\varepsilon = \frac{1}{2}$, the Cauchy Criterion fails for $\sum_{k=1}^{\infty} \frac{1}{k}$

Proposition (Linearity of Converging Series)

Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converges, then for $\alpha, \beta \in \mathbb{R}$

$\sum_{k=1}^{\infty} (\alpha \cdot a_k + \beta \cdot b_k)$ Converges with $\sum_{k=1}^{\infty} (\alpha \cdot a_k + \beta \cdot b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k$

Proof. We use linearity of sums, and of limits (when they exist)

$$\sum_{k=1}^{\infty} (\alpha \cdot a_k + \beta \cdot b_k) = \lim_{n\to\infty} \sum_{k=1}^{n} (\alpha \cdot a_k + \beta \cdot b_k) = \lim_{n\to\infty} \left(\alpha \cdot \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k\right) = \alpha \cdot \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \beta \cdot \lim_{n\to\infty} \sum_{k=1}^{n} b_k \quad (\text{since limit exists})$$

Theorem (Comparison test)

Suppose $0 \le a_k \le b_k$, for $k \ge N$ for some $N \in \mathbb{N}$

Then (i) If $\sum_{k=1}^{\infty} b_k$ Converges $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ Converges

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges $\Longrightarrow$ $\sum_{k=1}^{\infty} b_k$ diverges

Proof (i) Assume $\sum_{k=1}^{\infty} b_k$ Converges. Then for $n \ge N$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k \le \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} b_k$$

$$\le \sum_{k=1}^{N-1} a_n + \lim_{n\to\infty} \sum_{k=N}^{n} b_n = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} b_k$$

$$\le \underbrace{\sum_{k=1}^{N-1} a_k}_{\text{finite}} + \underbrace{\sum_{k=1}^{\infty} b_k}_{<\infty} \quad \left(\text{added in } \sum_{n=1}^{N-1} b_k \ge 0\right)$$

Also, $S_{n+1} - S_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} \ge 0$

$\Longrightarrow (S_n)_{n=1}^{\infty}$ is non-decreasing

Thus, by Monotone Convergence Theorem $\Longrightarrow$ $\sum_{k=1}^{\infty} a_n = \lim_{n\to\infty} S_n$ exists.

(ii) Assume $\sum_{k=1}^{\infty} a_k$ diverges. Since $S_n = \sum_{k=1}^{n} a_k$ is non-decreasing, we must have that $\sum_{k=1}^{\infty} a_k = \infty$ (i.e $S_n = \sum_{k=1}^{n} a_n$ eventually exceeds any real number >0)

Now, for $n \ge N$, we have $\sum_{k=1}^{n} b_k = \sum_{k=1}^{N-1} b_k + \sum_{k=N}^{n} b_k \ge \sum_{k=1}^{N-1} b_k + \sum_{k=N}^{n} a_k$

$$= \underbrace{\sum_{k=1}^{N-1} b_k - \sum_{k=1}^{N-1} a_k}_{\text{independent of } n} + \underbrace{\sum_{k=1}^{n} a_k}_{S_n}$$

N fixed

$$\xrightarrow[n\to\infty]{} \infty \Longrightarrow \sum_{k=1}^{\infty} b_n = \infty \text{ too} \quad \square$$

E.g. $\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$ $\log k \ge 2 \Longleftrightarrow k \ge e^2$. i.e $k = \lfloor e^2 \rfloor + 1$

$\Updownarrow$

$\frac{1}{(\log k)^k} \le \frac{1}{2^k} \Longrightarrow$ By Geo Series $\Longrightarrow \sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$

3.4

$$\frac{k^4+8}{k^3}$$

Corllary (Limit Comparsion Test)

If $a_n \geq 0$ and $b_k > 0$, and $\lim_{n\to\infty} \frac{a_k}{b_k} = L \geq 0$ exists

Then (i) if $L>0$ $\sum_{k=1}^{\infty} b_k$ Converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} a_k$ Converges

(ii) If $L=0$: $\sum_{k=1}^{\infty} b_k$ Converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ Converges

(ii') If $L=0$ $\sum_{k=1}^{\infty} a_k$ diverges $\Longrightarrow$ $\sum_{k=1}^{\infty} b_k$ diverges (Contrapositive of (ii))

Proof (i) We suppose $L>0$, Thus $\exists N_0 \in \mathbb{N}$ s.t.

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \quad \text{if } k \geq N$$

$$\Longleftrightarrow \quad \frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2} \Longleftrightarrow \frac{L}{2} b_k < a_k < \frac{3}{2}L$$

We have $\sum_{k=1}^{\infty} b_k$ Converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} \frac{L}{2} b_k = \frac{L}{2}\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} \frac{3L}{2} b_k = \frac{3L}{2}\sum_{k=1}^{\infty} b_k$ both Converge

We apply Comparison Test, twice

Eg. Let us Consider $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p \geq 2$, Recall that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ Converges

$$\frac{\frac{1}{k^p}}{\frac{1}{k(k+1)}} = \frac{k^2+k}{k^p} = \frac{1+\frac{1}{k}}{k^{p-2}} \xrightarrow{\text{as } k\to\infty} \begin{cases} 1 & , p=2 \\ 0 & , p>2 \end{cases}$$

Limit Comparson Test & (*) $\Longrightarrow$ $\sum_{k=1}^{\infty} \frac{1}{k^p}$ Converges if $p \geq 2$

Remark: The limit Comparsion Test is typically easier to Compute than Comparson Test, and more useful (i.e. You should remember this)

Corollary (Ratio Comparison Test)

If $a_k > 0$ and $b_k > 0$, and $\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k}$ for $k \geq N_0$, N_0 in $\mathbb{N}$

Then, $\sum_{k=1}^{\infty} b_k$ Converges $\Longrightarrow \sum_{k=1}^{\infty} a_k$ Converges

Remark: This is more difficult in Practice, then either Comparison Test or limit Comparison Test, we will see that it has strong theotical value

Proof: for $k \geq N$, $\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} \Longrightarrow \frac{a_{k+1}}{b_{k+1}} \le \frac{a_k}{b_k}$

$$\Longrightarrow \frac{a_k}{b_k} \le \frac{a_N}{b_N} = M \quad \text{for } k \geq N$$

$$\Longrightarrow a_k \le M b_k, \text{ for } k \geq N$$

Then, $\sum_{k=1}^{\infty} b_k$ Converges $\Longrightarrow \sum_{k=1}^{\infty} M b_k$ Converges

$\Longrightarrow$ (Comparison Test) $\sum_{k=1}^{\infty} a_k$ Converges $\square$

Theorem: (Ratio Test): Suppose $a_k > 0$ and that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$ exists

Then $r \geq 0$ and (i) if $r < 1 \Longrightarrow \sum_{k=1}^{\infty} a_k$ Converges

(ii) if $r > 1 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges

Remark: (i) Test is very easy to use, as no $\underbrace{\text{reference series}}_{\sum_{k=1}^{\infty} b_k}$ required

(ii) Case $r=1$ is ambiguous

E.g: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ Converges

Proof (Corollary of Ratio Test)

(i) Say $r<1$, Pick any s so $r<s<1$. Then there is N_0 in $\mathbb{N}$

$$\frac{a_{k+1}}{a_k} < r + (s-r) = s = \frac{s^{k+1}}{s^k}$$

We have that $\sum_{k=1}^{\infty} s^k$ Converges $(0<s<1)$ and hence by R.L.T $\sum_{k=1}^{\infty} a_k$ Converges too

(ii) Say $r>1$ Pick any s so $1<s<r$, then $\exists N_0 \in \mathbb{N}$ s.t. $\frac{a_{k+1}}{a_k} > r - \underbrace{(r-s)}_{>0} = s = \frac{s^{k+1}}{s^k}$

However, $\sum_{k=1}^{\infty} s^k$ diverges, if we had that $\sum_{k=1}^{\infty} a_k$ Converges, then R.L.T would imply $\sum_{k=1}^{\infty} s^k$ Converges Contradiction $\square$

E.g Consider $\sum_{k=0}^{\infty} \frac{(1000)^k}{\sqrt{k!}}$

Ratio Test: $\dfrac{\frac{(1000)^{k+1}}{\sqrt{(k+1)!}}}{\frac{1000^k}{\sqrt{k!}}} = \frac{1000\sqrt{k!}}{\sqrt{(k+1)!}} = \frac{1000}{\sqrt{k+1}} \xrightarrow{k\to\infty} 0 < 1$

$$\Longrightarrow \sum_{k=0}^{\infty} \frac{(1000)^k}{\sqrt{k!}} \text{ Converges}$$

3.6

Theorem (integral test)

Let $a_k > 0$, $k \in \mathbb{N}$, suppose a function $f: [1,\infty) \to \mathbb{R}$ s.t.
- $f(k) = a_k$ for $k \in \mathbb{N}$
- f is non-increasing

Then, $\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\int_1^\infty f(t)\,dt$ converges

Remark f non-increasing $\Rightarrow f$ integrable on each $[1,X]$, $X \geq 1$ (Assignment 1)

Proof f non-increasing If $t \in [1,\infty)$, find $k \in \mathbb{N}$ so $t \leq k$, then $f(t) \geq f(k) = a_k > 0$, hence $f(t) > 0$ for t in $[1,\infty)$

If $t \in [k, k+1]$, then $a_k = f(k) \geq f(t) \geq f(k+1) = a_{k+1}$

and hence $a_k \geq \int_k^{k+1} f(t)\,dt \geq a_{k+1}$ Since $k+1-k=1$

$$\Rightarrow \int_1^{n+1} f(t)\,dt = \sum_{k=1}^{n} \int_k^{k+1} f(t)\,dt \geq \sum_{k=1}^{n} a_{k+1} = \sum_{k=2}^{n+1} a_k \qquad (*)$$

If $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n+1} a_k$ converges, then for $X > 1$ ($\lceil X \rceil = \min\{n \in \mathbb{Z} : X \leq n\}$)

$$0 \leq \int_1^X f(t)\,dt \leq \int_1^{\lceil X \rceil} f(t)\,dt \leq \sum_{k=1}^{\lceil X \rceil} a_k \xrightarrow{X\to\infty} \sum_{k=1}^{\infty} a_k < \infty$$

Hence, $F(x) = \int_1^x f(t)\,dt$ is increasing, as $F'(x) = f(x) > 0$, and f is bounded

Conversely: $\int_1^\infty f(t)\,dt$ converges, Then for n in $\mathbb{N}$

$$0 \leq \sum_{k=1}^{n+1} a_k = a_1 + \sum_{k=2}^{n+1} a_k \leq a_1 + \int_1^{n+1} f(t)\,dt \xrightarrow[(\text{i.e. } X\to\infty)]{n\to\infty} a_1 + \int_1^\infty f(t)\,dt$$

and thus $S_{n+1} = \sum_{k=1}^{n+1} a_k$ is bounded and non-decreasing sequence, hence $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_{n+1}$ converges $\square$

Remark: Variant: We mildly (轻微) weaken assuption on f, above

If there is $M > 1$, So
- $f: [M,\infty) \to \mathbb{R}$ is non-decreasing
- $f(k) = a_k$ for $k \in \mathbb{N}$, $k \geq M$

Then $\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\int_1^\infty f(t)\,dt$ converges [exercise]

Corollary: If $p>0$, $\sum_{k=1}^{\infty}\frac{1}{k^p}$ Converges $\Leftrightarrow p>1$

proof: $f(t)=\frac{1}{t^p}$, which is decreasing $[1,\infty)$

$f(k)=\frac{1}{k^p}$ Integral test: $\sum_{k=1}^{\infty}\frac{1}{k^p} \Leftrightarrow \int_1^{\infty}\frac{dt}{t^p}$ Converges

$$\int_1^x \frac{dt}{t^p}=\int_1^x t^{-p}dt = \begin{cases}\frac{1}{1-p}(x^{1-p}-1), & p\neq 1\\ \log x, & p=1\end{cases} \xrightarrow{x\to\infty} \begin{cases}\infty, & p\leq 1\\ \frac{1}{p-1}, & p>1\end{cases} \quad \blacksquare$$

Indecisive Part of ratio test

$$\frac{\frac{1}{(k+1)^p}}{\frac{1}{k^p}}=\frac{k^p}{(1+k)^p}=\frac{1}{(\frac{1}{k}+1)^p}\xrightarrow[k\to\infty]{}1$$

Examples: $\sum_{k=1}^{\infty}\frac{k^3+1}{k^5+3k^3+1}$ Converges?

Integral Test: $f(t)=\frac{t^3+1}{t^5+3t^3+1} \longrightarrow$ does $\int_1^{\infty}f(t)dt$ Converge? Is f decreasing after some M?

$$\text{Ratio Test}: \frac{\frac{(k+1)^3+1}{(k+1)^5+3(k+1)^3+1}}{\frac{k^3+1}{k^5+3k^3+1}}=\frac{(k+1)^3+1}{k^3+1}\cdot\frac{k^5+3k^3+1}{(k+1)^5+3(k+1)^3+1}\xrightarrow[k\to\infty]{\text{(effort)}}1$$

Indecisive ☹

Limit Comparison Test: $\sum_{k=1}^{\infty}\frac{1}{k^2}$ $(2=5-3)$ Converges, from last corollary

$$\frac{\frac{k^3+1}{k^5+3k^3+1}}{\frac{1}{k^2}}=\frac{k^2(k^3+1)}{k^5+3k^3+1}=\frac{1+\frac{1}{k^3}}{1+\frac{3}{k^2}+\frac{1}{k^5}}\xrightarrow{k\to\infty}\frac{1}{1}=1$$

$\Longrightarrow \sum_{k=1}^{\infty}\frac{k^3+1}{k^5+3k^3+1}$ Converges

(ii) Does $\sum_{k=1}^{\infty} ke^{-k^2}$ Converge?

(1) Integral test: $f(t) = te^{-t^2}$, $f'(t) = e^{-t^2} - 2t^2e^{-t^2} = (1-2t^2)e^{-t^2} < 0$, if $t \geq 1$

$\Longrightarrow$ decreasing

$$\int_1^{\infty} te^{-t^2}dt = \lim_{x\to\infty}\int_1^x te^{-t^2}dt = \lim_{x\to\infty}\left(-\tfrac{1}{2}e^{-t^2}\right)\Big|_1^x = \lim_{x\to\infty}\tfrac{1}{2}(\tfrac{1}{e} - e^{-x^2}) = \frac{1}{2e} \Longrightarrow \sum_{k=1}^{\infty} ke^{-k^2} \text{ Converges}$$

(2) Ratio Test: $\dfrac{(k+1)e^{-(k+1)^2}}{ke^{-k^2}} = \dfrac{k+1}{k} \cdot e^{-2k-1} \xrightarrow{k\to\infty} 0 \Longrightarrow$ Series Converges

(3) Limit Comparison: $\sum_{k=1}^{\infty} e^{-k} = \dfrac{e}{1-e}$ (geometric series)

$\dfrac{ke^{-k^2}}{e^{-k}} = \dfrac{k}{e^{k^2-k}} \xrightarrow{k\to\infty} 0 \Longrightarrow$ Convergence of $\sum_{k=1}^{\infty} k\cdot e^{-k^2}$

$\left(\dfrac{t}{e^{t^2-t}} \sim \dfrac{1}{e^{t^2-t}(2t-1)} \xrightarrow[t\to\infty]{} 0 \right)$

(4) Comparison Test: Since $\lim_{k\to\infty} \dfrac{ke^{-k^2}}{e^{-k}} = 0$, then for some N_0 in $\mathbb{N}$

$\dfrac{ke^{-k^2}}{e^{-k}} < 1$ for $k \geq N$

$0 < ke^{-k^2} < e^{-k}$ for $k \geq N$

$\sum_{k=1}^{\infty} e^{-k}$ Converges $\Longrightarrow \sum_{k=1}^{\infty} ke^{-k^2}$ Converges.

3.9

Ratio Test

$$\frac{a_{k+1}}{a_k} \xrightarrow[\text{suppose}]{k\to\infty} r \begin{cases} >1 & \sum_{k=1}^{\infty} a_k \text{ diverges} \\ <1 & \sum_{a=1}^{\infty} a_k \text{ converges} \end{cases}$$

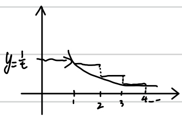
Ex. Euler's Constant : $\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right]$ exists

Recall: $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ $\Rightarrow$ $\lfloor t \rfloor \leq t \leq \lfloor t \rfloor + 1$, $t \geq 1$

$\Rightarrow \frac{1}{\lfloor t \rfloor + 1} \leq \frac{1}{t} \leq \frac{1}{\lfloor t \rfloor}$

$\Rightarrow 0 \leq \frac{1}{\lfloor t \rfloor} - \frac{1}{t}$

Consider $\int_1^n \left(\frac{1}{\lfloor t \rfloor} - \frac{1}{t} \right) dt$



$A_n \leq A_{n+1}$, as $0 \leq \frac{1}{\lfloor t \rfloor} - \frac{1}{t}$ for $t \geq 1$ i.e. $(A_n)_{n=1}^{\infty}$ is non-decreasing

$\Longrightarrow \frac{1}{\lfloor t \rfloor} - \frac{1}{t} \leq \frac{1}{\lfloor t \rfloor} - \frac{1}{\lfloor t \rfloor + 1}$

$A_n \leq \int_1^n \left(\frac{1}{\lfloor t \rfloor} - \frac{1}{\lfloor t \rfloor + 1} \right) dt$

$= \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} = 1 - \frac{1}{k} < 1$

$\Longrightarrow A_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log n$

$= \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right] - \frac{1}{n} \xrightarrow{n\to\infty} \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right]$

When $a_n > 0$ $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$ (Indetermined Case of Ratio Test)

Raabe's Test : Suppose $\lim_{k\to\infty} k\left(1 - \frac{a_{k+1}}{a_k}\right) = p \in \mathbb{R}$

Then, (i) if $p > 1 \Longrightarrow \sum_{k=1}^{\infty} a_k$ Converges

(ii) if $p < 1 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges

(iii) if $p = 1$ and $\left| k\left(1 - \frac{a_{k+1}}{a_k}\right) - 1 \right| \leq \frac{M}{k}$ for $M > 0 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges

Recall: Ratio Comparsion Test

If also $b_k > 0$, $\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$, $k \geq N$

$\sum_{k=1}^{\infty} b_k$ Converges $\Longrightarrow \sum_{k=1}^{\infty} a_k$

Contrapositive : $\sum_{k=1}^{\infty} a_k$ diverges $\Longrightarrow \sum_{k=1}^{\infty} b_k$ diverges

Raabe's Test:

Proof (I) Let $q>0$ in $\mathbb{R}$

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = 1-\frac{q}{k}+\frac{B_k}{k^2} \quad \text{where} \quad 0 \le B_k \le (q+1)q \quad \text{(i.e is bounded)}$$

Indeed: $\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = \frac{k^q}{(k+1)^q} = \frac{1}{(1+\frac{1}{k})^q} = (1+\frac{1}{k})^{-q}$, Let $f(x)=(1+x)^{-q}$, $f'(x)=-q(1+x)^{-q-1}$, $f''(x)=(q+1)q(1+x)^{-q-2}$

Taylor's Theorem about $a=0$: $f(x)=1-qx+\frac{q(q+1)}{(1+c_x)^{q+2}}\cdot x^2$. c_x between 0 and x

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = (1+\frac{1}{k})^{-q} = 1-\frac{q}{k}+\underbrace{\frac{(q+1)q}{(1+c_k)^{q+2}}}_{=B_k}\cdot\frac{1}{k^2}$$

$0<B_k\le q(q+1)$

$$\left[\lim_{k\to\infty} k\left(1-\frac{a_{k+1}}{a_k}\right)=P\right]$$

(II) We write
$$\frac{a_{k+1}}{a_k} = 1-\frac{P}{k}+\frac{P}{k}-1+\frac{a_{k+1}}{a_k}$$
$$= 1-\frac{P}{k}+\frac{1}{k}\big(\underbrace{P-k(1-\frac{a_{k+1}}{a_k})}_{=A_k}\big)$$
$$= 1-\frac{P}{k}+\frac{A_k}{k}$$

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty}\left[P-k\left(1-\frac{a_{k+1}}{a_k}\right)\right]$$
$$= P-\lim_{k\to\infty} k\left(1-\frac{a_{k+1}}{a_k}\right)=0 \quad \text{[by assumption]}$$

(III) Let us assume $P>1$, find q with $P>q>1$. then $\sum_{k=1}^{\infty}\frac{1}{k^q}$ Converges and (II) and (I) show

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} = \left[1-\frac{q}{k}+\frac{B_k}{k^2}\right]-\left[1-\frac{P}{k}+\frac{A_k}{k}\right]$$
$$= \frac{P-q+\frac{B_k}{k}-A_k}{k} \quad \text{where} \quad \lim_{k\to\infty}\left[\frac{B_k}{k}-A_k\right]=0$$

Hence, $\exists N_0\in\mathbb{N}$ s.t. $\left|\frac{B_k}{k}-A_k\right|<\frac{P-q}{2}$ for $k\ge N$

$$\Rightarrow -\frac{P-q}{2}<\frac{B_k}{k}-A_k<\frac{P-q}{2} \quad \text{for } k\ge N$$

So for $k\ge N$

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} > \frac{P-q}{2k} \Longrightarrow \frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} > \frac{a_{k+1}}{a_k}$$
. Thus, by Ratio Comparison Test, $\sum_{k=1}^{\infty} a_k$ Converges

(IV) If $P<1$, and find q so $P<q<1$, thus $\sum_{k=1}^{\infty}\frac{1}{k^q}$ diverges As in (III)

$$\frac{a_{k+1}}{a_k}-\frac{\frac{1}{(k+1)^P}}{\frac{1}{k^P}} = \frac{q-P+A_k-\frac{B_k}{k}}{k}$$

check!

and as $\lim_{k\to\infty}(A_k-\frac{B_k}{k})=0$, $\exists N_0\in\mathbb{N}$ so for $k\ge N$: $\frac{q-P}{2}<A_k-\frac{B_k}{k}<\frac{q-P}{2}$

So
$$\frac{a_{k+1}}{a_k}-\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}}>0 \Longrightarrow \frac{a_{k+1}}{a_k}>\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} \xrightarrow[\text{Ratio Comparison}]{} \sum_{k=1}^{\infty} a_k \text{ diverges}$$

3.11

<u>Raabe's Test</u>: Suppose $\lim_{k\to\infty} k(1-\frac{a_{k+1}}{a_k}) = p \in \mathbb{R}$

Then, (i) if $p>1 \implies \sum_{k=1}^{\infty} a_k$ converges

(ii) if $p<1 \implies \sum_{k=1}^{\infty} a_k$ diverges

(iii) if $p=1$ and $\left|k(1-\frac{a_{k+1}}{a_k})-1\right| \le \frac{M}{k}$ for $M>0 \implies \sum_{k=1}^{\infty} a_k$ diverges

[If $p=1$, and we don't have the 2nd condition $\implies$??]

Remark: The case $p=\infty$ also gives converges, Proof is similar to $p>1$ case [exercise]

(V) (Proof of (iii))

We suppose $p=1$, $\left|k(1-\frac{a_{k+1}}{a_k})-1\right| \le \frac{M}{k}$, $M>0$

(II) of Proof: $\frac{a_{k+1}}{a_k} = 1-\frac{1}{k}+\frac{1}{k}\underbrace{(1-k(1-\frac{a_{k+1}}{a_k}))}_{\ge -\frac{M}{k}} \ge 1-\frac{1}{k}-\frac{M}{k^2}$

Now $\sum_{k=\lfloor M\rfloor+2}^{\infty} \frac{1}{k-M-1}$ diverges. Integral test: $f(t)=\frac{1}{t-M-1}$. decreasing on $[\lfloor M\rfloor+1, \infty)$

$\int_{\lfloor M\rfloor+1}^{x} f(t)\,dt = \log(x-M-1) - \log(\lfloor M\rfloor+2-M-1) \xrightarrow[x\to\infty]{} \infty$. i.e limit does not exist.

For $k>M$, we have

$$\frac{\frac{1}{k-M}}{\frac{1}{k-M-1}} = \frac{k-M-1}{k-M} = 1-\frac{1}{k-M} = 1-\frac{1}{k}\left[\frac{1}{1-\frac{M}{k}}\right].$$

Look like A4, Q5 (a) :)

$$= 1-\frac{1}{k}\left[1+\frac{M}{k}+\frac{(\frac{M}{k})^2}{1-\frac{M}{k}}\right]$$

$$= 1-\frac{1}{k}-\frac{M}{k^2}-\underbrace{\frac{\frac{M^2}{k^3}}{1-\frac{M}{k}}}_{>0} \le 1-\frac{1}{k}-\frac{M}{k^2} \le \frac{a_{k+1}}{a_k}$$

Hence Ratio Comparison Test $\implies \sum_{k=1}^{\infty} a_k$ diverges □

Ex: Find $a, b \geq 0$ s.t. $\sum_{k=1}^{\infty} \frac{(a+1)(a+2)\cdots(a+k)}{(b+1)(b+2)\cdots(b+k)}$ Converge

$$\frac{a_{k+1}}{a_k} = \frac{\frac{(a+1)(a+2)\cdots(a+k)(a+k+1)}{(b+1)(b+2)\cdots(b+k)(b+k+1)}}{\frac{(a+1)(a+2)\cdots(a+k)}{(b+1)(b+2)\cdots(b+k)}} = \frac{a+k+1}{b+k+1} \xrightarrow{k\to\infty} 1$$ Ratio Test failed ∵

Raabe's Test: $k\left(1-\frac{a_{k+1}}{a_k}\right) = k\left(1-\frac{a+k+1}{b+k+1}\right) = k\left(\frac{b-a}{b+k+1}\right) \xrightarrow{k\to\infty} b-a$

If $b-a > 1 \Rightarrow$ Converges

If $b-a < 1 \Rightarrow$ diverges

If $b-a = 1$: $k\left(1-\frac{a_{k+1}}{a_k}\right) - 1 = \frac{k\overbrace{(b-a)}^{=1}}{b+k+1} - 1 = \frac{k}{b+k+1} - 1 = \frac{k-(b+k+1)}{b+k+1} = -\frac{b+1}{k+b+1}$

Compared earlier

Then, $\left|k\left(1-\frac{a_{k+1}}{a_k}\right) - 1\right| = \frac{b+1}{k+b+1} = \frac{1}{k}\left[\frac{b+1}{1+\frac{b+1}{k}}\right] < \frac{b+1}{k}$

So If $b-a = 1 \Longrightarrow$ diverges

Series $\sum_{k=1}^{\infty} a_n$ where we do not assume $a_k \geqslant 0$

Leibniz Alternating Series Test: Suppose

- $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$
- $\lim_{k\to\infty} a_k = 0$

Then, $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ Converges. Furthermore: $\sum_{n=1}^{\infty} (-1)^{k+1} a_k \in [0, a_1]$

Proof: We let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$, Then

$$S_{2n} \leq S_{2n} + \underbrace{a_{2n+1} - a_{2n+2}}_{\geqslant 0} = S_{2n+2}$$

$$= a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} - a_{2n+2}$$

$$= a_1 - \underbrace{(a_2 - a_3)}_{\geqslant 0} - \underbrace{(a_4 - a_5)}_{\geqslant 0} - \cdots - \underbrace{(a_{2n} - a_{2n+1})}_{\geqslant 0} - \underbrace{a_{2n+2}}_{\geqslant 0}$$

$\leq a_1$, Hence $0 \leq S_2 \leq S_{2n} \leq S_{2n+2} \leq a_1$ i.e. $(S_{2n})_{n=1}^{\infty}$ is non-negative, non-decreasing and bounded

MCT $\Longrightarrow$ $A = \lim_{n\to\infty} S_{2n} \leq a_1$ exists

Given $\varepsilon > 0$, $\exists N_0 \in N$, then $|A - S_{2k}| < \frac{\varepsilon}{2}$, $a_{2k+1} < \frac{\varepsilon}{2}$ whenever $k \geqslant N_0$

If $n \geqslant 2N+1$ and with $k = [\frac{n}{2}] \geqslant N$, we have $S_n = \begin{cases} S_{2k}, & \text{if } n \text{ is even} \\ S_{2k+1}, & \text{if } n \text{ is odd} \end{cases} = \begin{cases} S_{2k}, & n \text{ even} \\ S_{2k} + a_{2k+1}, & n \text{ odd} \end{cases}$

$$\text{Then, } |A - S_n| = \begin{cases} |A - S_{2k}|, & n \text{ even} \\ |A - S_{2k} - a_{2k+1}|, & n \text{ odd} \end{cases} < \begin{cases} \frac{\varepsilon}{2}, & n \text{ even} \\ \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, & n \text{ odd} \end{cases} \leq \varepsilon$$

Thus, we conclude $\lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k+1} a_k = A$, i.e. $A = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ $\square$

Corollary: Let $(a_n)_{n=0}^{\infty} \subset R$ satisfy

- a_k is eventually non-increasing, non-negative. $\exists N_0 \in N$ $a_k \geq a_{k+1} \geqslant 0$ if $k \geqslant N_0$
- $\lim_{k\to\infty} a_k = 0$

Then, (i) $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ Converges

(ii) Error estimate: if $m \geqslant N_0$ $\left|\sum_{k=m}^{\infty} (-1)^{k+1} a_k\right| \leq a_m$

Proof (i) Let $n > m \geqslant N$

$$\sum_{k=1}^{n} (-1)^{k+1} a_k = \sum_{k=1}^{m} (-1)^{k+1} a_k + \sum_{k=m+1}^{n} (-1)^{k+1} a_k$$

$$= \sum_{k=1}^{m} (-1)^{k+1} a_k + (-1)^m \sum_{k=m+1}^{n} (-1)^{k-m+1} a_k$$

$$\lim_{n\to\infty} \sum_{k=m}^{n} (-1)^{k-m+1} a_k = \lim_{n\to\infty} \underbrace{\sum_{l=1}^{n-m} (-1)^{l+1} a_{l+m}}_{\text{Sum term in L.A.S.T}} = \sum_{l=1}^{\infty} (-1)^{l+1} a_{l+m} \in [0, a_{m+1}]$$

Today's earlier note

A differentiable integral function without continuous derivative: an application of Alternating Series Test

Let
$$F(x) = \int_0^x \sin\left(\frac{1}{t}\right)\, dt.$$

Question. Is F differentiabe at 0?

Remark. If $x \neq 0$, then F.T.ofC. I tells us that $F'(x) = \sin\left(\frac{1}{x}\right)$.

Answer. We notice that $F(0) = 0$ and

$$F(-x) = \int_0^{-x} \sin\left(\frac{1}{t}\right)\, dt = \int_0^{x} \sin\left(\frac{1}{-u}\right)\, d(-u) = \int_0^{x} \sin\left(\frac{1}{u}\right)\, du = F(x).$$

Thus F is even. Hence we will asses $F'(0)$ by

$$\lim_{x\to 0} \frac{F(x)-F(0)}{x-0} = \lim_{x\to 0} \frac{F(x)}{x} \text{ and it suffices the learn } \lim_{x\to 0^+} \frac{F(x)}{x}.$$

To this end we fix $x > 0$, for now, and consider

$$\begin{aligned} F(x) &= \int_0^x \sin\left(\frac{1}{t}\right)\, dt = \lim_{u\to 0^+} \int_u^x \sin\left(\frac{1}{t}\right)\, dt \\ &= \lim_{n\to\infty} \int_{\frac{1}{n\pi}}^x \sin\left(\frac{1}{t}\right)\, dt \\ &= \lim_{n\to\infty} \sum_{k=k_x}^{n} \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \sin\left(\frac{1}{t}\right)\, dt + \int_{\frac{1}{k\pi}}^{x} \sin\left(\frac{1}{t}\right)\, dt \\ &= \sum_{k=k_x}^{\infty} \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \sin\left(\frac{1}{t}\right)\, dt + \int_{\frac{1}{k_x\pi}}^{x} \sin\left(\frac{1}{t}\right)\, dt \end{aligned}$$

where $\frac{1}{k_x\pi} \le x < \frac{1}{(k_x-1)\pi}$, so $\frac{1}{\pi x} \le k_x$ and $k_x - 1 < \frac{1}{\pi x}$, which means that

$$k_x = \left\lceil \frac{1}{\pi x} \right\rceil \le \frac{1}{\pi x} + 1 \qquad (*)$$

Now we let
$$a_k = \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left|\sin\left(\frac{1}{t}\right)\right|\, dt > 0$$

so
$$F(x) = \sum_{k=k_x}^{\infty} (-1)^k a_k + \int_{\frac{1}{k_x\pi}}^{x} \sin\left(\frac{1}{t}\right)\, dt. \qquad (\dagger)$$

We implicitly define

$$\frac{1}{u} + \pi = \frac{1}{t} \text{ so } t = \frac{u}{1+\pi u} \text{ and } dt = \frac{du}{(1+\pi u)^2}$$

so the transfomation $t \mapsto u$ takes $[\frac{1}{(k+2)\pi}, \frac{1}{(k+1)\pi}]$ to $[\frac{1}{(k+1)\pi}, \frac{1}{k\pi}]$ and we have

$$a_{k+1} = \int_{\frac{1}{(k+2)\pi}}^{\frac{1}{(k+1)\pi}} \left|\sin\left(\frac{1}{t}\right)\right| \, dt = \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left|\sin\left(\frac{1}{u} + \pi\right)\right| \frac{du}{(1+\pi u)^2} \leq \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left|\sin\left(\frac{1}{u}\right)\right| \, du = a_k.$$

We may now use the error estimate from Leibnitz's Alternating Series Test (L.A.S.T.):

$$\left|\sum_{k=k_x}^{\infty} (-1)^k a_k\right| \leq a_{k_x} = \int_{\frac{1}{(k_x+1)\pi}}^{\frac{1}{k_x\pi}} \left|\sin\left(\frac{1}{t}\right)\right| \, dt \qquad (\heartsuit)$$

We then use (†) then (∗) as follows:

$$\begin{aligned}
|F(x)| &\leq \int_{\frac{1}{(k_x+1)\pi}}^{\frac{1}{k_x\pi}} \left|\sin\left(\frac{1}{t}\right)\right| \, dt + \int_{\frac{1}{k_x\pi}}^{x} \left|\sin\left(\frac{1}{t}\right)\right| \, dt \\
&= \int_{\frac{1}{(k_x+1)\pi}}^{x} \left|\sin\left(\frac{1}{t}\right)\right| \, dt \leq \int_{\frac{1}{(k_x+1)\pi}}^{x} 1 \, dt \\
&= x - \frac{1}{(k_x+1)\pi} = \frac{(k_x+1)\pi x - 1}{(k_x+1)\pi} \\
&\leq \frac{2x}{(k_x+1)\pi}
\end{aligned}$$

Now since $\lim_{x\to 0^+} k_x = \lim_{x\to 0^+} \left\lceil \frac{1}{\pi x} \right\rceil = \infty$ we have

$$0 \leq \left|\frac{F(x)}{x}\right| \leq \frac{2}{(k_x+1)\pi} \xrightarrow{x\to 0^+} 0$$

which leads to

$$\boxed{\lim_{x\to 0^+} \frac{F(x)}{x} = 0 \text{ hence } F'(0) = 0.}$$

Recalling that F is even, we realize limit from the left is sufficient.

Bonus Problem. Let $H(x) = \int_0^x \left|\sin\left(\frac{1}{t}\right)\right| \, dt$. *Is H differentiable at 0?*

We remark that the critical estimate at $(\heartsuit)$ will not be available for this example.

Ex : Let $F(x)=\int_0^x \sin(\frac{1}{t})dt$. F.T. of C.I $x\neq 0$ $F'(x)=\sin(\frac{1}{x})$ as $x\longrightarrow \sin(\frac{1}{x})$ is continuous away from 0

Notice that the integral is not continuous at $x=0$, Can we evaluate $F'(0)=?$ Answer: $F'(0)=0$

Notice that $F(-x)=\int_0^{-x}\sin(\frac{1}{t})dt=\int_0^{x}\sin(\frac{1}{-u})(-1)du=\int_0^x \sin(\frac{1}{u})du=F(x)$, So F is even, Also $F(0)=0$

Want $\lim_{x\to0}\frac{F(x)-F(0)}{x-0}=\lim_{x\to0}\frac{F(x)}{x}$, We will assess $\lim_{x\to0^+}\frac{F(x)}{x}$

Set $x>0$. Since $t\longrightarrow\sin(\frac{1}{t})$ is bounded and continuous on $(0,x]$, $F(x)=\int_0^x\sin(\frac{1}{t})dt=\lim_{u\to0^+}\int_u^x\sin(\frac{1}{t})dt$

Hence, $F(x)=\lim_{n\to\infty}\int_{\frac{1}{(n+1)\pi}}^{x}\sin(\frac{1}{t})dt=\lim_{n\to\infty}\left[\sum_{k=k_x}^{n}\int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}\sin(\frac{1}{t})dt+\int_{\frac{1}{k_x\pi}}^{x}\sin(\frac{1}{t})dt\right]$ where k_x in $\mathbb{N}$ satisfy $\frac{1}{k_x\pi}\le x<\frac{1}{(k_x-1)\pi}$ $\Rightarrow$ $\frac{1}{x\pi}\le k_x$, $k_x-1\le\frac{1}{x\pi}$

$\Rightarrow$ $k_x=\lceil\frac{1}{x\pi}\rceil$ (integer ceiling)

$k_x=\lceil\frac{1}{x\pi}\rceil\geqslant\frac{1}{x\pi}$ (will use later)

Let $a_k=\int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}|\sin(\frac{1}{t})|dt>0$, So $\sum_{k=k_0}^{n}\int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}\sin(\frac{1}{t})dt=\sum_{k=k_x}^{n}(-1)^k a_k$

Now $a_{k+1}=\int_{\frac{1}{(k+2)\pi}}^{\frac{1}{(k+1)\pi}}|\sin(\frac{1}{t})|dt$ Let $u=\frac{\frac{1}{k\pi}-\frac{1}{(k+1)\pi}}{\frac{1}{(k+1)\pi}-\frac{1}{(k+2)\pi}}\cdot(t-\frac{1}{(k+2)\pi})+\frac{1}{(k+1)\pi}=\frac{k+2}{k}(t-\frac{1}{(k+2)\pi})+\frac{1}{(k+1)\pi}\Rightarrow\frac{du}{dt}=\frac{k+2}{k}$

$$=\frac{k}{k+2}\int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}|\sin(\frac{1}{u})|du<a_k$$

Hence, $\sum_{k=k_x}^{\infty}(-1)^k a_k$ converges with $|\sum_{k=k_x}^{\infty}(-1)^k a_k|\le a_{k_x}$ Error estimate from last

Thus, $F(x)=\underbrace{\sum_{k=k_x}^{\infty}\int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}\sin(\frac{1}{t})dt}_{(-1)^k a_k}+\int_{\frac{1}{k_x\pi}}^{x}\sin(\frac{1}{t})dt$

Thus $\left|\frac{F(x)}{x}\right|\le\frac{1}{x}\cdot\left(\left|\sum_{k=k_x}^{\infty}(-1)^k a_k\right|+\left|\int_{\frac{1}{k_x\pi}}^{x}\sin(\frac{1}{t})dt\right|\right)$

$\le\frac{1}{x}\left(a_{k_x}+\int_{\frac{1}{k_x\pi}}^{x}|\sin(\frac{1}{t})|dt\right)$

$=\frac{1}{x}\left(\int_{\frac{1}{(k_x+1)\pi}}^{\frac{1}{k_x\pi}}\underbrace{|\sin(\frac{1}{t})|}_{\le1}dt+\int_{\frac{1}{k_x\pi}}^{x}\underbrace{|\sin(\frac{1}{t})|}_{\le1}dt\right)$

$=\frac{1}{x}\left(\frac{1}{k_x\pi}-\frac{1}{(k_x+1)\pi}+x-\frac{1}{k_x\pi}\right)$, $k_x=\lceil\frac{1}{x\pi}\rceil\le\frac{1}{x\pi}$, $\frac{1}{(k_x+1)\pi}\geqslant\frac{1}{(\frac{1}{x\pi}+1)\pi}=\frac{x}{1+x\pi}$

$=\frac{1}{x}\left(x-\frac{1}{(k_x+1)\pi}\right)$

$\le1-\frac{1}{1+x\pi}$

$=\frac{1+x\pi-1}{1+x\pi}$

$=\frac{x\pi}{1+x\pi}\xrightarrow{x\to0^+}0$ Squeeze Theorem $\Rightarrow\lim_{x\to0^+}\left|\frac{F(x)}{x}\right|=0$

Remark (i) If $x \neq 0$, then one can show that

$$F(x) = \int_0^x \sin\left(\frac{1}{t}\right) dt = x^2 \cos\left(\frac{1}{x}\right) - 2\int_0^x f(t)\,dt \qquad f(t) = \begin{cases} t\cos\left(\frac{1}{t}\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$$

Not hard to compute : $F'(0) = 0$ manual on $x^2\cos\left(\frac{1}{x}\right)$ FT of CI on $\int_0^x f(t)\,dt$

(ii) $H(x) = \int_0^x \left|\sin\left(\frac{1}{t}\right)\right| dt$ Question: Is H diff'ble at $x=0$?

$$= \underbrace{\sum_{k=k_x}^{\infty} \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left|\sin\left(\frac{1}{t}\right)\right| dt}_{\leq \frac{1}{k\pi} - \frac{1}{(k+1)\pi} = \frac{1}{k(k+1)\pi}} + \int_{\frac{1}{k_x\pi}}^{x} \left|\sin\left(\frac{1}{t}\right)\right| dt$$

Bonus Question

3.13

Absolute Convergence

Def'n : $\sum_{k=1}^{\infty} a_k$ Converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ Converges

Remark: If $a_k \geq 0$, absolute Converge $\Longleftrightarrow$ Converge

Prop : $\sum_{k=1}^{\infty} a_k$ Converges absolutely $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ Converges

Proof: Cauchy Criterion Given $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\sum_{k=m}^{n} |a_k| < \varepsilon$ whenever $n > m \geq N_0$

Hence. $\left|\sum_{k=m}^{n} a_k\right| \underset{\text{triangle} \leq}{\leq} \sum_{k=m}^{n} |a_k| < \varepsilon$ whenever $n > m \geq N$, Therefore, $\sum_{k=1}^{\infty} a_k$ Converges

Prop (Arbitrary Rearrangement)

If $\sum_{k=1}^{\infty} a_k$ Converges absolutely, and Let $\sigma : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto (arbitrary rearrangement of $\mathbb{N}$)

Then, $\sum_{k=1}^{\infty} a_{\sigma(k)}$ Converges, to the value of $\sum_{k=1}^{\infty} a_k$

Proof: We start with Cauchy Criterion Given $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$. s.t. $\sum_{k=m}^{n} |a_k| < \frac{\varepsilon}{2}$ whenever $n > m \geq N_0$

We next chose N_σ s.t. $\{1, \dots, N_0\} \subseteq \{\sigma(1), \sigma(2), \dots, \sigma(N_\sigma)\}$

[ontoness of σ is required] Thus we let $n > N_\sigma \geq N_1$ then $\left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{\sigma(k)}\right|$

$$= \left|\sum_{k=N_0+1}^{n} a_k - \underbrace{\sum_{\substack{k=1 \\ \sigma(k) \notin \{1,\dots,N_0\}}}^{n} a_{\sigma(k)}}_{\sum_{k \in \{1,\dots,n\} \setminus \{\sigma^{-1}(1), \dots, \sigma^{-1}(N_0)\}} a_{\sigma(k)}}\right|$$

$$\leq \sum_{k=N_0+1}^{n} |a_k| + \sum_{k=N_0+1}^{\max\{\sigma(j)\,:\, j=1,\dots,N_\sigma\}} |a_k|$$

$$\text{by } (*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

It follows that $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{k\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = \sum_{k=1}^{\infty} a_{\sigma(k)}$ $\square$

Remark

(i) We say that $\sum_{k=1}^{\infty} a_n$ is Conditionally Convergent if it's Convergent, but not absolutely Convergent. E.g $\sum_{k=1}^{n} \frac{(-1)^k}{k}$ (L.A.S.T. $\Rightarrow$ Convergence)

in fact $= -\log 2$ [A4]

(ii) If $\sum_{k=1}^{\infty} a_k$ is Conditionally Convergent, and $\alpha \in \mathbb{R}$ (any α will do)

$\exists\ \sigma_\alpha : \mathbb{N} \to \mathbb{N}$ s.t $\sum_{k=1}^{\infty} a_{\sigma(k)} = \alpha$

Proposition (Cauchy Product Formula). Let $\sum_{k=0}^{\infty} a_k$, $\sum_{l=0}^{\infty} b_k$ each be absolutely Convergent

Then, $\sum_{k=1}^{\infty} a_k \cdot \sum_{l=0}^{\infty} b_l = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right)$

Proof Let $N_0 \in \mathbb{N}$. So for $n > m \geq N_0$, we have

$$\sum_{k=m}^{n} |a_k| < \sqrt{\varepsilon}, \quad \sum_{l=m}^{n} |b_l| < \sqrt{\varepsilon}$$

For $n \in \mathbb{N}$, $\sum_{k=0}^{n} a_n \cdot \sum_{l=0}^{n} b_l = \sum_{k=0}^{n} \sum_{l=0}^{n} a_n b_l$

Now let $n \geq 2N_0 + 2 \Rightarrow \frac{n}{2} \geq N_0 + 1 \Rightarrow \lfloor \frac{n}{2} \rfloor \geq N_0$

$$\left| \sum_{k=0}^{n} a_k \sum_{l=0}^{n} b_k - \sum_{k=0}^{n} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) \right| = \left| \sum_{\substack{k,l=1 \\ k+l>n}}^{n} a_k b_l \right| \leq \sum_{\substack{k,l=1 \\ k+l>n}}^{n} |a_k||b_l| \leq \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n} |a_k| \cdot \sum_{l=\lfloor \frac{n}{2} \rfloor}^{n} |b_l| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$$

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{l=0}^{\infty} b_l = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cdot \sum_{l=0}^{n} b_l = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j \cdot b_{k-j} \right) \quad \square$$

X NOPE

Def'n Let $f : [1, \infty) \to \mathbb{R}$, be integrable on $[1, X]$ $X > 1$

We say that $\int_1^{\infty} f$ Converges absolutely if $\int_1^{\infty} |f|$ Converges

Comparison Test: let f satisfy that assumption above, if $\exists\ g : [1, \infty) \to \mathbb{R}$:
- $|f(t)| \leq g(t)$
- $\int_1^{\infty} g$ Converges

Then $\int_1^{\infty} f$ Converges (absolutely) Proof: Almost Same as last Proof.

Prop: With f Satisfying first assumption above, $\int_1^{\infty} f$ Converges absolutely $\Rightarrow \int_1^{\infty} f$ Converges

Proof: (Cauchy Criterion) Given $\varepsilon > 0$, $\exists\ M > 1$ s.t. $\int_u^v |f| < \varepsilon$ whenever $v > u \geq M$. Hence $\left| \int_u^v f \right| \leq \int_u^v |f| < \varepsilon$ if $v > u \geq M$

Thus, $F(x) = \int_1^x f$ Converges as $x \to \infty$ $\square$

Cauchy product, revisted

Proposition.. *Let*

$$\sum_{k=0}^{\infty} a_k, \quad \sum_{k=0}^{\infty} b_k$$

each be absolutely converging series. Then

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} c_k \text{ where } c_k = \sum_{j=0}^{k} a_j b_{j-k}.$$

Proof. We let $A = \sum_{k=1}^{\infty} |a_k|$ and $B = \sum_{k=1}^{\infty} |b_k|$ so $\left|\sum_{k=0}^{\infty} a_k\right| \leq A$ and $\left|\sum_{k=0}^{\infty} b_k\right| \leq B$.

Let us apply the Cauchy Criterion. Given $\varepsilon > 0$ et use find N in $\mathbb{N}$ for which

$$\sum_{k=m}^{n} |a_k| < \frac{\varepsilon}{4B} \text{ and } \sum_{k=m}^{n} |b_k| < \frac{\varepsilon}{4A} \text{ whenever } n > m \geq N.$$

Notice that for $m \geq N$ we also have

$$\left|\sum_{k=0}^{\infty} a_k - \sum_{k=1}^{m} a_k\right| = \lim_{n\to\infty}\left|\sum_{k=0}^{n} a_k - \sum_{k=1}^{m} a_k\right| \leq \lim_{n\to\infty} \sum_{k=m+1}^{n} |a_k| \leq \frac{\varepsilon}{4B}$$

and, likewise

$$\left|\sum_{k=0}^{\infty} b_k - \sum_{k=0}^{m} b_k\right| \leq \frac{\varepsilon}{4A}.$$

Hence given $n \geq 2N+1$ so $\frac{n}{2} \geq N + \frac{1}{2}$, we have that

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_k \cdot \sum_{\ell=0}^{n} b_\ell - \sum_{k=0}^{n}\left(\sum_{j=0}^{k} a_j b_{j-k}\right)\right| &= \left|\sum_{\substack{k,\ell=1,\dots,n\\ k+l>n}} a_k b_\ell\right| \leq \sum_{\substack{k,\ell=1,\dots,n\\ k+l>n}} |a_k||b_\ell| \\
&\leq \sum_{k=1}^{n} |a_k| \sum_{\ell=\lfloor \frac{n}{2}\rfloor}^{n} |b_\ell| + \sum_{k=\lfloor \frac{n}{2}\rfloor}^{n} |a_k| \sum_{k=1}^{n} |b_\ell| \\
&< A\frac{\varepsilon}{4A} + \frac{\varepsilon}{4B}B = \frac{\varepsilon}{2}
\end{aligned}$$

Thus for $n \geq 2N+1$ we have

$$\begin{aligned}
\left|\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k - \sum_{k=0}^{n} c_k\right| &\leq \left|\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k - \sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k\right| + \left|\sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k - \sum_{k=0}^{n} c_k\right| \\
&< \left|\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k - \sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{n} b_k\right| + \left|\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k\right| + \frac{\varepsilon}{2} \\
&\leq A\left|\sum_{k=0}^{\infty} b_k - \sum_{k=0}^{n} b_k\right| + \left|\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k\right| \sum_{k=0}^{n} |b_k| + \frac{\varepsilon}{2} \\
&\leq A\frac{\varepsilon}{4A} + \frac{\varepsilon}{4B}B + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence we have shown the desired result. $\square$

Remark. Notice that

$$\sum_{k=0}^{\infty} |a_k| \cdot \sum_{k=0}^{\infty} |b_k| = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} |a_j||b_{j-k}|\right)$$

where

$$|c_k| = \left|\sum_{j=0}^{k} a_j b_{j-k}\right| \leq \sum_{j=0}^{k} |a_j||b_{j-k}|$$

so the Comparison Test shows that $\displaystyle\sum_{k=0}^{\infty} c_k$ is absolutely convergent.

Math 148, Section 002 (Spronk), Winter 2020

Lecture notes for March 23.

Intergal Test – revisted.

The integral test can help resolve some of the most subtle series.

Example. Does $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(\log k)^{\log k}}$ converge?

Now let $f:[2,\infty)\to\mathbb{R}$ be given by $f(t)=\frac{1}{(\log t)^{\log t}}=e^{-\log t\log(\log t)}$. Notice that

$$f'(t)=e^{-\log t\log(\log t)}\left[-\frac{\log(\log t)}{t}-\frac{1}{t}\right]<0\quad\Rightarrow\quad f\text{ is decreasing.}$$

Hence we have satisfied hypotheses of, and may use, the *Integral Test*:

$$\sum_{k=2}^{\infty}\frac{1}{(\log k)^{\log k}}\text{ converges}\quad\Leftrightarrow\quad\int_2^{\infty}e^{-\log t\log(\log t)}\,dt\text{ converges.}$$

Like the series, this daunting improper integral does not seem to yield to any obvious comparison. Hence let us use change of variables $u=\log t$ so $e^u=t$ and hence $e^u\,du=dt$, and for $x>2$ we have

$$\int_2^{x}e^{-\log t\log(\log t)}\,dt=\int_{\log 2}^{\log x}e^{-u\log u}e^u\,du=\int_{\log 2}^{\log x}\frac{e^u}{u^u}\,du\xrightarrow{x\to\infty}\int_{\log 2}^{\infty}\frac{e^u}{u^u}\,du.$$

[Keep in mind, that this limit is either finite non-negative or ∞.] It may be possible to find a nice comparison with $g:[\log 2,\infty)=[\log 2,1]\cup[1,\infty)\to\mathbb{R}$. given by $g(u)=\frac{e^u}{u^u}=e^{u-u\log u}$, to see that the limit is finite. [Any suggestions?] However, let us notice that

$$g'(u)=e^{u-u\log u}[1-(\log u-1)]=-e^{u-u\log u}\log u<0,\text{ for }u>1\quad\Rightarrow\quad g\text{ is decreasing on }[1,\infty).$$

Hence we may again use the *Integral Test*:

$$\int_1^{\infty}\frac{e^u}{u^u}\,du\text{ converges}\quad\Leftrightarrow\quad\sum_{k=1}^{\infty}\left(\frac{e}{k}\right)^k\text{ converges.}$$

Let us try the *Ratio Test* on the series:

$$\frac{\left(\frac{e}{k+1}\right)^{k+1}}{\left(\frac{e}{k}\right)^k}=\frac{ek^k}{(k+1)^k(k+1)}=\frac{e}{\left(1+\frac{1}{k}\right)^k(k+1)}\xrightarrow{k\to\infty}0<1$$

since $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k=e$. Hence $\sum_{k=1}^{\infty}\left(\frac{e}{k}\right)^k$ converges, and we can deduce that $\sum_{k=2}^{\infty}\frac{1}{(\log k)^{\log k}}$ converges.

[Notice, as well, for $k \geq 3$ we have comparison $\left(\frac{e}{k}\right)^k \leq \left(\frac{e}{3}\right)^k$, and hence it follows from the *Comparison Test* against geometric series, that $\sum_{k=1}^{\infty} \left(\frac{e}{k}\right)^k$ converges. There is a lesson, here: $\frac{e^u}{u^u} \leq \left(\frac{e}{3}\right)^u = e^{-(\log 3 - 1)u}$ for $u \geq 3$, and the *Intergal Comparison Test* would have sufficed, earlier. This answers the question posed earlier.]

Question. Will Raabe's test work on $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\log k)^{\log k}}$? [Check that the Ratio Test in inconclusive.]

Take-away point. *The Integral Test is useful, and works in both directions.*

Convergence of functions.

Goal. We wish to show statements like

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{or} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

which, as we know, must mean that

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \quad \text{or} \quad \sin x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

More generally we wish to consider

$$f(x) = \sum_{k=0}^{\infty} f_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f_k(x).$$

Indeed, in the first case above $f_k(x) = \frac{x^k}{k!}$, and in the second, $f_k(x) = \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

We can abstract this idea a bit further. We let $s_n(x) = \sum_{k=0}^{n} f_k(x)$, and then we are really interested in statements like

$$f(x) = \lim_{n\to\infty} s_n(x).$$

This may seem belaboured, but we shall see that notions of convergence vary widely for functions.

Definition. (Pointwise convergence.) *Let $f_1, f_2, \ldots$ and f be functions on an interval I. We say that*

$$\lim_{n\to\infty} f_n = f \ \underline{\textit{pointwise}} \text{ on } I, \text{ if } \lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \text{ in } I.$$

This is clearly the minimum we would expect. It is worth investigating its properties by way of examples.

Examples. **(i)** (Instability of differentiability/continuity.) Let $g : [-\pi, \pi] \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 0 & \text{if } x \leq -\frac{\pi}{2} \\ \frac{1}{2}[\sin x + 1] & \text{if } -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} < x \end{cases}.$$

Notice that

$$g'(x) = \begin{cases} 0 & \text{if } x \leq -\frac{\pi}{2} \\ \frac{1}{2}\cos x & \text{if } -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \end{cases}$$

and hence g' exists everywhere ($\pm\frac{\pi}{2}$ must be checked manually) and is continuous.

We then let $f_n(x) = [g(x)]^n$, i.e. $f_n = g^n$. Then each f_n is differentiable with continuous derivative, thanks to the Chain Rule. However, since $0 \leq g(x) < 1$ for $x < \frac{\pi}{2}$ we see that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} [g(x)]^n = \begin{cases} 0 & \text{if } x < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} \leq x. \end{cases}$$

Hence *a pointwise limit of differentiable/continuous functions need <u>not</u> be continuous.*

[Play with this yourself. Go to `https://www.desmos.com/calculator` (internet search "desmos"). Type in

$$\begin{aligned} &y = 0\left\{x < -\frac{\pi}{2}\right\} \text{ (syntax: } \mathtt{y = 0\{x < -pi/2\}}\text{), then hit+} \\ &y = \frac{1}{2}(\sin x + 1)^{100}\left\{-\frac{\pi}{2} < x < \frac{\pi}{2}\right\}, \text{ then hit+} \\ &y = 1\left\{x > \frac{\pi}{2}\right\} \end{aligned}$$

Play with the exponent 100 to get a feel for how this converges.]

(ii) (Instability of integrals.) Let $f_n : [0, 2] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < \frac{1}{n} \\ -n^2\left(x - \frac{1}{n}\right) + n = 2n - n^2 x & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 2 \end{cases}$$

[As above, it may instructive to see how `desmos` depicts this with $n = 1$, then $n = 3$, etc.]

We compute that

$$\int_0^2 f_n(x)\,dx = n^2 \int_0^{\frac{1}{n}} x\,dx + n\int_{\frac{1}{n}}^{\frac{2}{n}} (2 - nx)\,dx = 1.$$

However, since $f(0) = 0$, and for $x > 0$ we have $f_n(x) = 0$ whenever $\frac{2}{n} \leq x$, i.e. $\lceil \frac{2}{x} \rceil \leq n$, we see that

$$\lim_{n\to\infty} f_n(x) = 0.$$

In particular notice that

$$\lim_{n\to\infty} \int_0^2 f_n(x)\, dx = 1 \neq 0 = \int_0^2 \left[\lim_{n\to\infty} f_n(x)\right] dx.$$

Hence *integrals of pointwise limits need not limits of integrals.*

(iii) (Limits need not even be integrable.) We can enumerate $\mathbb{Q} \cap [0,1]$. Indeed consider the following labelling of these elements

$$\begin{array}{cccccc} 0 & 1 & 1/2 & 1/4 & 3/4 & \dots \\ 1/3 & 2/3 & 1/6 & 5/6 & 1/12 & \dots \\ 1/5 & 2/5 & 3/5 & 4/5 & 1/10 & \dots \\ 1/7 & 2/7 & \dots & \dots & \dots & \dots \end{array} \quad = \quad \begin{array}{cccccc} q_1 & q_3 & q_6 & q_{10} & \dots & \dots \\ q_2 & q_5 & q_9 & \dots & \dots & \dots \\ q_4 & q_8 & \dots & \dots & \dots & \dots \\ q_7 & \dots & \dots & \dots & \dots & \dots \end{array}$$

[For example, 1/15 is 4 places along the 3rd row; 1/14, 1/21 and 1/35 are some ways along the 4th row.]

We let for any set $E \subset [0,1]$, $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$

Let $f_n = \chi_{\{q_1,\dots,q_n\}} : [0,1] \to \mathbb{R}$. Notice that each f_n is piecewise continuous, hence integrable, with $\int_0^1 f_n = 0$. However, $\lim_{n\to\infty} f_n(x) = \chi_{\mathbb{Q}}(x)$, and is not integrable!

Take-away point. *Pointwise convergence is highly unstable.* Next lecture we shall introduce *uniform convergence*, which is much better.

Math 148, Section 002 (Spronk), Winter 2020

Lecture notes for March 25.

Uniform Convergence – three theorems.

LAST TIME:

Definition. (Pointwise convergence.) *Let $f_1, f_2, \ldots$ and f be functions on an interval I. We say that*

$$\lim_{n\to\infty} f_n = f \ \underline{\textit{pointwise}} \text{ on } I, \text{ if } \lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \text{ in } I.$$

This disappointing notion of convergence does not respect continuity (hence not differentiability), nor integrals.

The following is more subtle, but more robust.

Definition. (Uniform convergence.) *Let $f_1, f_2, \ldots$ and f be functions on an interval I. We say that*

$$\lim_{n\to\infty} f_n = f \ \underline{\textit{uniformly}} \text{ on } I$$

if, given $\varepsilon > 0$, there is N in $\mathbb{N}$ for which

$$|f_n(x) - f(x)| < \varepsilon \text{ for every } x \text{ in } I, \text{ whenever } n \geq N.$$

This may be restated as follows. Given $\varepsilon > 0$, there is N in $\mathbb{N}$ for which

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \text{ for every } x \text{ in } I, \text{ whenever } n \geq N.$$

Hence for $n \geq N$, we have

$$\{(x, f_n(x)) : x \in I\} \subset \{(x, y) : f(x) - \varepsilon < y < f(x) + \varepsilon : x \in I\}$$

in other words the graph $y = f_n(x)$ sits inside a certain "ε-bubble" about the graph $y = f(x)$, whenever $n \geq N$.

[Play some more with `desmos`, which is mentioned last lecture. Depict

$$y = x^2$$
$$y = x^2 - \frac{1}{10}$$
$$y = x^2 + \frac{1}{10}.$$

Even with this rather large $\varepsilon = \frac{1}{10}$, you will have to enlarge to see the ε-bubble about the graph $y = x^2$. Change x^2 to $\sin x$, to get a new perspective.]

Exercise: Consider the three examples of sequences of functions from last lecture. Observe how none of them converge uniformly. (For two of these sequences, this will be evident form the two theorems below.)

The next two theorems show how uniform continuity is very stable.

Theorem. (Uniform convergence and integrals.) *Let $f_1, f_2, \ldots$ and f be functions on $[a, b]$, such that*

- *each of $f, f_1, f_2, \ldots$ are integrable on $[a, b]$, and*
- $\lim\limits_{n\to\infty} f_n = f$ *uniformly on $[a, b]$.*

Then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. We use uniform convergence: given $\varepsilon > 0$ there N be so

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b - a + 1} \text{ for every } x \text{ in } [a, b] \text{, whenever } n \geq N.$$

Thus, if $n \geq N$, we use linearity and order properties of integrals to see that

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f| \leq \int_a^b \frac{\varepsilon}{b - a + 1} < \varepsilon$$

which shows that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. $\square$

Remark. Above, we assume integrability of f. In the next result we make no extra assumptions about f.

Theorem. (Uniform convergence and continuity.) *Let $f_1, f_2, \ldots$ and f be functions on an interval I, such that*

- *each of $f_1, f_2, \ldots$ is continuous on I, and*
- *$\lim_{n\to\infty} f_n = f$ uniformly on I.*

Then

$$f \text{ is continuous on } I.$$

Proof. We will leverage continuity of certain elements f_n into learning the same for f. Intuition: "if there is a continuous function g whose graph lies in the "ε-bubble" about that of f, then g's continuity can be leveraged to get near continuity of f. If for each ε I can find such g, f will be continuous.

Fix x_0 in I and $\varepsilon > 0$. Then uniform convergence provides N in $\mathbb{N}$ for which

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for every } x \text{ in } I, \text{ whenever } n \geq N.$$

Next we let $\delta > 0$ satisfy the definition of continuity of f_N at x_0:

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3} \text{ whenever } x \in I,\ |x - x_0| < \delta.$$

Now we can put this together. Let $x \in I$ with $|x - x_0| < \delta$. Then

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows that f is continuous at x_0. This being true for any x_0 in $[a, b]$, we see that f is continuous on $[a, b]$. $\square$

Remark/Exercise. Remind yourself of the definition of uniform continuity. Show that if each f_n, above, is uniformly continuous, then so too is f. (Interestingly, this proof is mildly simpler than the one above.) Recall that uniform continuity is automatic if $I = [a, b]$, but not if I is not closed and bounded.

The next result returns us to series.

Theorem. (Weierstrass M-Test.) *Let* $f_1, f_2, \ldots$ *be functions on an interval* I*, such that*

$$\text{there are } M_1, M_2, \ldots \text{ such that each } \sup_{x\in I}|f_k(x)| \le M_k \text{ and } M = \sum_{k=1}^{\infty} M_k \text{ converges.}$$

Then there is a function $f : I \to \mathbb{R}$ *such that*

$$\sum_{k=1}^{\infty} f_k = \lim_{n\to\infty}\sum_{k=1}^{n} f_k = f \text{ uniformly on } I.$$

In particular, if each f_k *is continuous, then so too is* f*.*

Proof. For each x in I, $|f_k(x)| \le M_k$ so $\sum_{k=1}^{\infty} f_k(x)$ converges absolutey, thanks to the *Comparison Test*. Hence we define defines $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for x in I.

Given $\varepsilon > 0$ there is N in $\mathbb{N}$ such that for $m \ge N$ we have

$$\varepsilon > \left|M - \sum_{k=1}^{m} M_k\right| = \left|\sum_{k=m+1}^{\infty} M_k\right| = \sum_{k=m+1}^{\infty} M_k.$$

Hence $n \ge N$ we have for any x in I that

$$\begin{aligned}\left|f(x) - \sum_{k=1}^{m} f_k(x)\right| &= \left|\sum_{k=m+1}^{\infty} f_k(x)\right| = \left|\lim_{n\to\infty}\sum_{k=m+1}^{n} f_k(x)\right| \\ &\le \lim_{n\to\infty}\sum_{k=m+1}^{n}|f_k(x)| \le \lim_{n\to\infty}\sum_{k=m+1}^{n} M_k \\ &= \sum_{k=m+1}^{\infty} M_k < \varepsilon.\end{aligned}$$

Notice that the above estimate is for every x in I, and hence we get $f = \lim_{m\to\infty}\sum_{k=1}^{m} f_k = \sum_{k=1}^{\infty} f_k$ uniformly on I.

If each f_k is continuous, then each $\sum_{k=1}^{m} f_k$ is continuous, and the Theorem on *Uniform convergence and continuity* show that f must be, too. $\square$

Summary.

Uniform convergence and integrals: on finite length intervals, the integral of the limit is the limit of integrals.

Uniform convergence and continuity: uniform limits of continuous functions are continuous.

Weierstrass M-Test: if the bounds of a sequence of functions are summable (i.e. have converging series) then the series converges uniformly.

MATH 148, SECTION 002 (SPRONK), WINTER 2020

Lecture notes for March 27.

Power series and Taylor series.

LAST TIME:

Uniform convergence and continuity: uniform limits of continuous functions are continuous.

Weierstrass M-Test: if the bounds of a sequence of functions are summable (i.e. have converging series) then the series converges uniformly.

Definition. A power series about a in $\mathbb{R}$ is any function defined in a neighbourhood of a of the form

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \tag{$\heartsuit$}$$

where $(a_k)_{k=0}^{\infty} \subset \mathbb{R}$. This being a series, the determination of when it converges is an issue. This is where the *Root Test* really comes into its own. This series converges whenever

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} \cdot |x-a| = \limsup_{k\to\infty} \sqrt[k]{|a_k(x-a)^k|} < 1$$

and diverges whenever the above lim sup is greater than 1. Hence this motivates us to define the important radius of convergence:

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

whose value we shall take as ∞ if $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 0$, and as 0 if $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \infty$. We thus learn the following.

Crucial Fact. *Give a power series $f(x)$ as in ($\heartsuit$) with radius of convergence R, we have that*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \begin{cases} \text{converges if} & |x-a| < R \\ \text{diverges if} & |x-a| > R \end{cases}.$$

The behavior at $x - a = \pm R$ can vary widely, as we shall see in examples (quite possibly on A6).

Role of Ratio Test. We saw in A5, Q2, that

$$\text{if } r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \text{ exists, then } r = \lim_{k\to\infty} \sqrt[k]{|a_k|}.$$

Hence we conclude that the radius of convergence satisfies

$$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}, \text{ if the limit exists.}$$

This holds even if the limit is ∞.

Examples. **(i)** We consider power series about 0:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^p}$$

where $p > 0$. Then

$$\lim_{k\to\infty} \frac{\frac{1}{k^p}}{\frac{1}{(k+1)^p}} = \lim_{k\to\infty} \frac{(k+1)^p}{k^p} = 1.$$

Hence we have radius of convergence $p = 1$.

(ii) We consider power series about 2:

$$f(x) = \sum_{k=0}^{\infty} k!(x-2)^k.$$

Here we have

$$\lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

This series clearly converges when $|x-2| = 0$, i.e. $x = 2$. If $|x-2| > 0$, i.e. $x \neq 2$, then this series diverges. Hence this is a degenerate case: it is not defined on a neighbourhood of 2, but only at 2 itself.

(iii) We consider power series about 0:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{2^k}$$

We could apply either Root or Ratio Tests to see that $R = 2$, in this case. Or, we may observe this as the *geometric series*

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x} \text{ if } |x| < 2.$$

Of course, if $|x| \geq 2$, this series diverges.

Exercise. Suppose $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ has radius of convergence R_1, $g(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ has radius of convergence R_2. Show that each of

$$f(x) + g(x) = \sum_{k=0}^{\infty}(a_k + b_k)(x-a)^k, \qquad f(x)g(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)(x-a)^k$$

holds whenever $|x - a| < \min\{R_1, R_2\}$; hence these series have radius of convergence $R \geq \min\{R_1, R_2\}$.

Theorem. (Convergence of power series.) *Let $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ be a power series with radius of convergence $R > 0$. Then for any $0 < r < R$ we have that*

$$f(x) = \lim_{n\to\infty}\sum_{k=0}^{n} a_k(x-a)^k \text{ uniformly on } [a-r, a+r].$$

In particular, f is continuous on $(a - R, a + R)$ (which is all of $\mathbb{R}$ if $R = \infty$).

Proof. The heart of this proof is the *Weierstrass M-Test.* However, we must do a bit of development, first.

Since $r < R$ we exploit the method by which we devised R, rather than its actual definition, to get that

$$L = \limsup_{k\to\infty} \sqrt[k]{|a_k|}\cdot r = \limsup_{k\to\infty}\sqrt[k]{|a_k|}\cdot |a - (a \pm r)| < 1.$$

Hence there is $s < 1$ for which

$$L = \limsup_{k\to\infty}\sqrt[k]{|a_k|}\cdot r < s.$$

Using the definition of $\limsup$ we see that there N in $\mathbb{N}$ such that

$$\sup_{k\geq n}\sqrt[k]{|a_k|}\cdot r < L + (s - L) = s \text{ whenever } n \geq N$$

and hence

$$k \geq N \quad \Rightarrow \quad \sqrt[k]{|a_k|}\cdot r < s \quad \Rightarrow \quad |a_k|r^k < s^k.$$

Now we are in good shape. Let $f_k(x) = a_k(x-a)^k$, so the last line tells us that

$$|f_k(x)| = |a_k(x-a)^k| \leq |a_k|r^k < s^k \text{ for } |x-a| \leq r \text{ and } k \geq N$$

where $0 < s < 1$. Let

$$M_k = \begin{cases} |a_k|r^k & \text{if } k = 0, 1, \ldots, N-1 \\ s^k & \text{if } k \geq N \end{cases}.$$

Then the *Comparison Test* tells us that $\sum_{k=0}^{\infty} M_k$ converges. Hence the *Weierstrass M-Test* tells us that $f = \sum_{k=0}^{\infty} f_k$ converges uniformly on $[a - r, a + r]$. Continuity on $[a - r, a + r]$ follows from *Uniform convergence and continuity.* Notice that if $x \in (a - R, a + R)$, then $x \in [a - r, a + r]$ for any $r < \min\{(a + R) - x, x - (a - R)\}$. $\square$

We now come to some important examples.

Taylor series. Suppose that each derivative $f^{(k)}$ exists in a neighbourhood of a. We define the Taylor series of f about a by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Of course, we wish to make this meaningful.

Proposition. *Suppose f as above admits $r > 0$ for which the remainder term admits uniform bound*

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1} \right| \leq M_n \text{ for } |x-a| \leq r \text{ where } \lim_{n\to\infty} M_n = 0.$$

Then the radius of convergence of the Taylor series satisfies $R > r$, and

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ for } |x-a| \leq r.$$

Proof. We do as we did in A4. We have for x in $[a-r, a+r]$ that

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| = |R_n(x)| \leq M_n \overset{n\to\infty}{\to} 0$$

which, by squeeze principle means that

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

uniformly on $[a-r, a+r]$. □

With complex analysis this can be vastly improved. This is a topic for PM332 or PM352.

Summary. *Power series functions* are defined on an interval determined by *radius of convergence*. Radius of convergence can be determined via the Root Test, and very often the Ratio Test.

Taylor polynomials converge to a *Taylor series*, provided that there is uniform control of remainders.

守夜冠军杯

Math 148, Section 002 (Spronk), Winter 2020

Lecture notes for March 30.

Uniform Convergence and derivatives – derivatives of power series.

LAST WEEK:

Uniform convergence and integrals: on finite length intervals, the integral of the limit is the limit of integrals.

Uniform convergence and continuity: uniform limits of continuous functions are continuous.

Weierstrass M-Test: if the bounds of a sequence of functions are summable (i.e. have converging series) then the series converges uniformly.

In other words, uniform convergence behaves nicely with integrals and continuity.

Theorem. (Convergence of power series.) *Let $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ be a power series with radius of convergence $R > 0$. Then for any $0 < r < R$ we have that*

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(x-a)^k \text{ uniformly on } [a-r, a+r].$$

In particular, f is continuous on $(a-R, a+R)$ (which is all of $\mathbb{R}$ if $R = \infty$).

Question. Does uniform convergence behave nicely with respect to derivatives?

In brief, the answer is no.

Examples. (i) Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} -x & \text{if } x \leq -\frac{1}{2n} \\ \frac{1}{n} - \sqrt{\frac{1}{2n^2} - x^2} & \text{if } -\frac{1}{2n} < x \leq \frac{1}{2n} \\ x & \text{if } \frac{1}{2n} < x \end{cases}.$$

[As usual, it is nice to sketch this. Do it for $n = 3$ on `desmos` (if you do it for any larger n you will have to enlarge your picture).] We notice that f_n is differentiable with

$$f_n'(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{2n} \\ \frac{x}{\sqrt{\frac{1}{2n^2} - x^2}} & \text{if } -\frac{1}{2n} < x < \frac{1}{2n} \\ 1 & \text{if } \frac{1}{2n} \leq x \end{cases}$$

where we are forced to reason manually at $x = \pm\frac{1}{2n}$.

Notice that $\lim_{n\to\infty} f_n(x) = |x|$ uniformly for x in $\mathbb{R}$; in fact $|f_n(x) - |x|| \le \frac{1-\frac{1}{\sqrt{2}}}{n}$, so a simple squeeze argument suffices.

The absolute value function is well-known to not be differentiable at 0, even though each $f_n'(0) = 0$.

(ii) We notice that $\left|\frac{\sin(4^k x)}{2^k}\right| \le \frac{1}{2^k}$. Hence the Weierstrauss M-Test shows that

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(4^k x)}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin(4^k x)}{2^k} \text{ uniformly}$$

and hence is continuous on $\mathbb{R}$. However, each $f_n(x) = \sum_{k=1}^{n} \frac{\sin(4^k x)}{2^k}$ admits derivative

$$f_n'(x) = \sum_{k=1}^{n} 2^k \cos(4^k x).$$

These derivatives are unbounded at 0, in fact unbounded at any $\frac{\ell}{2^m}\pi$ where $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. Hence we have that

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sum_{k=1}^{n} 2^k \cos(4^k x) \text{ diverges at infinitely many points.}$$

It is thus impossible to write $f'(x) = \lim_{n\to\infty} f_n'(x)$ for general x. [It is unliklely that f is anywhere differentiable ... but that's another story.]

Hence we are led to expect that we need to know more if differentiability and derivatives are to be respected by limits.

Aside. For any $(b_k)_{k=0}^{\infty} \subset [0, \infty)$, there is a subsequence $(b_{k_j})_{j=1}^{\infty}$ so

$$\limsup_{k\to\infty} b_k = \lim_{j\to\infty} b_{k_j}.$$

Indeed, we let $k_1 = 1$ and inductively find for $j = 1, 2, 3, \dots$ a k_{j+1} so

$$k_{j+1} > k_j \quad \text{and} \quad \sup_{\ell \geq k_j+1} b_\ell \geq b_{k_{j+1}} > \begin{cases} \left[\sup_{\ell \geq k_j+1} b_\ell\right] - \frac{1}{j} & \text{if } \sup_{\ell \geq k_j+1} b_\ell < \infty \\ j & \text{if } \sup_{\ell \geq k_j+1} b_\ell = \infty. \end{cases}$$

[Notice, under sup we have $k_j + 1$, not k_{j+1}.] Then squeeze principle provides that

$$\lim_{j\to\infty} b_{k_j} = \lim_{k\to\infty} \sup_{\ell \geq k} b_\ell = \limsup_{k\to\infty} b_k.$$

Application: Power series are good. Given the examples above, consider it remarkable that power series are stable for operations of calculus. Much of it boils down to the following.

Lemma. *Let $(a_k)_{k=0}^{\infty} \subset \mathbb{R}$. Then the power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \quad \textit{and} \quad g(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$$

admit the same radius of convergence.

Proof. It is premature to deem g as the derivative of f, but we are so close.

We first observe that convergence of a series is not effected by multiplication by a real number $x - a$, so

$$g(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1} \text{ converges } \Leftrightarrow\ (x-a)g(x) = \sum_{k=1}^{\infty} k a_k (x-a)^k \text{ converges.}$$

Secondly, we observe that

$$\lim_{k\to\infty} \sqrt[k]{k} = \lim_{k\to\infty} e^{\frac{\log k}{k}} = e^{\lim_{k\to\infty} \frac{\log k}{k}} = e^0 = 1.$$

We finally have that

$$\limsup_{k\to\infty} \sqrt[k]{|k a_k|} = \lim_{j\to\infty} \sqrt[k_j]{k_j} \cdot \sqrt[k_j]{|a_{k_j}|} = \lim_{j\to\infty} \sqrt[k_j]{|a_{k_j}|} = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$$

where the subsequence is chosen to realize the $\limsup_{k\to\infty} \sqrt[k]{|a_k|}$, as above. $\square$

Theorem. (Derivatives and integrals of power series.) *Let* $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *have radius of convergence* $R > 0$. *Then*

(i) *for* $x \in (a-R, a+R)$, $\displaystyle\int_a^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}$, *and*

(ii) f *is differentiable on* $(a-R, a+R)$ *with* $\displaystyle f'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$.

Proof. (i) Since $f_n(x) = \sum_{k=0}^{n} a_k(x-a)^k$, above, converge uniformly on $[a, x]$ (or $[x, a]$) for each x in $(a-R, a+R)$, we can use *Uniform convergence and integrals* to see that

$$\begin{aligned}\int_a^x f(t)\,dt = \int_a^x \lim_{n\to\infty} \sum_{k=0}^{n} a_k(t-a)^k\,dt &= \lim_{n\to\infty} \int_a^x \sum_{k=0}^{n} a_k(t-a)^k\,dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k+1}(x-a)^{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}.\end{aligned}$$

(ii) The fucntions f_n above satisfy $f_n'(x) = \sum_{k=1}^{n} k a_k(x-a)^{k-1}$. Let

$$g(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$$

as in the Lemma, above. Then by *Convergence of power series* we have that

$$\lim_{n\to\infty} f_n' = g \text{ uniformly on } [a-r, a+r] \text{ for } 0 < r < R.$$

Since each $f_n(a) = a_0$, we apply F.T.ofC. II to each f_n', and then *Uniform convergence and integrals* to see for x in $(a-R, a+R)$ that

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left[a_0 + \int_a^x f_n'(t)\,dt\right] = a_0 + \int_a^x g(t)\,dt.$$

But then, by F.T.ofC. I, we see that

$$f'(x) = g(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}. \qquad \square$$

Corollary. *Let* $f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *be as above. Then*

$$a_k = \frac{f^{(k)}(a)}{k!} \text{ for each } k = 0, 1, 2, \ldots$$

In particular, the power series representation for f *is unique on* $(a-R, a+R)$.

Proof. A simple induction shows that

$$f(a) = a_0,\ f'(a) = a_1,\ f''(a) = 2a_2, \ldots,\ f^{(k)}(a) = k!a_k. \qquad \square$$

Summary. Uniform convergence does not respect differentiability.

However, power series functions are differentiable, with expected derivatives. They also admit expected integrals/antiderivatives.

Math 148, Section 002 (Spronk), Winter 2020

Lecture notes for April 1. (No foolin'!)

Taylor series. (For real this time!)

LAST WEEK (March 27):

Proposition. *Suppose f is in infinitely differentiable in a neighbourhood of a and admits $r > 0$ for which the remainder terms admit uniform bounds:*

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1} \right| \le M_n \text{ for } |x-a| \le r \text{ where } \lim_{n\to\infty} M_n = 0.$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ for } |x-a| \le r.$$

Hence the Taylor series has radius of convergence $R \ge r$.

Let us see how this plays out in examples.

Examples. **(i)** $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ on $\mathbb{R}$, centred at $a = 0$.

Indeed, consider any $r > 0$. Then we have remainder term

$$0 < R_n(x) = \frac{e^{c_x}}{(n+1)!}x^{n+1} \le \frac{e^r}{(n+1)!}r^{n+1} \text{ if } |x| \le r.$$

We note that for $n > N > r$ we have

$$\frac{e^r}{(n+1)!}r^{n+1} \le e^r \frac{r^N}{N!}\left(\frac{r}{N+1}\right)^{n+1-N} \xrightarrow{n\to\infty} 0.$$

[Actually, this decays very quickly: the Ratio Test show that $\sum_{n=1}^{\infty} \frac{r^{n+1}}{(n+1)!}$ converges, which implies that $\lim_{n\to\infty} \frac{r^{n+1}}{(n+1)!} = 0$.]

It is now easy to form other power series representations with same radius of convergence $R = \infty$:

$$e^{5x} = \sum_{k=0}^{\infty} \frac{5^k}{k!}x^k, \quad e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}x^{2k}, \quad e^{x-1} = \sum_{k=0}^{\infty} \frac{1}{k!}(x-1)^k.$$

These are in fact the Taylor series about 0, 0, respectively 1; see the last Corollary of last lecture.

(ii) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$, $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ on $\mathbb{R}$ centered at $a = 0$.

Here we observe, for the cosine case, that for $0 < r$ we have

$$|R_{2n}(x)| = \left| \frac{\cos^{(2n+1)}(c_x)}{(2n+1)!} x^{2n+1} \right| \leq \frac{r^{2n+1}}{(2n+1)!} \overset{n\to\infty}{\longrightarrow} 0$$

with reasoning similar to that above. The case for sine is very similar (see A5).

Notice that if we write

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{\ell=0}^{\infty} a_\ell x^\ell$$

we find that $a_\ell = 0$ for each even ℓ.

Notice, furthermore, that we can use differentiation of series:

$$\cos x = \sin' x = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{(2k+1)!} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

(iii) Geometric series: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ on $(-1, 1)$, centered at $a = 0$.

Here we did not require Taylor's Theorem, though it may be manually checked that $\left(\frac{d}{dx}\right)^k \frac{1}{1-x} = \frac{k!}{(1-x)^k}$, and hence evaluates to $k!$ at $x = 0$. Hence this is the Taylor series (which we should have known from last Corollary of last lecture).

We can use this to recentre. Let us consider centre $a = -1$.

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{\frac{1}{2}}{1 - \frac{x+1}{2}} = \sum_{k=0}^{\infty} \frac{(x+1)^k}{2^{k+1}}.$$

Notice that this series admit radius of convergence $R = 2$.

We can use integration an differentiation of series for $|x| < 1$, e.g.:

$$-\log(1-x) = \int_0^x \frac{dt}{1-t} = \int_0^x \sum_{k=0}^{\infty} t^k \, dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k = \sum_{k=1}^{\infty} k x^{k-1} = \sum_{k=0}^{\infty} (k+1) x^k$$

Let us take a slightly different perspective to the result which we quoted at the beginning of this lecture.

Proposition. (Endpoints.) *Suppose the Taylor series of f about a has radius of convergence $0 < R < \infty$, and on $[a, a+R]$ (and/or on $[a-R, a]$) we have that*

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1} \right| \leq M_n \text{ where } \lim_{n\to\infty} M_n = 0.$$

Then

$$\text{on } (a-R, a+R] \text{ (and/or on } [a-R, a+R)).$$

Moreover, convergence is uniform on $[a, a+R]$ (and/or on $[a-R, a]$).

Proof. We proceed nearly exactly as in the lecture of March 27 (i.e. as in A4). We have for x in $[a, a+R]$ (and/or on $[a-R, a]$) that

$$\left| f(x) - \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq M_n \overset{n\to\infty}{\to} 0$$

which, by squeeze principle means that

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

uniformly on $[a, a+R]$ (and/or on $[a-R, a]$). $\square$

Example. For $f(x) = -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$, above we should revisit the derivative sequence

$$f'(x) = \frac{1}{1-x},\ f''(x) = \frac{1}{(1-x)^2}, \ldots,\ f^{(n)}(x) = \frac{(n-1)!}{(1-x)^n}.$$

Thus

$$0 \leq R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1} = \frac{x^{n+1}}{(n+1)(1-c_x)^{n+1}} \leq \frac{1}{n+1} \text{ for } -1 \leq x \leq 0$$

since $\frac{1}{(1-c_x)^{n+1}} \leq 1$ as $-1 \leq c_x \leq x \leq 0$. Notice that this estimate will *not* work for $0 \leq x \leq 1$, though we have convergence for $(-1, 0]$ by usual power series results. Hence we see that

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \text{ for } x \text{ in } [-1, 1).$$

Newton's binomial series. We recall the *Binomial Theorem*: if $n \in \mathbb{N}$ then

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

We will, in principle, generalize this to $(1+x)^\alpha$, where α in $\mathbb{R}$.

Let us now let $\alpha \in \mathbb{R}$ and define $f_\alpha(x) = (1+x)^\alpha$ for $x > -1$. Notice that $f_\alpha(x) = e^{\alpha \log(1+x)}$ for general α, which is why we must restrict the domain. We compute

$$f'_\alpha(x) = \alpha(1+x)^{\alpha-1},\ f''_\alpha(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}, \dots, f^{(k)}_\alpha(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}.$$

[Notice that if $\alpha \in \mathbb{N}$, then this process will eventually terminate in zeros.] We thus get Taylor polynomial about 0 given by

$$P_n(x) = 1 + \sum_{k=1}^{n} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k.$$

It will be convenient to define the <u>generalized binomial coefficients</u>:

$$\binom{\alpha}{0} = 1,\ \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \text{ for } k = 1, 2, \dots$$

so we may write

$$P_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$$

We now make a simplifying assumption:

$$0 < \alpha < 1.$$

Hence for $k \geq 1$ we have

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} = (-1)^{k-1}\frac{\alpha(1-\alpha)\dots(k-1-\alpha)}{k!} = (-1)^{k-1}\left|\binom{\alpha}{k}\right| \quad (\spadesuit)$$

We will try the Ratio Test to determine radius of convergence:

$$\frac{\left|\binom{\alpha}{k}\right|}{\left|\binom{\alpha}{k+1}\right|} = \frac{\frac{\alpha(1-\alpha)\dots(k-1-\alpha)}{k!}}{\frac{\alpha(1-\alpha)\dots(k-1-\alpha)(k-\alpha)}{(k+1)!}} = \frac{k+1}{k-\alpha} \xrightarrow{k\to\infty} 1$$

Hence the power series $g(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$ has $R = 1$.

On the other hand, if $0 < r < 1$, and $|x| \le r$ then

$$|R_n(x)| = \left|\frac{f_\alpha^{(n+1)}(c_x)}{(n+1)!}x^{n+1}\right| = \left|\binom{\alpha}{n+1}(1+c_x)^{\alpha-n-1}x^{n+1}\right| = \left|\binom{\alpha}{n+1}\right| r^{n+1}$$
$$= \frac{\alpha(1-\alpha)\dots(n-\alpha)}{(n+1)!}r^{n+1} \le r^{n+1} \overset{n\to\infty}{\longrightarrow} 0$$

and this limit is uniform on $[-r, r]$. Hence we conclude for $0 < \alpha < 1$ that

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \text{ for } |x| < 1. \qquad (\clubsuit)$$

Extend to α in $\mathbb{R}$. We use integration and differentiation of series to see for $0 < \alpha < 1$, $-1 < x < 1$ that

$$\frac{1}{\alpha+1}[(1+x)^{\alpha+1} - 1] = \int_0^x (1+t)^\alpha\,dt = \int_0^x \sum_{k=0}^{\infty}\binom{\alpha}{k}t^k\,dt = \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{\alpha}{k}x^{k+1}$$
$$\alpha(1+x)^{\alpha-1} = \frac{d}{dx}(1+x)^\alpha = \frac{d}{dx}\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k\right] = \sum_{k=1}^{\infty}k\binom{\alpha}{k}x^{k-1}$$

The formulas

$$\frac{\alpha+1}{k+1}\binom{\alpha}{k} = \binom{\alpha+1}{k+1}, \quad \frac{k}{\alpha}\binom{\alpha}{k} = \binom{\alpha-1}{k-1}$$

along with elementary manipulations of series show that

$$(1+x)^{\alpha+1} = \sum_{k=0}^{\infty}\binom{\alpha+1}{k}x^k, \quad (1+x)^{\alpha-1} = \sum_{k=0}^{\infty}\binom{\alpha-1}{k}x^k.$$

Then, a simple induction shows for any n in $\mathbb{N}$ that

$$(1+x)^{\alpha+n} = \sum_{k=0}^{\infty}\binom{\alpha+n}{k}x^k, \quad (1+x)^{\alpha-n} = \sum_{k=0}^{\infty}\binom{\alpha-n}{k}x^k. \qquad (1)$$

(2) For $\alpha = -1$ we have geometric series $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, where we notice that $(-1)^k = \binom{-1}{k}$. We can take derivatives of this series, as in (1) above.

(3) For n in $\mathbb{N}$ we have the Binomial Theorem. Here $\binom{n}{k} = 0$ if $k \ge n$.

We collect (1), (2) and (3) together to see that ($\clubsuit$) holds for any α in $\mathbb{R}$.

Endpoints $x = \pm 1$ for case $0 < \alpha$. Notice that for $\alpha > 0$ but $\alpha \notin \mathbb{N}$ that

$$k\left[1 - \frac{\left|\binom{\alpha}{k+1}\right|}{\left|\binom{\alpha}{k}\right|}\right] = k\left[1 - \frac{k-\alpha}{k+1}\right] = k \cdot \frac{1+\alpha}{k+1} \xrightarrow{k\to\infty} 1+\alpha > 1 \tag{$*$}$$

Hence by Raabe's Test, $\sum_{k=0}^{\infty} \binom{\alpha}{k}$ is absolutely convergent. Thus, as $\left|\binom{\alpha}{k}x^k\right| \leq \left|\binom{\alpha}{k}\right|$ for x in $[-1, 1]$, Hence the Weierstrass M-Test tells us that

$$g(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ converges uniformly to a continuous function on } [-1, 1]$$

Then (♣) tells us that $g(x) = (1+x)^\alpha$ for x in $(-1, 1)$. Hence continuity tells us that

$$2^\alpha = (1+1)^\alpha = \lim_{x\to 1^-}(1+x)^\alpha = \lim_{x\to 1^-} g(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \text{ and}$$

$$0 = (1-1)^\alpha = \lim_{x\to -1^+}(1+x)^\alpha = \lim_{x\to -1^+} g(x) = \sum_{k=0}^{\lceil\alpha\rceil}(-1)^k\left|\binom{\alpha}{k}\right| + (-1)^{\lceil\alpha\rceil}\sum_{\lceil\alpha\rceil+1}^{\infty}\left|\binom{\alpha}{k}\right|$$

where we have used to obvious analogue of (♠) to gain the last formula. In particular, for $\alpha > 0$ we have that

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \text{ for } -1 \leq x \leq 1. \tag{♣+}$$

Endpoints $x = \pm 1$ for case $-1 < \alpha < 0$. Here the computation $(*)$ gives value $1+\alpha$ which satisfies $0 < 1+\alpha < 1$. Also, (♠) becomes $\binom{\alpha}{k} = (-1)^k\left|\binom{\alpha}{k}\right|$ and hence

$$\sum_{k=0}^{\infty}\binom{\alpha}{k} = \sum_{k=0}^{\infty}(-1)^k\left|\binom{\alpha}{k}\right|$$

converges conditionally by *Raabe's Alternating Series Test*. Furthermore, we have

$$|R_n(x)| = \left|\binom{\alpha}{n+1}\frac{x^{n+1}}{(1+c_x)^{\alpha-n-1}}\right| \leq \left|\binom{\alpha}{n+1}\right| \text{ for } 0 \leq x \leq 1.$$

where $\lim_{n\to\infty}\left|\binom{\alpha}{n+1}\right| = 0$ (part of Raabe's Alternating Series Test). Hence by the last Proposition, on Endpoints, we conclude for $-1 < \alpha < 0$ that

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \text{ for } -1 < x \leq 1. \tag{♣−}$$

Exercise. Show that if $\alpha \leq -1$, then at neither of the endpoints ± 1 does the binomial series converge.

Summary. Taylor series must be understood as approximation by Taylor polynomials. Hence the Taylor Remainder Formula always plays a role.

In many cases, the natural radius of convergence for the Taylor series defines the interval on which "function = series" (!!). Also, typically, if the series converges at an endpoint, shown either

– via Taylor's theorem, or

– by way a Weierstrass M-Test argument, like for the endpoints of binomial series for $\alpha > 0$, above.

(!!) Next lecture, I will show an example where this is not true.

Math 148, Section 002 (Spronk), Winter 2020

Lecture notes for April 3.

Last remarks on power series.

LAST WEEK:

The following is our only way to see that a function can be represented by Taylor series.

Proposition. *Suppose f as above admits $r > 0$ for which the remainder term admits uniform bound*

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}\right| = \left|\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt\right| \leq M_n \text{ for } |x-a| \leq r$$

where $\lim_{n\to\infty} M_n = 0$. *Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ for } |x-a| < r.$$

Surprising example. (Just because a Taylor series exists, doesn't mean that it has meaning.) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Claim 1. *For each n in $\mathbb{N}$, there is a polynomial p_n with $p_n(0) = 0$ such that*

$$f^{(n)}(x) = e^{-\frac{1}{x^2}} p_n\left(\frac{1}{x}\right) \text{ for } x \neq 0.$$

We prove this by induction. First see that

$$f'(x) = e^{-\frac{1}{x^2}}\left(-\frac{2}{x^3}\right) = e^{-\frac{1}{x^2}}\left[-2\left(\frac{1}{x}\right)^3\right]$$

so $p_1(t) = -2t^3$. Suppose the result holds for n. Then

$$f^{(n+1)}(x) = \frac{d}{dx}\left[e^{-\frac{1}{x^2}} p_n\left(\frac{1}{x}\right)\right] = e^{-\frac{1}{x^2}}\left[-\frac{2}{x^3}p_n\left(\frac{1}{x}\right) - \frac{1}{x^2}p_n'\left(\frac{1}{x}\right)\right].$$

Indeed, if $p(\frac{1}{x}) = a_1\frac{1}{x} + a_2\frac{1}{x^2} + \cdots + a_m\frac{1}{x^m}$, then $\frac{d}{dt}p(\frac{1}{x}) = -a_1\frac{1}{x^2} - 2a_2\frac{1}{x^3} - \cdots - ma_m\frac{1}{x^{m+1}} = -\frac{1}{x^2}p'(\frac{1}{x})$. Thus $p_{n+1}(t) = -2t^3p_n(t) - t^2p_n'(t)$.

Claim 2. $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$ *for any* n *in* $\mathbb{N}$.

We use L'hopital's rule many times

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = \lim_{x\to 0} \frac{\frac{1}{x^n}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{\frac{1n}{x^{n+1}}}{e^{\frac{1}{x^2}}\frac{1}{x^3}} = \lim_{x\to 0} \frac{\frac{n}{x^{n-2}}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{\frac{n(n-2)}{x^{n-4}}}{e^{\frac{1}{x^2}}} = \cdots = \lim_{x\to 0} \begin{cases} \frac{n(n-2)\ldots 2}{e^{\frac{1}{x^2}}} & \text{if } n \text{ is even} \\ \frac{n(n-2)\ldots 3\cdot 1x}{e^{\frac{1}{x^2}}} & \text{if } n \text{ is odd} \end{cases}$$

and the last limit is 0.

Claim 3. $f^{(n)}(0) = 0$ *for each* n *in* $\mathbb{N}$.

We have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = 0$$

by Claim 2. We apply induction and Claim 1 to see

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} p_n\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} e^{-\frac{1}{x^2}} \frac{p(x)}{x} = 0.$$

Since $\frac{p(x)}{x}$ is the sum of terms of the form $\frac{a}{x^n}$, we can apply Claim 2.

Conclusion. We have that f is infinitely differentiable on $\mathbb{R}$ with $f^{(n)}(0) = 0$ for each n in $\mathbb{N}$. Hence we get Taylor series about $a = 0$ given by $\sum_{k=0}^{\infty} 0x^k$. This admits infinite radius of convergence. Yet

$$f(x) = \sum_{k=0}^{\infty} 0x^k = 0 \text{ only for } x = 0.$$

What's going on?

We have failed to attempt to check the remainders

$$|R_n(x)| = \left| \frac{e^{-\frac{1}{c_x^2}} p_{n+1}\left(\frac{1}{c_x}\right)}{n!} x^{n+1} \right| = \left| \frac{1}{n!} \int_0^x e^{-\frac{1}{t^2}} p_{n+1}\left(\frac{1}{t}\right) (x-t)^n \, dt \right|$$

Given the Proposition from last week, we must conclude that $|R_n(x)|$ are not nicely bounded in any neighbourhood of 0.

A related example. $f(x) = \begin{cases} e^{-\frac{1}{(x-1)^2}} & \text{if } |x| > 1 \\ 0 & \text{if } |x| \leq 1 \end{cases}$ is infinitely differentiable on all of $\mathbb{R}$. Clearly $f^{(k)}(0) = 0$ for all k in $\mathbb{N}$. But the Taylor series about $a = 0$ satisfies

$$f(x) = \sum_{k=0}^{\infty} 0x^k \text{ only for } -1 \leq x \leq 1, \text{ even though radius of convergence satisfies } R = \infty.$$

The following is what we hope happens to power series at endpoints.

Abel's Limit Theorem. *Suppose* $\sum_{k=0}^{\infty} a_k$ *converges. Then*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ defines a continuous function on } (-1,1].$$

Proof. The *nth Term Test* tells us that $\lim_{n\to\infty} |a_n| = 0$. Thus $0 \le |a_k| \le M$ for some M and hence it follows that $\limsup_{k\to\infty} \sqrt[k]{|a_k|} \le 1$. Thus *Convergence of Power Series* tells us that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges, and define a continuous function on $(-1,1)$. Hence it remains to check that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k. \tag{$*$}$$

We begin with a finite sum rearrangement formula, known as *Abel summation*: for $n = 0,1,2,\ldots$, we let $S_n = \sum_{k=0}^{n} a_k$ and set $S_{-1} = 0$, and get

$$\begin{aligned}\sum_{k=0}^{n} a_k x^k &= \sum_{k=0}^{n} (S_k - S_{k-1}) x^k = \sum_{k=0}^{n} S_k x_k - \sum_{k=0}^{n} S_{k-1} x^k \\ &= \sum_{k=0}^{n} S_k x_k - \sum_{k=0}^{n-1} S_k x^{k+1} = S_n x^n + (1-x) \sum_{k=0}^{n-1} S_k x^k\end{aligned}$$

Since we have assumed that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = \sum_{k=0}^{\infty} a_k := S$$

we have for $|x| < 1$ that $\lim_{n\to\infty} S_n x^n = 0$. Thus

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k = \lim_{n\to\infty} \left[S_n x^n + (1-x) \sum_{k=0}^{n-1} S_k x^k \right] = (1-x) \sum_{k=0}^{\infty} S_k x^k.$$

Let us put this to good use. Given $\varepsilon > 0$ we let

- N in $\mathbb{N}$ be so $|S_k - S| \le \dfrac{\varepsilon}{2}$ for $k \ge N$; and
- let $\delta = \dfrac{\varepsilon}{2\left[\sum_{k=1}^{N} |S_k - S|\right]}$.

We recall that

$$(1-x) \sum_{k=0}^{\infty} x^k = 1 \text{ for } -1 < x < 1.$$

Then for $1-\delta < x < 1$ we have

$$\begin{aligned}
|f(x) - S| &= \left|(1-x)\sum_{k=0}^{\infty} S_k x^k - (1-x)\sum_{k=0}^{\infty} S x^k\right| \\
&= (1-x)\left|\sum_{k=0}^{\infty}(S_k - S)x^k\right| \le (1-x)\sum_{k=0}^{\infty}|S_k - S|x^k \\
&= (1-x)\sum_{k=0}^{N}|S_k - S|x^k + (1-x)\sum_{k=N+1}^{\infty}|S_k - S|x^k \\
&\le (1-x)\sum_{k=0}^{N}|S_k - S| + (1-x)\sum_{k=N+1}^{\infty}\frac{\varepsilon}{2}x^k \\
&= (1-x)\sum_{k=0}^{N}|S_k - S| + \frac{\varepsilon}{2}x^{N+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

which shows $(*)$. □

Corollary. (Endpoints.) *Suppose that* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *has radius of convergence* $0 < R < \infty$ *and the series* $\sum_{k=0}^{\infty} a_k(-R)^k$, *respectively* $\sum_{k=0}^{\infty} a_k R^k$, *converges. Then*

$$\lim_{x\to(a-R)^+}\sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} a_k(-R)^k, \textit{ respectively } \lim_{x\to(a+R)^-}\sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} a_k R^k.$$

Proof. We shall prove only the left endpoint case. Consider the power series in variable t

$$\sum_{k=0}^{\infty} a_k(-R)^k t^k$$

which satisfies the hypotheses of Abel's limit theorem. Thus we have

$$\sum_{k=0}^{\infty} a_k(-R)^k = \lim_{t\to1^-}\sum_{k=0}^{\infty} a_k(-R)^k t^k = \lim_{t\to1^-}\sum_{k=0}^{\infty} a_k(-Rt)^k = \lim_{x\to(a-R)^+}\sum_{k=0}^{\infty} a_k(x-a)^k$$

where the last inequality is facilitated by variable substitution $x-a=-Rt$. □

Examples. **(i)** We have for $-1 < x < 1$ that

$$-\log(1-x) = \int_0^x \frac{dt}{1-t} = \int_0^x \sum_{k=1}^{\infty} x^k \, dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

with $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ convergent, thanks to Leibniz's alternating series test. Then the Corollary shows that

$$-\log 2 = -\lim_{x\to -1^+} \log(1-x) = \lim_{x\to -1^+} \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

(ii) If $\alpha > 0$, it was shown last lecture that

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ for } -1 < x < 1.$$

Also, Raabe's Test was applied to show that $\displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k}$ converges absolutely. Hence

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (\pm 1)^k = \lim_{x\to \pm 1^{\mp}} \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \lim_{x\to \pm 1^{\mp}} (1+x)^{\alpha} = \begin{cases} 2^{\alpha} & \text{if } \pm 1 = 1 \\ 0 & \text{if } \pm 1 = -1. \end{cases}$$

(iii) Like (ii), above, we have for $-1 < \alpha < 0$ that $\displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k}$ is a conditionally convergent alternating series. Hence the right endpoint $x = 1$ works, and we see that $2^{\alpha} = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k}$, as above.

Summary. We absolutely require some knowledge of the Taylor remainder to see that "function = Taylor series" on an interval ... well, unless we are manipulating one of our reference series.

Abel's Limit Theorem is a a way of handling endpoints. However, this is really a means of last resort, as we have buried much of the analysis into a clever and tricky proof. [At least we can see why Abel is famous.]

A Guide to Series Comparison Tests

General Results

nth Term Test. (Weakest test.) $\sum_{k=1}^{\infty} a_k$ *converges* $\Rightarrow \lim_{n\to\infty} a_n = 0$.

Cauchy Criterion. (Important.) $\sum_{k=1}^{\infty} a_k$ *converges* $\Leftrightarrow$ *given* $\varepsilon > 0$ *there is* N *in* $\mathbb{N}$ *so* $|\sum_{k=m}^{n} a_k| < \varepsilon$, *whenever* $N \leq M < n$.

Results on non-negative series.

We assume, for now, that $a_k \geq 0$ for $k = 1, 2, \dots$

Monotone Convergence. (Underlying fact.) *The sequence* $(S_n)_{n=1}^{\infty}$, $S_n = \sum_{k=1}^{n} a_k$ *is non decreasing. Hence* $\sum_{k=1}^{\infty} a_k$ *converges* $\Leftrightarrow \sup_{n\in\mathbb{N}} S_n < \infty$.

Comparison Test. (Most important.) *Suppose* $b_k \geq 0$ *for each* k *and there is* N *in* $\mathbb{N}$ *so*

$$a_k \leq b_k \text{ for each } k \geq N.$$

(i) $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges; and*

(ii) $\sum_{k=1}^{\infty} a_k$ *diverges* $\Rightarrow \sum_{k=1}^{\infty} b_k$ *diverges.*

Application of Monotone Convergence. Notice that (ii) is the contrapositive of (i).

Limit Comparison Test. (Useful in practice.) *Suppose* $b_k > 0$ *and*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L \text{ exists.}$$

(i) *If* $L > 0$ *then* $\sum_{k=1}^{\infty} b_k$ *converges* $\Leftrightarrow \sum_{k=1}^{\infty} a_k$ *converges; and*

(ii) *If* $L = 0$, *then* $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges.*

Ratio Comparison Test. (Useful in theory.) *Suppose* $a_k, b_k > 0$ *and there is* N *in* $\mathbb{N}$ *so*

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} \text{ for each } k \geq N$$

(i) $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges; and*

(ii) $\sum_{k=1}^{\infty} a_k$ *diverges* $\Rightarrow \sum_{k=1}^{\infty} b_k$ *diverges.*

Ratio Test. (Very Important, easy to compute.) *Suppose* $a_k > 0$ *and*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r \text{ exists.}$$

(i) $r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *converges; and*

(ii) $r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges.*

Really is Ratio Comparison Test against geometric series, $\sum_{k=1}^{\infty} s^k$.

Root Test. (Very Important for power series, harder to compute.) *Let* $R = \limsup_{k\to\infty} \sqrt[k]{a_k}$.

(i) $R < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *converges; and*

(ii) $R > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges.*

Is Comparison Test against geometric series is one direction, nth Term test in the other.

Integral Test. (Very important.) *Suppose that* $f : [1, \infty) \to \mathbb{R}$ *and there is* N *in* $\mathbb{N}$ *so that*

$$f \text{ is integrable on } [1, N], \text{ non-decreasing on } [N, \infty), \text{ and } f(k) = a_k \text{ for } k \geq N.$$

Then $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges* $\Leftrightarrow \displaystyle\int_1^{\infty} f$ *converges.*

This gives important p-series, $\sum_{k=1}^{\infty} \frac{1}{k^p}$.

Raabe's Test. (When ratio test indecisive, and no obvious comparisons in sight.) *Suppose* $a_k > 0$ *and*

$$p = \lim_{k\to\infty} k\left(1 - \frac{a_{k+1}}{a_k}\right) \text{ exists.}$$

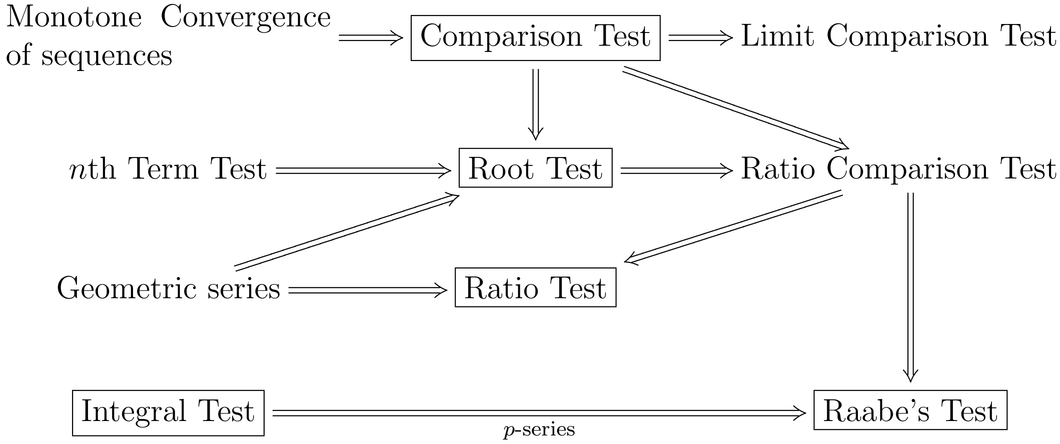
(i) $p > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *converges;*

(ii) $p < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges; and*

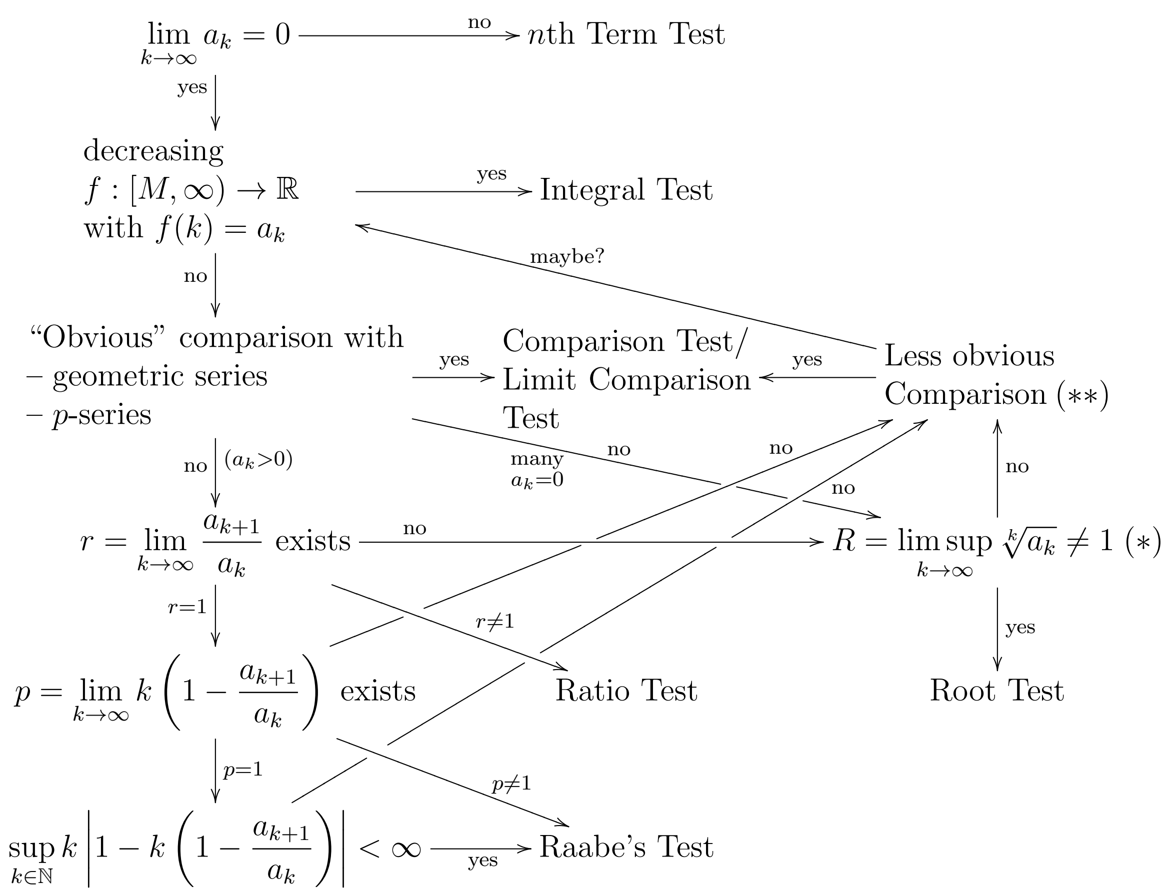
(iii) $p = 1$ *and* $\sup_{k\in\mathbb{N}} k \left|1 - k\left(1 - \frac{a_{k+1}}{a_k}\right)\right| < \infty \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges.*

Really Ratio Comparison Test against p-series. Also succeeds when $p = \infty$.

How results are proved:



An imperfect guide on use of these tests in practice: $a_k \geq 0$



($*$) computation likely difficult; ($**$) possibly really difficult

Results on general series.

Absolute Convergence. $\sum_{k=1}^{\infty} |a_k|$ converges. Can use any test of last section on $|a_k|$.

Cauchy Product. $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ absolutely convergent. Then

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right).$$

May fail if absolute convergence assumption dropped.

Leibniz's Alternating Series Test. *If*

- a_k *eventually decreasing, non-negative:* $a_k \geq a_{k+1} \geq 0$ *when* $k \geq N$
- $\lim_{k\to\infty} a_k = 0$

then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges.*

Raabe's Alternating Series Test. *If* $a_k > 0$ *and*

$$p = \lim_{k\to\infty} k \left(1 - \frac{a_{k+1}}{a_k} \right) \text{ exists}$$

(i) $p > 1 \Rightarrow \sum_{k=1}^{\infty} (-1)^k a_k$ *converges absolutely;*

(ii) $0 < p \leq 1 \Rightarrow \sum_{k=1}^{\infty} (-1)^k a_k$ *converges (conditionally if* $p < 0$*); and*

(iii) $0 < p \Rightarrow \sum_{k=1}^{\infty} (-1)^k a_k$ *diverges.*

Case $p = 1$ occurs sometimes with absolute convergence; case $p = 0$ must be inspected by other means.