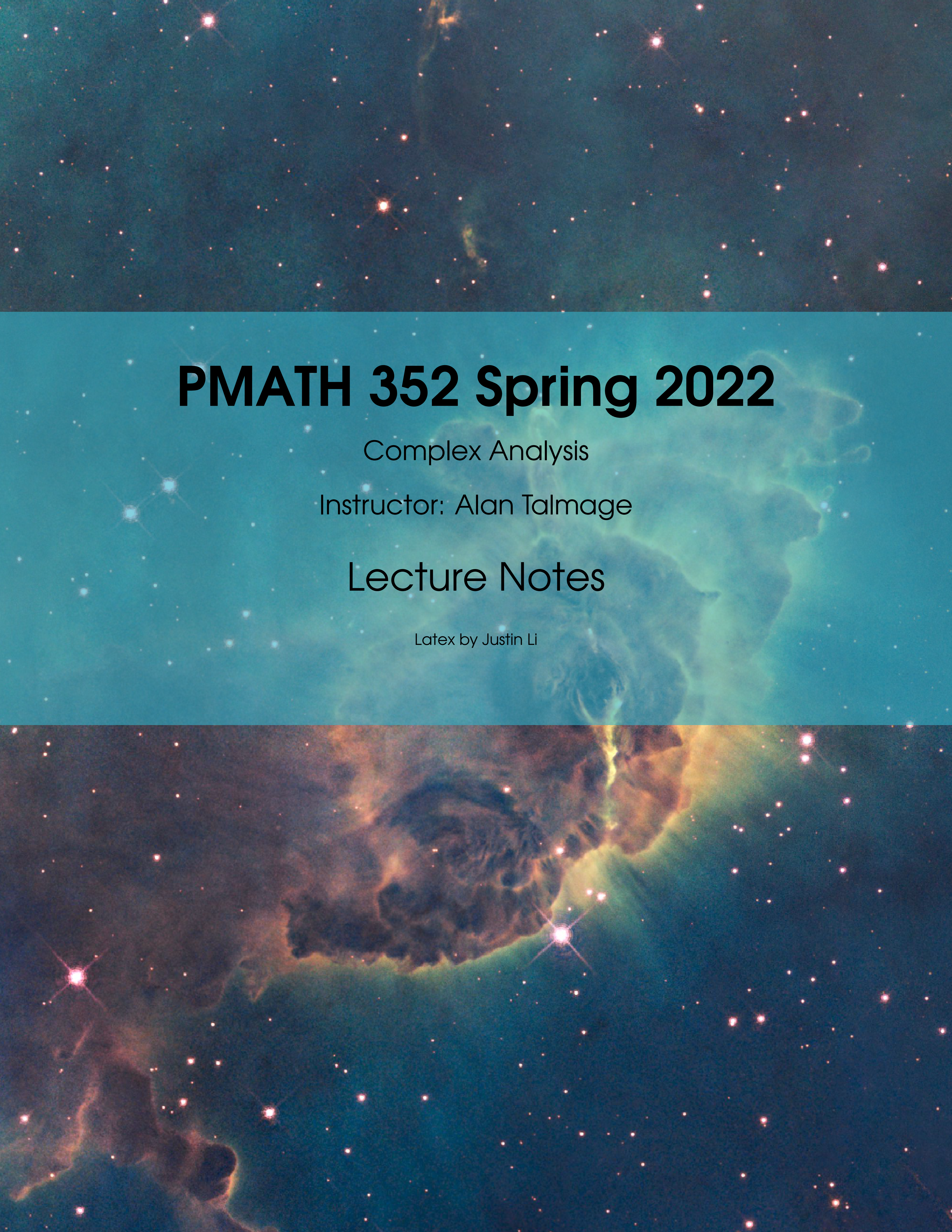

PMATH 352 Spring 2022

Complex Analysis

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Lecture Notes

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Contents

1. Complex Number and Functions

1.1 Complex Number

Definition 1.1.1 — Complex Number.
The **complex number** is defined as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \quad \textbf{where } i^2 = -1$$

Note: There is no prior distinction between i and $-i$, then all behavior in $\mathbb{C}$ should be invariant under the map $i \longleftrightarrow -i$

Question: Is this a good definition?
For any $a, b, c, d \in \mathbb{R}$, we have

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \textbf{closed under addition}$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i \quad \textbf{closed under multiplication}$$

For $a, b, c, d \in \mathbb{R}$ with $c, d \neq 0$, we have

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i \quad \textbf{closed under division}$$

The complex number are closed under its operations, then it's a good defintion.

Definition 1.1.2 — Conjugation.
The **conjugation** in $\mathbb{C}$ is defined as

$$\overline{a + bi} = a - bi$$

where $a, b \in \mathbb{R}$.

Moreover, we have

$$(a+bi)(\overline{a+bi}) = (a+bi)(a-bi) = a^2+b^2$$

■ **Remark 1.1** There is a conical bijection between $\mathbb{C}$ and $\mathbb{R}^2$, that is

$$a+bi \longleftrightarrow (a,b)$$

we usually can write it as $\mathbb{C} \cong \mathbb{R}^2$

Definition 1.1.3 — Norm. The **norm** in $\mathbb{C}$ is defined as

$$|a+bi| = \sqrt{(a+bi)(\overline{a+bi})} = \sqrt{a^2+b^2}$$

■ **Remark 1.2** The i^n for $n \in \mathbb{N}$ has a cycle 4, that is

$$\underbrace{i=i \quad i^2=-1 \quad i^3=-i \quad i^4=1}_{cycle 4} \quad i^5=i \quad ....$$

■ **Remark 1.3** The polar coordinate is defined differently in $\mathbb{C}$ and $\mathbb{R}^2$.
For $(x,y) \in \mathbb{R}^2$ we have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2} = \|(x,y)\| \qquad \theta = \arctan\frac{y}{x}$$

For $a+bi \in \mathbb{C}$ we have

$$a = r\cos\theta \qquad b = r\sin\theta \qquad r = \sqrt{a^2+b^2} = |a+bi| \qquad \theta = \arctan\frac{b}{a}$$

Behaviors in $\mathbb{R}$ and $\mathbb{C}$ function operation:
Let $x \in \mathbb{R}$ and $z = a+bi$, then

$$x^3+2x+1 \qquad \Longrightarrow \qquad z^3+2z+1$$

For $z, w \in \mathbb{C}$, we insist that $\frac{d}{d}e^z = e^z$ and $e^{w+z} = e^w \cdot e^z$, then we have

$$e^x \qquad \Longrightarrow \qquad e^z = e^{a+bi} = e^a \cdot e^{bi}$$

Note that i is a constant, so for $y \in \mathbb{R}$ we have

$$\frac{d}{d(iy)}e^{iy} = \frac{1}{i} \cdot \underbrace{\frac{d}{dy}e^{iy}}_{=ie^{iy}} = e^{iy} \qquad \textbf{and} \qquad \frac{d^2}{dy^2}e^{iy} = \frac{d}{dy}(ie^{iy}) = i \cdot \frac{d}{dy}e^{iy} = i^2 e^{iy} = -e^{iy}$$

Therefore, the $f(y) = e^{iy}$ satisfies $\frac{d^2}{dy^2} = -f$, then by **ODE** the f must be:

$$f(y) = A\sin y + B\cos y \qquad \textbf{and} \qquad f'(y) = A\cos y - B\sin y$$

but we can see that $f(0) = B = e^{i0} = 1$, $f'(0) = ie^{i0} = i$. This gives us that

$$f(y) = e^{iy} = \cos y + i\sin y$$

Therefore, we have

$$e^{a+bi} = e^a e^{bi} = e^a(\cos b + i\sin b) = e^a\cos b + ie^a\sin b$$

1.2 Basic Topology, Limit, Continuity and Differentiability

Definition 1.2.1 — Distance.
The **distance** between two points $w, z \in \mathbb{C}$ is

$$|w-z| \qquad \textbf{(or} \quad |z-w|\textbf{)}$$

Thus $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic as metric spaces.

Definition 1.2.2 — Open Set and Closed Set.
An **open set** $S \subseteq \mathbb{C}$ is a set such that for every $z \in S$, there exists $\varepsilon > 0$ s.t.

$$|z-w| < \varepsilon \qquad \Longrightarrow \qquad w \in S$$

A set $S \subseteq \mathbb{C}$ is **closed** if $\mathbb{C} \setminus S$ is open.

Definition 1.2.3 — Limit.
Let $f : \mathbb{C} \to \mathbb{C}$, we say

$$\lim_{z \to w} f(z) = L$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|z-w| < \delta$, then

$$|f(z) - f(w)| < \varepsilon$$

■ **Example 1.1** Calculate the following:

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

Let's try to approach in different directions.
Let $z = x \in \mathbb{R}$, then

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{\bar{x}}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

Let $z = iy$ for $y \in \mathbb{R}$, then we have

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{\overline{iy}}{iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1$$

so the limit does not exist. ■

Definition 1.2.4 — Continuity.
A function is **continuous at point** z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A function is **continuous on a set** S if it's continuous at each point of S

■ **Example 1.2** Let $f : \mathbb{C} \to \mathbb{C}$ where $f(z) = z^2$. Consider $\Delta z = z - z_0$, then

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} z^2 = \lim_{\Delta z \to 0} (z_0 + \Delta z)^2 = \lim_{\Delta z \to 0} z_0^2 + 2z_0 \Delta z + \Delta z^2 = z_0^2$$

so f is continuous everywhere. ■

Definition 1.2.5 — Correctness, Path-Correctness and Domain.
A set S is **connected** if S there is no open sets S_1, S_2 with $S_1 \cap S_2 = \emptyset$ such that

$$S \subseteq S_1 \cup S_2$$

A set S is **path-connected** if $\forall z_1, z_2 \in S$, there exists a path from z_1 to z_2 lying in S where a **path** is the image of $[0,1]$ under a continuous

A **domain** is a path-connected open set.

■ **Example 1.3** Consider a set $S \subseteq \mathbb{R}^2$ where

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0 \right\} \cup \{(0,y) : y \in \mathbb{R}\}$$

is a connected set. ■

Proposition 1.2.1
Path connected $\implies$ connected

Proof: Let X be a path-connected set and $x_0 \in X$. For each $y \in X$, find a continuous map $f : [0,1] \to X$ such that

$$f(0) = x_0 \quad \textbf{and} \quad f(1) = y$$

Since an interval is connected and the image of continuous map preserve correctness, then $f([0,1])$ is connected. Therefore, y belongs to the largest connected set that contains x_0, so X is connected.

Definition 1.2.6 — Differentiability.
Let $f : \mathbb{C} \to \mathbb{C}$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \textbf{exists}$$

we say that f is differentiable at z_0 and that the limit is its derivative $f'(z_0)$

■ **Example 1.4** Let $f(z) = z^2$, then we have

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0 \Delta z + \Delta z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z_0 + \Delta z = 2z_0$$

■

■ **Example 1.5** Let $f(z) = |z|$, then we have

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z| - |z_0|}{\Delta z}$$

Let's consider $g(a,b) = \sqrt{a^2+b^2}$, then we have

$$\frac{\partial g}{\partial a} = \frac{a}{\sqrt{a^2+b^2}} \quad (*) \qquad \textbf{and} \qquad \frac{\partial g}{\partial b} = \frac{b}{\sqrt{a^2+b^2}} \quad (**)$$

Now we try $\Delta z = x \in \mathbb{R}$ and $z_0 = a + bi$ where $a, b \in \mathbb{R}$, then

$$\lim_{x \to 0} \frac{|x+a+bi| - |a+bi|}{x} = \lim_{x \to 0} \frac{\sqrt{(a+x)^2+b^2} - \sqrt{a^2+b^2}}{x} = \frac{a}{\sqrt{a^2+b^2}} \qquad \textbf{by (*)}$$

Similarly, if we let $\Delta z = yi$ for $y \in \mathbb{R}$, then by **(**)** we have

$$\lim_{y \to 0} \frac{\sqrt{a^2+(b+y)^2} - \sqrt{a^2+b^2}}{yi} = \frac{1}{i}\frac{\partial g}{\partial y} = \frac{1}{i}\frac{b}{\sqrt{a^2+b^2}} = -\frac{bi}{\sqrt{a^2+b^2}}$$

Therefore, $f(z) = |z|$ is **differentiable nowhere** ■

Proposition 1.2.2
Let $f, g : \mathbb{C} \to \mathbb{C}$ and $c \in \mathbb{C}$, then we have

$$(f+g)(z)' = f'(z) + g'(z) \qquad (cf)'(z) = cf'(z)$$

$$(fg)'(z) = f(z)g'(z) + f'(z)g(z) \qquad (f \circ g)'(z) = f'(g(z))g'(z)$$

Definition 1.2.7 — Real Part and Imaginary Part.
Let $z \in \mathbb{C}$ with $z = a + bi$ where $a, b \in \mathbb{R}$, then a and b are call **real and imaginary parts** of z respectively denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

■ **Example 1.6** Let $f(z) = \mathrm{Re}(z)$, what's the differentiability of f?
We look at

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Let $h = h_x \in \mathbb{R}$, then

$$\lim_{h_x \to 0} \frac{\mathrm{Re}(a + h_x + bi) - \mathrm{Re}(a+bi)}{h_x} = \lim_{h_x \to 0} \frac{a + h_x - a}{h_x} = 1$$

Let $h = ih_y$ for $h_y \in \mathbb{R}$, then

$$\lim_{h_y \to 0} \frac{\mathrm{Re}(a + ih_y + bi) - \mathrm{Re}(a+bi)}{ih_y} = \lim_{h_y \to 0} \frac{a - a}{ih_y} = 0$$

Therefore, $f(z) = \mathrm{Re}(z)$ is differentiable nowhere. Similarly, $\mathrm{Im}(z)$ is differentiable nowhere. ■

Remark 1.4

$$\operatorname{Im}(z) = \operatorname{Re}(-iz) \qquad \textbf{and} \qquad \bar{z} = \operatorname{Re}(z) - i\operatorname{Im}(z)$$

differentiable nowhere.

Intuition: Differentiable functions are those that acts on z and are blind to $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$

Example 1.7 The function $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = \bar{z}$ is a reflection by real-axis ■

Example 1.8 The function $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2$, if we write $z = a + bi$, then

$$f(z) = z^2 = (a+bi)^2 = (a^2 - b^2) + 2abi$$

Recall $z = a + bi = re^{i\theta}$, so let $z = re^{i\theta}$, then

$$f(z) = z^2 = r^2 e^{i(2\theta)}$$

■

Question: What is $i^{\frac{1}{2}}$?

$$i = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i \qquad \Longrightarrow \qquad i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

1.3 Modulus and Argument

Definition 1.3.1 — Modulus.
Let $z = re^{i\theta}$ with $0 \le r \in \mathbb{R}$ and $\theta \in \mathbb{R}$. The **modulus** (or **magnitude** or **absolute value**) of z is $r = |z|$

Definition 1.3.2 — Argument.
Let $z = re^{i\theta}$ with $0 \le r \in \mathbb{R}$ and $\theta \in \mathbb{R}$. The **argument** of z is $\theta = \arg(z)$

Remark 1.5 The argument of z is **not** unique. That's because

$$i = e^{i\frac{\pi}{2}} = e^{i(\frac{\pi}{2}+2\pi)} = e^{i(\frac{\pi}{2}+4\pi)} = ... = e^{i(\frac{\pi}{2}+n\pi)} \quad \textbf{for } n \in \mathbb{Z}$$

We also note that

$$i = e^{i(\frac{\pi}{2}+2\pi)} \qquad \Longrightarrow \qquad i^{\frac{1}{2}} \in \left\{ e^{i(\frac{\pi}{2}+2\pi)}, e^{i(\frac{\pi}{2}+4\pi)} \right\} = \left\{ e^{\frac{i\pi}{4}}, e^{\frac{i5\pi}{4}} \right\}$$

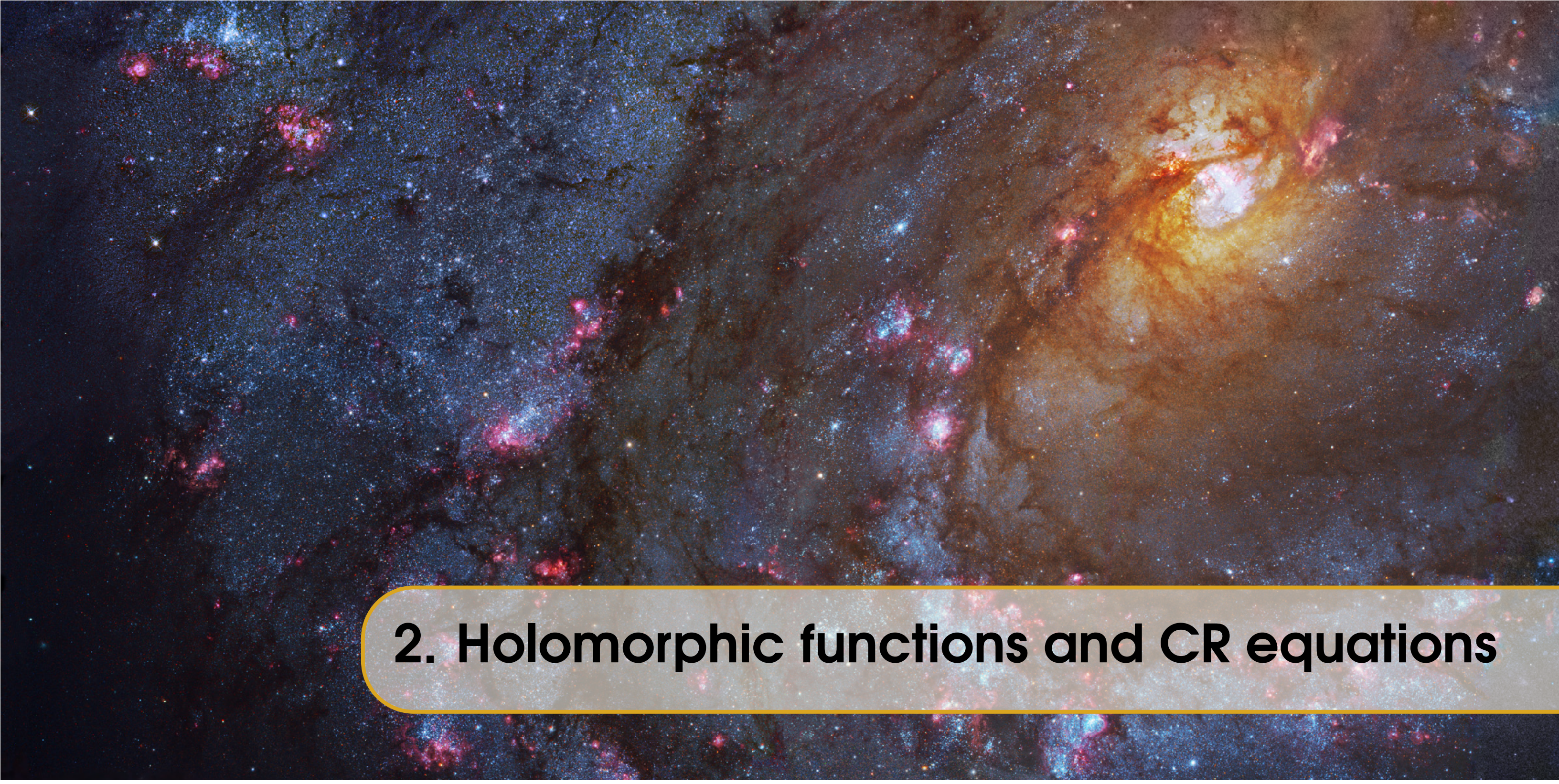
Proposition 1.3.1

More generally, if $n > 0$ and $n \in \mathbb{Z}$, $z = re^{i\theta}$, then

$$z^{\frac{1}{n}} = (re^{i\theta})^{\frac{1}{n}} = \left\{ r^{\frac{1}{n}} e^{i\frac{\theta}{n}}, r^{\frac{1}{n}} e^{i\frac{\theta+2\pi}{n}}, r^{\frac{1}{n}} e^{i\frac{\theta+4\pi}{n}}, \ldots., r^{\frac{1}{n}} e^{i\frac{\theta+2(n-1)\pi}{n}} \right\}$$

so any non-zero $z \in \mathbb{C}$, has **exactly** n distinct n^{th} roots:

$$r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} \qquad \textbf{where } 0 \leq k < n$$



2. Holomorphic functions and CR equations

2.1 Holomorphic Function

Definition 2.1.1 — Holomorphic.
If $f : \mathbb{C} \to \mathbb{C}$ is differentiable on a domain D, we say f is **holomorphic** on D. Also called (complex) analytic regular.

Sloppy terminology warning: A function is said holomorphic at point z_0 if it is holomorphic on some open set containing z_0

Proposition 2.1.1
Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic on a domain D and let $z \in D$, then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \textbf{extsts}$$

Consider $h = h_x \in \mathbb{R}$, then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h_x) - f(z)}{h_x}$$

and let $z = x + iy$, then

$$f'(z) = \lim_{h \to 0} \frac{f(x+h_x+iy) - f(x+iy)}{h_x}$$

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then

$$\begin{aligned} f'(x+iy) &= \lim_{h_x \to 0} \frac{u(x+h_x,y) + iv(x+h_x,y) - u(x,y) - iv(x,y)}{h_x} \\ &= \lim_{h_x \to 0} \frac{u(x+h_x,y) - u(x,y)}{h_x} + i \cdot \lim_{h_x \to 0} \frac{v(x+h_x,y) - v(x,y)}{h_x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= f'(x+iy) \end{aligned}$$

Now let $h = ih_y$ where $h_y \in \mathbb{R}$, then

$$\begin{aligned} f'(x+iy) &= \lim_{h_y \to 0} \frac{u(x,y+h_y) + iv(x,y+h_y) - u(x,y) - iv(x,y)}{ih_y} \\ &= \lim_{h_y \to 0} \frac{u(x,y+h_y) - u(x,y)}{ih_y} + \cdot \lim_{h_y \to 0} \frac{iv(x,y+h_y) + iv(x,y)}{ih_y} \\ &= \frac{1}{i} \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

Therefore, we have

$$f'(x+iy) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \Longrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textbf{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the **Cauchy-Riemann equations**. If $f = u + iv$ is holomorphic on D, then u and v satisfy the **CR** equations on D.

■ **Example 2.1** Let f be holomorphic on a domain D and let $v(x,y) = \text{Im}(f) = xy$ on D, find $u(x,y)$. Let $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = x \quad \textbf{and} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = y$$

so that

$$\frac{\partial u}{\partial x} = x \qquad \frac{\partial u}{\partial y} = -y \qquad \Longrightarrow \qquad u = \frac{1}{2}x^2 + C_1(y) \qquad u = -\frac{1}{2}y^2 + C_2(x)$$

This gives us that

$$u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + C$$

so we have

$$f(x+iy) = \left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + C\right) + xyi$$

Note that

$$\frac{(x+iy)^2}{2} = \frac{x^2 - y^2 + 2xyi}{2} = \left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + C\right) + xyi$$

then

$$f(z) = \frac{z^2}{2} + C \quad \textbf{and} \quad f'(z) = z$$

where $z = x + iy$ ■

■ **Example 2.2** Let f be holomorphic on a domain D and let $\text{Re}(f) = x^2y$. Let $z = x + iy$ and $f(x+iy) = u(x,y) + iv(x,y)$, $u(x,y) = x^2y$. Then we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2xy \quad \Longrightarrow \quad v = \int 2xydy = xy^2 + \underbrace{C_1(x)}_{\text{cause contradiction}}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = x^2 \quad \Longrightarrow \quad v = \int x^2 dx = \frac{1}{3}x^3 + \underbrace{C_2(y)}_{\text{cause contradiction}}$$

this implies that there is no such v exists, so this is a contradiction, there is no such function f. ■

Proposition 2.1.2

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

■ **Example 2.3** Let $f(z) = e^z$, so consider

$$f(x+iy) = e^x e^{iy} = e^x \cdot (\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y)$$

Then

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -e^x\sin(y) = \frac{\partial v}{\partial x}$$

■

Theorem 2.1.3 Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ has continuous particle derivatives at (x_0, y_0) which satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0). Then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic at $z_0 = x_0 + iy_0$

Proof: Let $D \subseteq \mathbb{C}$ be domain with $z_0 \in D$. Let $z = x + iy \in D$. Then

$$u(x,y) = u(x_0,y_0) + (x - x_0)\left(\frac{\partial u}{\partial x}(x_0,y_0) - \varepsilon_1(x,y)\right) + (y - y_0)\left(\frac{\partial u}{\partial y}(x_0,y_0) - \varepsilon_2(x,y)\right)$$

where $\varepsilon_1, \varepsilon_2$ are continuous at (x_0, y_0) and $\varepsilon_1(x_0,y_0) = \varepsilon_2(x_0,y_0) = 0$. Similarly, we have

$$v(x,y) = v(x_0,y_0) + (x - x_0)\left(\frac{\partial v}{\partial x}(x_0,y_0) - \varepsilon_3(x,y)\right) + (y - y_0)\left(\frac{\partial v}{\partial y}(x_0,y_0) - \varepsilon_4(x,y)\right)$$

so

$$f(x+iy) = u(x,y) + iv(x,y) = f(z_0) + (z - z_0)\left(\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon(z)\right)$$

where

$$\varepsilon(z) = \frac{x - x_0}{z - z_0}(\varepsilon_1 + i\varepsilon_3) + \frac{y - y_0}{z - z_0}(\varepsilon_2 + i\varepsilon_4)$$

satisfying ε continuous at z_0 with $\varepsilon(z_0) \to 0$. Then

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon(z) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = f'(z_0)$$

■ **Example 2.4** Let $f(z) = \bar{z}$ so $f(x+iy) = x - iy$ so $u = x$ and $v = -y$. Note that

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial y}$$

This holds nowhere, so f is holomorphic **nowhere** ■

■ **Example 2.5** Let $f(z) = \frac{1}{z}$ so

$$f(x+iy) = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\left(\frac{y}{x^2+y^2}\right)$$

Then we have

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad -\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

so f is holomorphic everywhere except $\{0\}$

Note: $f(z) = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$ **(*)** ■

Proposition 2.1.4 Let $f, g : \mathbb{C} \to \mathbb{C}$ holomorphic at z_0 with $g(z_0) \neq 0$, then

$$\frac{f}{g} \quad \textbf{is holomorphic at } z_0$$

The converse is false because of the Example 1.13, so let f be holomorphic, h is not holomorphic, then $\frac{fh}{h}$ is holomorphic at z while $h(z) \neq 0$.

■ **Example 2.6** Let $f(z) = \frac{1}{z}$ and $z = re^{i\theta}$, then

$$f(re^{i\theta}) = \frac{1}{re^{\theta}} = \frac{1}{r}e^{-i\theta}$$

Let $z = re^{i\theta}$, $z_0 = r_0e^{i\theta_0}$ and $f(z)$ be holomorphic and let $f(z) = u(r,\theta) + iv(r,\theta)$ we have

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

First fix $\theta = \theta_0$ and $r \to r_0$, then

$$\begin{aligned} f'(re^{i\theta_0}) &= \lim_{r\to r_0} \frac{u(r,\theta_0) + iv(r,\theta_0) - u(r_0,\theta_0) - iv(r_0,\theta_0)}{re^{i\theta_0} - r_0e^{i\theta_0}} \\ &= e^{-i\theta_0} \lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r-r_0} + i\frac{v(r,\theta_0) - v(r_0,\theta_0)}{r-r_0} \\ &= e^{-i\theta_0} \cdot \left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right) \end{aligned}$$

Next let $r = r_0$ and $\theta \to \theta_0$, so similarly we have

$$
\begin{aligned}
f'(r_0 e^{i\theta}) &= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left[\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + i \frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta - \theta_0} \right] \left(\frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} \right) \\
&= \frac{1}{r} \frac{1}{ie^{i\theta}} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) \\
&= \frac{1}{r} e^{-i\theta} \left(\frac{\partial v}{\partial \theta} - i \frac{\partial u}{\partial \theta} \right) \\
&= f'(re^{i\theta}) \\
&= e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)
\end{aligned}
$$

Therefore, we have

$$
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}
$$

Polar Form of CR-Equation ■

■ **Example 2.7** Consider

$$
z^{\frac{1}{n}} = r^{\frac{1}{n}} \cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)
$$

so we have

$$
u(r,\theta) = r^{\frac{1}{n}} \cos\left(\frac{\theta}{n}\right) \qquad v(r,\theta) = \sin\left(\frac{\theta}{n}\right)
$$

so that

$$
\frac{\partial u}{\partial r} = \frac{1}{n} r^{\frac{1}{n}-1} \cos\left(\frac{\theta}{n}\right) = \frac{1}{r} \frac{\partial v}{\partial \theta}
$$

and

$$
\frac{\partial v}{\partial r} = \frac{1}{n} r^{\frac{1}{n}-1} \sin\left(\frac{\theta}{n}\right) = -\frac{1}{r} \frac{\partial u}{\partial \theta}
$$

■

Definition 2.1.2 — Branch and Principal Branch.
Let's define
$$\text{Arg}(re^{i\theta}) = \theta \qquad -\pi \leq \theta \leq \pi$$
This is a **branch** of the multvalued function arg. In particular, Arg is called the **principal branch**

Other breanches:
$$f(re^{i\theta}) = \theta \qquad \pi < \theta \leq 3\pi = \text{Arg}(re^{i\theta}) + 2\pi$$
so we have the value
$$\text{Arg}(z) + 2\pi m \quad \textbf{for} \quad m \in \mathbb{Z}$$
is a **branch** of arg(z)

Consider the branches $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$ for $-\pi < \theta\pi$
Question: Where is this function holomorphic?
Answer: This function is not continuous along $\text{Arg}(z) = \pi$, but it's holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, this is called **branch cut**

■ **Example 2.8** The function $f(re^{i\theta}) = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$ for $-\pi < \theta \leq \pi$ has a branch cut along $\mathbb{R}_{\leq 0}$ but holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

We use branches in $\mathbb{R}$ as well: $\sqrt{4} = 2$ because $\sqrt{\cdot}$ means principal branch of the function $x^{\frac{1}{n}}$

We can put a branch cut somewhere else:
$$f(re^{i\theta}) = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \quad \textbf{for} \quad 0 < \theta < 2\pi$$
■

■ **Remark 2.1** Some times we can define a branch cut that is not a straight line.

■ **Example 2.9** Let's consider the log function, that is
$$\log(z) = \log\left(re^{i\theta}\right) = \log(r) + \log\left(e^{i\theta}\right) = \log(r) + i\theta = \ln|z| + i\text{Arg}(\theta) + 2\pi im = \ln|z| + i\text{Arg}(z) + 2\pi im$$
for $m \in \mathbb{Z}$. Use log for the multivalued function and log for the principal branch. That is
$$\log(z) = \log|z| + i\text{Arg}(z) = \log|z| + 2\pi im$$
for $m \in \mathbb{Z}$ ■

■ **Example 2.10** Let's consider $\log\left(z^{\frac{1}{2}}\right)$, so we have
$$\log\left(z^{\frac{1}{2}}\right) = \log\left(r^{\frac{1}{2}} e^{i\frac{\theta}{2}}\right) = \ln\left|r^{\frac{1}{2}}\right| + i\frac{\theta}{2} = \frac{1}{2}(\ln|z| + i\theta) = \frac{1}{2}\log(z)$$
■

■ **Remark 2.2** Note that for $w,z \in \mathbb{C}$:

$$\log(wz) \neq \log(w) + \log(z)$$

in general. We can see that

$$\log\left(e^{\frac{2\pi i}{3}} e^{\frac{2\pi i}{3}}\right) = -\frac{2\pi i}{3} \neq \frac{4\pi}{3} = \frac{2\pi i}{3} + \frac{2\pi i}{3}$$

That is because of the branches we use.

■ **Remark 2.3**

$$\cos(y) = \frac{e^{iy}+e^{-iy}}{2} \qquad \sin(y) = \frac{e^{iy}-e^{-iy}}{2i}$$

are holomorphic.
Moreover

$$\frac{d}{dz}\sin(z) = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$$

and

$$\frac{d}{dz}\cos(z) = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{-e^{iz}+e^{-iz}}{2} = -\sin(z)$$

■ **Remark 2.4** If two holomorphic funcions are equal on **"enough"** of a set, they must agree on their domains.

Definition 2.1.3 — Tangent Function.

$$\tan(z) = \frac{1}{2} \cdot \frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$$

and all trig identities carry over in the obvious way.

■ **Remark 2.5** **Trig function are not bounded**. Consider $\cos(iy)$ for $y \in \mathbb{R}$, then

$$|\cos(iy)| = \left|\frac{e^{i(iy)}+e^{-i(iy)}}{2}\right|$$

so we have $|\cos(iy)| \to \infty$ as $y \to \pm\infty$. The sin function is similar. Therefore, cos, sin are unbounded in $\mathbb{C}$.

■ **Remark 2.6**

$$\sinh(z) = \frac{e^{z}-e^{-z}}{2} \qquad \cosh(z) = \frac{e^{z}+e^{-z}}{2} \qquad \textbf{and} \qquad \sinh(iz) = i\sin(z) \qquad \cosh(iz) = \cos(z)$$

Question: What is i^i?
Since $\log(i) = \log e^{i\frac{\pi}{2}} = \frac{i\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$, then

$$i^i = \left(e^{\log(i)}\right)^i = e^{i\log(i)} = e^{i\cdot\left(\frac{i\pi}{2}+2\pi k\right)} = e^{-\frac{\pi}{2}-2\pi k}$$

Define

$$z^2 = e^{w\log(z)} = \frac{d}{dz}(z^w) = \frac{d}{dz}e^{w\log(z)} = e^{w\log(z)}\frac{d}{dz}(w\log(z) = w\cdot\frac{1}{z}\cdot e^{w\log(z)} = \frac{w}{z}\cdot z^w = w\cdot z^{w-1}$$

as expected.

Question: How many values does z^w have?
For $k \in \mathbb{Z}$ we have

$$z^w = e^{w\log(z)} = e^{w(\log(z)+2\pi ik)} = e^{w\log(z)} \cdot e^{2\pi ikw}$$

When is $e^{2\pi ikw} = e^{2\pi inw}$?
Then are equal when for some $n \in \mathbb{Z}$

$$e^{2\pi ikw} = e^{2\pi inw + 2\pi in}$$

Consider $e^z = 1$, then $e^z = 1e^{i\cdot 0} = e^{x+iy} = e^{x+iy} = e^x e^{iy}$ implies $e^x = 1$ and $y = 2\pi n$. Now we can see that $kw = nw + m$ for some $m \in \mathbb{Z}$, then

$$w = \frac{m}{k-n}$$

for $n,m,k \in \mathbb{Z}$, so the powers z^w repeats if and only if $w \in \mathbb{Q}$. Now if $w = \frac{p}{q}$, so $z^2 = z^{\frac{p}{q}} = (z^p)^{\frac{1}{q}}$ has q distinct values. If $z \neq 0$, we have

$$z^w = \begin{cases} 1 & \textbf{if } w \in \mathbb{Z} \\ q & \textbf{if } w = \frac{p}{q} \in \mathbb{Q} \\ \infty & \textbf{otherwise} \end{cases}$$

Proposition 2.1.5 — Rotalion Approximation.
Let f be holomorphic at z_0, so that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

The modolus and argument must converge individually.

$$\left|f'(z_0)\right| = \left|\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}\right| \qquad \Longrightarrow \qquad |f(z)-f(z_0)| \approx \left|f'(z_0)\right||z-z_0|$$

and

$$\arg(f'(z_0)) = \lim_{z\to z_0} \arg\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$

for some branch of arg holomorphic near z_0, $f(z_0), f'(z_0)$ so

$$\arg(f'(z_0)) = \arg(f(z)-f(z_0)) - \arg(z-z_0) \qquad \Longrightarrow \qquad \arg(f(z)-f(z_0)) \approx \arg(f'(z_0)) + \arg(z-z_0)$$

near z_0 we have

$$f(x) \approx f(z_0) + e^{i\arg(f'(z_0))}\left|f'(z_0)\right|(z-z_0)$$

this is a roralion of $z-z_0$ by $\arg(f'(z_0))$ and a scaling by $|f'(z_0)|$

■ **Example 2.11** Consider $f(z) = z^2$ so $f(re^{i\theta}) = r^2e^{i2\theta}$. Let $z_0 = 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, so $f(z_0) = 2i$ and $f'(z_0) = 2\sqrt{2}e^{i\frac{\pi}{4}}$ so for small $h = z-z_0$ we have

$$f(z_0+h) \approx f(z_0) + e^{i\arg(f'(z_0))}$$

■

■ **Remark 2.7**
If f is differentiable on an open interval (a,b) and $f'(x) = 0$ on (a,b), then f is constant on (a,b)

Theorem 2.1.6
If f is holomorphic on a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant on D

Proof:

$$f'(z) = 0 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

so u, v are constant on any horizontal or vertical line segment in D, but D is a domain so it's open and path connected. Then any two points in D can be connected by a path of horizontal and vertical line segment, so u and v are constant on D, that means $f = u + iv$ is constant on D.

Example 2.12 Find a branch of $(z^2-1)^{\frac{1}{2}}$ holomorphic on $|z| > 1$
Note that the principal branch of $z^{\frac{1}{2}}$ does not work:

$$e^{\frac{1}{2}\log\left(z^2-1\right)}$$

Its branch cut is where $z^2 - 1 \in \mathbb{R}$ with $z^2 - 1 \leq 0$. But let $z = 2i$, so $z^2 - 1 = -5 \leq 0$.

Consider the principal branch of $f(z) = z(1-\frac{1}{z^2})^{\frac{1}{2}}$, its branch cut lies wherever $1 - \frac{1}{z^2} \leq 0$ in $\mathbb{R}$, which is $\frac{1}{z^2} \geq 1$ in $\mathbb{R} \iff z^2 \leq 1$ in $\mathbb{R} \implies |z| < 1$ ■

2.2 Smooth Curve

Definition 2.2.1 — Smooth Curve.
A **smooth curve** in $\mathbb{C}$ is the image of the function $r : [a,b] \to \mathbb{C}$ satisfying:

1. r is continuous and differentiable on $[a,b]$
2. $r' \neq 0$ on $[a,b]$
3. r is one to one

Remark 2.8 The definition of **smooth curve** results **gaps, sharp corners, pausing, retracing and self-intersection** are not **smooth curve**.

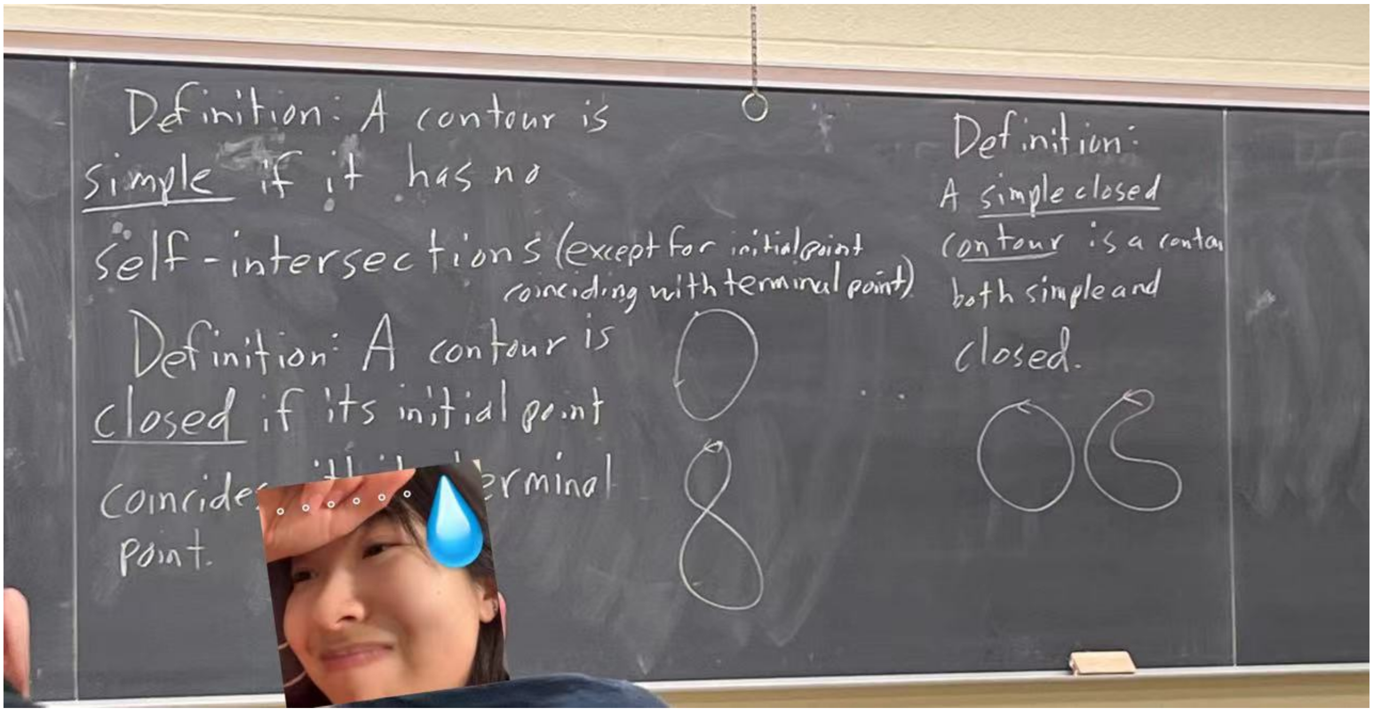
Definition 2.2.2 — Directed Smooth Curve.
A **directed smooth curve** is a smooth curve with a fixed direction. i.e. The points on the curve are ordered and and r must trace them in order.

Definition 2.2.3 — Contour.
A **contour** is a directed piecewise smooth curve. i.e. $\Gamma = \bigcup_{i=1}^{n} C_i$ where each C_i is directed smooth curve and the terminal point of C_i is the initial point of C_{i+1}.

Definition 2.2.4
A contour is **simple** if it has no self-intersection

Definition 2.2.5
A contour is **closed** if its initial point coincides with its terminal point.

Definition 2.2.6 A **simple closed contour** is a contour both simple and closed.



■ **Example 2.13** Let $\Gamma : r_1(t) = z_0t + z_1(1-t)$ for $t \in [0,1]$ where z_0, z_1 are constants.
Note: Parametrizations are **not** unique. That means we have

$$r_2(t) = z_0(2t) + z_1(1-2t) \quad \textbf{for } t \in [0, \frac{1}{2}] \qquad r_3(t) = z_0t^2 + z_1(1-t^2) \quad \textbf{for } t \in [0,1]$$

■

■ **Example 2.14** Consider $\Gamma : r(t) = R \cdot e^{it} + z_0$ for $t \in [0, 2\pi]$ ■

Definition 2.2.7 If $C_r(z_0) = r \cdot e^{it} + z_0$ for $t \in [0, 2\pi]$ is the circular contour with radius r and center z_0, traversed counterclockwise

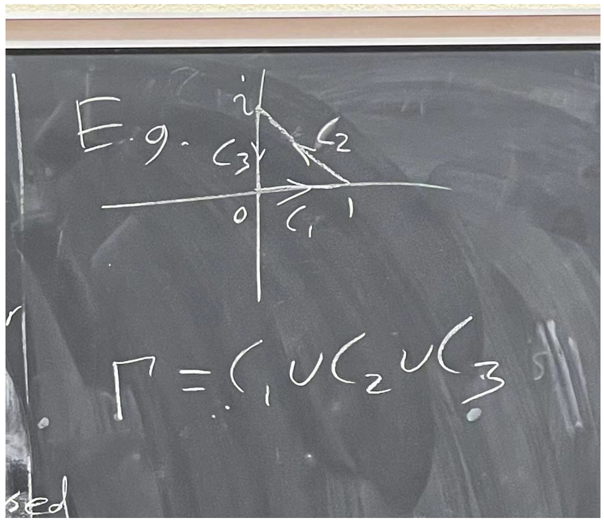
■ **Example 2.15** Consider $\Gamma = C_1 \cup C_2 \cup C_3$ where

$$C_1 : \ r_1(t) = t \qquad C_2 : \ r_2(t) = ti + (1 = t) \qquad C_3 : \ r_3(t) = (1-t)i$$

where $t \in [0,1]$ so we can parametrize Γ by

$$r(t) = \begin{cases} r_1 & t \in [0,1] \\ r_2 & t \in [0,1] \\ r_3 & t \in [0,1] \end{cases}$$

■



Theorem 2.2.1 — Jordan Curve Theorem.
A simple closed contour divides $\mathbb{C}$ into two disjoint regions, a bounded interior and an unbounded exterior.

Definition 2.2.8
A simple closed contour is **positively oriented** if its interior is to the left when traversed, **negatively oriented** clockwise otherwise.

3. Integration and Series

3.1 Integration

Definition 3.1.1 — Partition. Let Γ be a directed with initial point w_0 and terminal point w_1. A **partition** of Γ is a set of points where

$$w_0 = z_0,\ z_1, z_2, \ldots, z_n = w_1$$

such that for all $0 \leq i < n$, z_{i+1} is further along Γ than z_i.

Definition 3.1.2 — Mesh.
The **mesh** of a partition is the largest distance between two consecutive points z_i, z_{i+1} along Γ

Definition 3.1.3 — Riemann Sum.
Let Γ lie on a domain D and let $f : D \to \mathbb{C}$. The **Riemann sum** of f with respect to P_n is

$$S_f(P_n) = \sum_{i=1}^{n} f(z_i) \cdot (z_i - z_{i-1})$$

Definition 3.1.4
f is **integrable** along Γ if

$$\lim_{\textbf{mesh}(P_n) \to 0} S(P_n) \quad \textbf{exists}$$

Definition 3.1.5 — integral.
If f is integrable along Γ the **integral** of f along Γ is

$$\int_\Gamma f = \lim_{\textbf{mesh}(P_n) \to 0} S(P_n)$$

■ **Remark 3.1** This definition does not reference a parametrization of Γ, thus the integral is independent of the choice of parametrization of the curve.

Let Γ parametrized by $r : [a,b] \to \Gamma$, then

$$\lim_{\mathbf{mesh}(P_n)\to 0} \sum_{i=0}^{n-1} f(z_i)(z_{i+1} - z_i)$$

Let $t_0, t_1, ..., t_n$ be the partition of $[a,b]$ s.t. $r(t_i) = z_i$ and $0 \le i \le n$. This gives us

$$\lim_{\mathbf{mesh}(P_n)\to 0} \sum_{i=0}^{n-1} f(r(t_i))\Delta z_i$$

where $\Delta t_i = t_{i+1} - t_i$ and $\Delta z_i = z_{i+1} - z_i$, so that

$$\lim_{\mathbf{mesh}(P_n)\to 0} \sum_{i=0}^{n-1} f(r(t_i))\Delta z_i = \lim_{\Delta t_i \to 0} \sum_{i=0}^{n-1} f(r(t_i))r'(t_i)\Delta t_i = \int_a^b f(r(t))r'(t)dt$$

That is

$$\int_\Gamma f(z)dz = \int_a^b f(r(t))r'(t)dt \qquad \textbf{very important}$$

■ **Remark 3.2** Let's define the integral over a contour. First we consider

$$\Gamma = \Gamma_1 + \Gamma_2 + ... + \Gamma_n$$

where the Γ_i are smooth directed curves, to be

$$\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + ... + \int_{\Gamma_n} f$$

The contour integral immediately satisfies the followings

$$\int_\Gamma f + g = \int_\Gamma f + \int_\Gamma g \qquad \int_{\Gamma_1+\Gamma_2} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f \qquad \int_\Gamma c \cdot f = c \cdot \int_\Gamma f$$

for some constant $c \in \mathbb{C}$

■ **Example 3.1** Let $\Gamma : r(t) = e^{it}$ where $t \in [0,\pi]$, then by **Remark 1.14** we have

$$\int_\Gamma z dz = \int_0^\pi e^{it}(ie^{it})dt = i\int_0^\pi e^{2it}dt = i\left[\frac{1}{2i}e^{2it}\right]_0^\pi = \frac{1}{2}(e^{2\pi i} - e^0) = 0$$

also we have

$$\int_\Gamma z^2 dz = \int_0^\pi (e^{it})^2(ie^{it})dt = i\int_0^\pi e^{3it}dt = \frac{1}{3}(e^{3i\pi} - e^{3i\cdot 0}) = -\frac{2}{3}$$

■

■ **Example 3.2** Let $C_1(0) = r(t) = e^{it}$ for $t \in [0,2\pi]$, then by **Remark 1.14** we have

$$\int_{C_1(0)} z dz = \int_0^{2\pi} e^{it}(ie^{it})dt = \frac{1}{2}(e^{4\pi i} - e^0) = 0$$

and

$$\int_{C_1(0)} = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = \int_0^{2\pi} idt = 2\pi i$$

■

■ **Example 3.3** Let's define $\Gamma_1 : r_1(t) = t$ with $t \in [0,1]$ and $\Gamma_2 : r_2(t) = 1 + it$ with $t \in [0,1]$, then by **Remark 1.14** we have

$$\begin{aligned}\int_\Gamma z^2 dz &= \int_{\Gamma_1} z^2 dz + \int_{\Gamma_2} z^2 dz \\ &= \int_0^1 t^2(1)dt + \int_0^1 (1+it)^2(i)dt \\ &= -\frac{2}{3} + \frac{2}{3}i\end{aligned}$$

■

■ **Example 3.4** Let $C_1(z_0) : r(t) = z_0 + e^{it}$ for $t \in [0,2\pi]$, then by **Remark 1.14** we have

$$\int_{C_1(z_0)} (z-z_0)^n dz = \int_0^{2\pi} (z_0 + e^{it} - z_0)^n (ie^{it})dt = \int_0^{2\pi} e^{nit} \cdot ie^{it}dt = i\int_0^{2\pi} e^{i(n+i)t}dt$$

by solving the integral we get

$$\int_{C_1(z_0)} (z-z_0)^n dz = i\int_0^{2\pi} e^{i(n+i)t}dt = \begin{cases} \frac{1}{n+1}(e^{2\pi i(n+1)} - e^0) & \textbf{if } n \neq -1 \\ 2\pi i & \textbf{otherwise} \end{cases} = \begin{cases} 0 & \textbf{if } n \neq -1 \\ 2\pi i & \textbf{otherwise} \end{cases}$$

■

Definition 3.1.6 — Length.
The **length** of a contour Γ parametrized by $r : [a,b] \to \Gamma$ with

$$\int_a^b |r'(t)| dt$$

Theorem 3.1.1
Let f be integrable on Γ and $|f(z)| \leq M$ on Γ, then

$$\left|\int_\Gamma f(z)dz\right| = \left|\int_a^b f(r(t))r'(t)dt\right| \leq \int_a^b |f(r(t))r'(t)|dt = \int_a^b |f(r(t))||r'(t)|dt$$

Definition 3.1.7 — Primitive.
A function F is **primitive** (or antiderivative) for a function f on a domain D if F is holomorphic on D and for all $z \in D$ with $F'(z) = f(z)$

Let f have a primitive F on D and Γ lie on D, consider

$$\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz$$

Let $r : [a,b] \to \Gamma$ parametrize Γ, then

$$\int_\Gamma F'(z)dz = \int_a^b F'(r(t))r'(t)dt = \int_a^b \frac{dF}{dr}(r(t))\frac{dr}{dt}dt = \int_{r(a)}^{r(b)} \frac{dF}{dr}dr = F(r(b)) - F(r(a))$$

by the fundamental theorem of calculus in $\mathbb{R}$

Theorem 3.1.2
Fundamental Theorem of Calculus in $\mathbb{C}$: If f has a primitive F on a domain D and Γ lies in D with initial point z_0 and terminal point z, then

$$\int_\Gamma f(z)dz = F(z_1) - F(z_0)$$

■ **Example 3.5** Let $f(z) = z$ so it has primitive $F(z) = \frac{1}{2}z^2$ on all of $\mathbb{C}$. Then for Γ containing from z_0 to z_1,

$$\int_\Gamma zdz = \frac{1}{2}z^2 \,|_{z_0}^{z_1}$$

If $z_1 = 1+i$ and $z_0 = 0$, so we get

$$\int_\Gamma zdz = \frac{1}{2}z^2 \,|_0^{1+i} = i$$

■

■ **Example 3.6** Let $f(z) = \frac{1}{z}$ has a primitive $\log(z)$, that is any branch of $\log(z)$ is a primitive of $\frac{1}{z}$ on its domain. so $\log(z)$ is primitive of $\frac{1}{z}$ on the domain $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then

$$\int_\Gamma \frac{1}{z}dz = \log(i) - \log(-i) = \log\left(e^{\frac{i\pi}{2}}\right) - \log\left(e^{\frac{-i\pi}{2}}\right) = i\pi$$

but for $f(z) = \frac{1}{z}$ has no primitive valid on all of $C_1(0)$ ■

Corollary 3.1.3

If f has a primitive on a domain D and Γ is a closed contour lying in D, then

$$\int_\Gamma f(z)dz = 0$$

Proof: Note that

$$\int_\Gamma f = F(z_1) - F(z_0) = F(z_0) - F(z_0) = 0$$

that is **primitive** implies $\oint f = 0$

Lemma 3.1.4

Let f be continuous on a domain D and let

$$\oint_\Gamma f = 0$$

for any closed Γ lying in D. Then given Γ_1, Γ_2 in D with the same initial and terminal points, then

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f$$

Proof: Note that $\Gamma_1 + (-\Gamma_2)$ is closed, so

$$\int_{\Gamma_1+(-\Gamma_2)} f = 0 = \int_{\Gamma_1} f - \int_{\Gamma_2} f = 0$$

Lemma 3.1.5

Let f be continuous on a domain D such that for Γ_1, Γ_2 in D sharing initial and terminal point

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f$$

then f has a primitive on D

Proof: Fix $z_0 \in D$ and define

$$F(z) = \int_\Gamma f(z)dz$$

where Γ is a contour lying in D with initial point z_0 and terminal point z. This is well-defined by path-independent (and path-connectedness of D). Now consider

$$F'(z) = \lim_{|\Delta z|\to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = \lim_{|\Delta z|\to 0} \frac{\int_E f(z)dz}{\Delta z}$$

where E is the line segment running from z to $z + \Delta z$, that is $E : r(t) = z + t\Delta z$ for $t \in [0,1]$. Then we have

$$F'(z) = \lim_{\Delta z\to 0} \frac{\int_E f(z)dz}{\Delta z} = \lim_{|\Delta z|\to 0} \frac{1}{\Delta z}\int_0^1 f(z+t\Delta z)\Delta z dt = \lim_{|\Delta z|\to 0}\int_0^1 f(z+t\Delta z)dt$$

since f is continuous, so

$$\lim_{\Delta z \to 0} f(z + t\Delta z) = f(z)$$

so for all $\varepsilon > 0$ $\exists \delta > 0$ s.t. as $\Delta z < \delta$, then $f(z + t\Delta z) - f(z) < \varepsilon$. Now we can see that

$$0 \leq \lim_{|\Delta z| \to 0} \int_0^1 f(z + t\Delta z) dt \leq \lim_{|\Delta z| \to 0} \int_0^1 f(z) + \varepsilon dt = \lim_{\varepsilon \to 0} f(z) + \varepsilon = f(z)$$

Theorem 3.1.6
Let f be continuous on a domain D, **TFAE**:

1. f has a primitive on D
2. For all closed contours Γ lying in D, $\int_\Gamma f = 0$
3. For any two contours Γ_1, Γ_2 in D that sharing intial and terminal points, then

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f$$

Definition 3.1.8 — Cauchy Sequence.
A **Cauchy sequence** is a sequence $\{z_n\}_{n=1}^{\infty}$ such that $\forall \varepsilon > 0$ $\exists N > 0$ such that $n_1, n_2 > N$

$$|z_{n_1} - z_{n_2}| < \varepsilon$$

Lemma 3.1.7
A Cauchy seuqence in a compact set $S \subseteq \mathbb{R}^n$ converges to a point in S

Lemma 3.1.8 Any closed and bounded subset of $\mathbb{R}^n$ is compact

Lemma 3.1.9 Let f be holomorphic at z_0, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)(z - z_0)$$

for some $\varepsilon(z)$ satisfying $\lim_{z \to z_0} \varepsilon(z) = 0$

Proof: Let

$$\varepsilon(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

then take $\lim_{z \to z_0}$.

Theorem 3.1.10 — Goursat's Theorem.
Let f be the holomorphic on a domain D and let T be a triangle lying D with interior in D then

$$\int_T f(z)dz = 0$$

Proof: Divide T into four triangles by connecting the midpoints of its sides. Now

$$\int_T f = \int_{T_1} f + \int_{T_2} f + \int_{T_3} f + \int_{T_4} f$$

there exists a T_i such that

$$\left|\int_T f\right| \leq 4\left|\int_{T_i} f\right|$$

Note that $length(T_1) \leq \frac{1}{2}length(T)$ and $diam(T_1) \leq \frac{1}{2}diam(T)$. Repeat this process, yielding $T = T^{(0)}, T^{(1)}, \ldots$ such that

$$\left|\int_T f\right| \leq 4^n\left|\int_{T^{(n)}} f\right|$$

with $length(T^{(n)}) \leq \frac{1}{2^n}length(T)$ and $diam(T^{(n)}) \leq \frac{1}{2^n}diam(T)$.
Let z_n be a point in the interior of $T^{(n)}$ for each n, then $\{z_n\}$ is a Cauchy sequence. Then $\lim_{n\to\infty} z_n = w$ where w lies in the interior of each $T^{(n)}$. Since f is hlomorphic at w, then

$$f(z) = f(w) - f'(w)(z-w) + \varepsilon(z)(z-w)$$

$\lim_{z\to w} \varepsilon(z) = 0$. Now consider

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} f(w) + f'(w)(z-w) + \varepsilon(z)(z-w)dz$$

Note that $f(w)$ has primitive $zf(w)$ and $f'(w)(z-w)$ has primitive $\frac{1}{2}f'(w)(z-w)^2$ so

$$\int_{T^{(n)}} f(w) + f'(w)(z-w)dz = 0$$

so that

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \varepsilon(z)(z-w)dz$$

Let's define $\varepsilon_n = \sup_{z\in T^{(n)}} |\varepsilon(z)|$ and then

$$|z-w| \leq diam(T^{(n)}) \leq \frac{1}{2^n}diam(T)$$

and

$$length(T^{(n)}) \leq \frac{1}{2^n}length(T^{(n)})$$

so that

$$\left|\int_{T^{(n)}} f(z)dz\right| = \left|\int_{T^{(n)}} \varepsilon(z)(z-w)dz\right| \leq \varepsilon_n diam(T^{(n)})length(T^{(n)}) \leq \varepsilon_n \frac{1}{4^n}diam(T)length(T)$$

thus

$$\left|\int_T f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right| \leq 4^n \varepsilon_n \frac{1}{4^n} diam(T) length(T) = \varepsilon_n diam(T) length(T)$$

let $n \to \infty$ and $\varepsilon_n \to 0$, we get

$$\left|\int_{T^{(n)}} f(z)dz\right| \to 0 \qquad \Longrightarrow \qquad \int_{T^{(n)}} f(z)dz = 0$$

which completes the proof.

Corollary 3.1.11
The **Goursat's Theorem** also works for retangles and polygons.

Corollary 3.1.12
If f is holomorphic on an open disk, then f has a primitive on that disk

Proof: Choose $z_0 \in D$ and define

$$F(z) = \int_\Gamma f(z)dz$$

so that

$$F(z+h) - F(z) = \int_{\Gamma_n} f(z)dz + \underbrace{\int_\Delta f + \int_\Box f}_{=0}$$

so that

$$\frac{d}{dz}\int_{\Gamma_n} f(z)dz = f(z)$$

as in last lecture.

■ **Example 3.7** f holomorphic on domain D does not imply f has primitive on D.

Let $f(z) = \frac{1}{z}$ is holomorphic on

$$\{z \in \mathbb{C} : 1 < |z| < 2\}$$

and

$$\int_{C_1(0)} \frac{1}{z} dz = 2\pi i$$

■

Definition 3.1.9 — Homotopic.
Let Γ_1, Γ_2 be two contours in a domain D with the same initial and terminal point. Γ_1 is **homotopic** (or continuously deformable) if there exists $r : [0,1]^2 \to \mathbb{C}$ satisfying:

1. r is continuous on $[0,1]^2$
2. For a fixed s, $r(s,t)$ is a parametrization of a contour in D with initial and terminal point shared with Γ_1, Γ_2
3. $r(0,t)$ parmetrizes Γ_1, $r(1,t)$ parmetrizes Γ_2

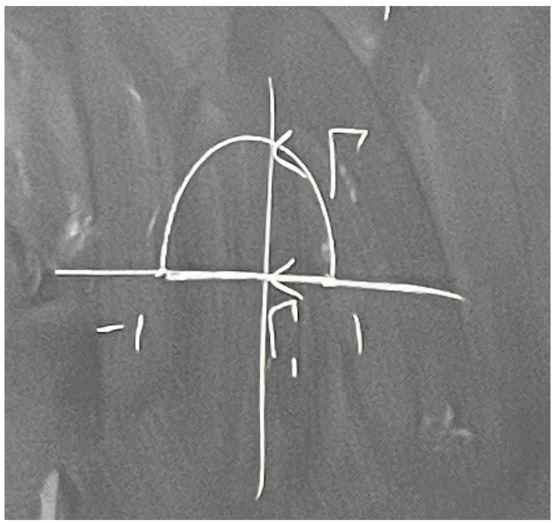
Definition 3.1.10 — Simply Connected.
A domain D is **simply connected** if any two contours in D sharing initial and terminal point are homotopic to each other.

3.2 Cauchy's Theorem and its Integration Formula

Theorem 3.2.1 — Cauchy's Theorem.
Let f be holomorphic on a simply connected domain D and let Γ be a closed contour in D, then

$$\int_\Gamma f = 0$$

Proof: Γ is homotopic and triangle.



■ **Example 3.8** Let $f(z) = z^2$, since z^2 is entire so by Cauchy's Theorem,

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz = \int_1^{-1} x^2 dx = -\frac{2}{3}$$

■

■ **Example 3.9** Let $f(z) = \frac{1}{z^2-1}$, so f is holomorphic on $\mathbb{C} \setminus \{1, -1\}$, then

$$\int_{C_2(0)} f(z)dz = \int_{C_\varepsilon(-1)} f(z)dz + \int_{C_\varepsilon(1)} f(z)dz$$

for $\varepsilon \in (0, 2)$. Note that

$$\frac{1}{z^2-1} = \frac{1}{2} \cdot \left(\frac{1}{z-1} - \frac{1}{z+1} \right)$$

then

$$\int_{C_\varepsilon(1)} f(z)dz = \frac{1}{2} \cdot \left(\int_{C_\varepsilon(1)} \frac{1}{z-1} dz - \underbrace{\int_{C_\varepsilon(1)} \frac{1}{z+1} dz}_{=0} \right) = \frac{1}{2} \cdot 2\pi i = \pi i$$

Similarly, we have

$$\int_{C_\varepsilon(-1)} f(z)dz = \frac{1}{2}\cdot\left(\underbrace{\int_{C_\varepsilon(-1)}\frac{1}{z-1}dz}_{=0} - \int_{C_\varepsilon(-1)}\frac{1}{z+1}dz\right) = \frac{1}{2}\cdot -2\pi i = -\pi i$$

so

$$\int_{C_2(0)} f(z)dz = 0$$

■

■ **Example 3.10** Say f has a taylor series at z_0:

$$f(z) = f(z_0) + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

Then we get

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz &= \frac{1}{2\pi i}\int_\Gamma\left(\frac{f(z_0)}{z-z_0} + \underbrace{a_1 + a_2(z-z_0)+\dots}_{holomorphic}\right)dz \\
&= \frac{f(z_0)}{2\pi i}\int_\Gamma \frac{1}{z-z_0}dz \\
&= \frac{f(z_0)}{2\pi i}\cdot 2\pi i \\
&= f(z_0)
\end{aligned}$$

■

Theorem 3.2.2 — Cauchy Integral Formula.
Let f be function holomorphic on a domain $\Omega \subseteq \mathbb{C}$, Γ is a jordan curve (closed contour) contained in Ω and whose interior in contained in Ω. Let $z_0 \in$ the interior of Γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$$

Proof: WE can replace Γ with $C(r) = \{z \in \mathbb{C} : |z-z_0| = r\}$ for small enough r, then

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz &= \frac{1}{2\pi i}\int_{C(r)} \frac{f(z)}{z-z_0}dz \\
&= \frac{1}{2\pi i}\int_{C(r)} \frac{f(z_0)}{z-z_0}dz + \int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0}dz \\
&= f(z_0) + \int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0}
\end{aligned}$$

We will show that

$$\int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0} = 0$$

Since

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and $\left|\frac{f(z)-f(z_0)}{z-z_0}\right|$ is bounded on $C(r)$ and its interior S_0, so

$$\int_{C(r)} \frac{f(z) - f(z_0)}{z - z_0} dz \to 0$$

as $r \to 0$. so since this integral is independent of r and it must equal 0, which complesthes the proof.

■ **Example 3.11** Say g is holomorphic on $0 < |z| < R$, which of the following implies that

$$\int_{C(r)} g(z) dz = 0$$

(a) g is holomorphic at 0
(b) g is identically 0 on $0 < |z| < R$
(c) $|g|$ is bounded on $0 < |z| < R$
(d) $g(z) = 2\pi i$ on $0 < |z| < R$
(e) g is defined and continuouos at 0
(f) $\lim_{z \to 0} g(z) = \infty$

Answer: (a)(b)(c)(d)(e) ■

■ **Example 3.12** Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$, compute
(a) $\int_\Gamma \frac{\cos(z)}{z} dz$

(b) $\int_\Gamma \frac{e^z}{z-2} dz$

(c) $\int \frac{\cos(2\pi z)}{2z-1} dz$

By **Cauchy Integral Thm/Formula**, (a)(b) are 0. ■

Proposition 3.2.3 — Cauchy Integral Formula for Derivaives.
Note that $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$, then the **CIF**:

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w} dz \quad \Longrightarrow \quad \frac{d}{dw} f(w) = \frac{1}{2\pi i} \frac{d}{dw} \int_\Gamma \frac{f(z)}{z-w} dz = \frac{1}{2\pi i} \int_\Gamma \frac{d}{dw} \frac{f(z)}{z-w} dz$$

Taking devaritive again:

$$f'(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-w)^2} dz \quad \Longrightarrow \quad \frac{2}{2\pi i} \int_\Gamma \frac{f(z)}{z-w} dz$$

Taking devaritive n times:

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-w)^{n+1}}dz$$

so f is **infinitely** differentiable!

Proposition 3.2.4
Let f be holomorphic on Ω with $D = \{|z-z_0| < R\} \subseteq \Omega$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$$

where $M = \max_{|z|=r}|f(z)|$.

Proof: Take $\Gamma = \{|z-z_0| = r\}$ then apply **prop 3.1.15**, then we have

$$\left|f^{(n)}(z_0)\right| \leq \left|\frac{n!}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \left|\frac{n!}{2\pi i}\int_{\Gamma}\frac{M}{r^n}dz\right| = \left|\frac{n!}{2\pi i}\frac{M}{r^n}\cdot 2\pi i\right| = \frac{n!M}{r^n}$$

3.3 Liouvlle Theorem and Maximum Modulus Principle

Theorem 3.3.1 — Liouivlle.
A bounded **entire (holomorphic on** $\mathbb{C}$**)** function is constant.

Proof: By the **CIF** for derivatives we have $|f'(z)| \leq \frac{n!M}{r}$ for any $r > 0$, so since M canbe taken independent of r, we get $|f'(z_0)| = 0$ for all z_0 so f is constant.

Theorem 3.3.2 — Maximum Modulus Principle.
A non-constant holomorphic function on a domain Ω cannot chieve its supremum on Ω. More precisely, for all $z_0 \in \Omega$, there is some $z_1 \in \Omega$ with $|f(z_1)| > |f(z_0)|$

Theorem 3.3.3
Every non-constant complex polynomials has a root in $\mathbb{C}$

Proof: Let p be a complex polynomial with no root in $\mathbb{C}$, we will show that p is constant. Then we have $\frac{1}{p}$ is **entire**, let

$$m(r) = \max_{|z|=r}|p(z)|$$

then $m(r)$ increases as $r \to \infty$, so $g(r) = \min_{|z|=r}|p(z)|$ decreases as $r \to \infty$, but $\lim_{z\to\infty}|p(z)| = \infty$ is p is not constant, which they couldn't true at the same time, so it's a contradiction. That means p is constant.

■ **Example 3.13**
If f is entire and non-constant, which of the followings are true?

(a) Image$(f) = f(\mathbb{C})$ must intersect the upper half plane.
(b) Image(f) must intersect every straight line.
(c) Image(f) must intersect every non-empty open set.
(d) Image(f) must contain every point.

Answer: (a)(b)(c) ■

■ **Example 3.14**
If p is a polynomial satisfying $|p(z)| \leq |e^z|$ for all z, what is $p(z)$?

Answer: Only $p(z) = 0$ by taking negative z with $|z|$ large. ■

Theorem 3.3.4 — Maximum Modulus Priciple.
Let f be holomorphic on an open set Ω. If f achieves its maximum on Ω, then f is a constant. That is, if there is some $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$, then f is constant.

Proof: Let z_0 be a local max of $|f|$ on Ω, let

$$D = \{|z - z_0| \leq r\} \subseteq \Omega$$

be a disc around z_0. Then the Cauchy integral formula says for $C(r) = \partial D = \{|z - z_0| = r\}$:

$$\begin{aligned} f(z) = \frac{1}{2\pi i}\int_{C(r)} \frac{f(z)}{z - z_0} dz &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} d(z_0 + re^{i\theta}) \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(z) d\theta \end{aligned}$$

This gives us that

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z)| d\theta \leq \max_{z \in C(r)} |f(z)|$$

with equality **iff** $|f|$ is constant on $C(r)$ with $|f(z_0)| = \max_{z \in C(r)} |f(z)|$ because r is arbitrary (as long as $D \subseteq \Omega$), we will show f is constant on D. Write $f = u + iv$ then $u^2 + v^2$ is constant on D.

$$2uu_x + 2vv_x = 0 \qquad 2uu_y + 2vv_y = 0 \qquad \textbf{since} \quad u_x = v_y \quad \textbf{and} \quad u_y = -v_x$$

then we get $-2uv_x + 2vu_x = 0$. That is to solve

$$\underbrace{\begin{bmatrix} u & v \\ v & -u \end{bmatrix}}_{\det = -u^2 - v^2} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so either $u^2 + v^2 = 0$ or $u_x = v_x = 0$. The $u^2 + v^2 = 0$ implies $f = 0$ is a constant or $u_x = v_x = u_y = v_y = 0$ implies f is constant. Therefore, f is constant on D, since D is arbitrary so f is constant on Ω.

3.4 Morera's Theorem

Theorem 3.4.1 — Morera's Theorem.
If f is continuous on a domain Ω with

$$\int_\Gamma f(z)dz = 0$$

for all simple closed curves $\Gamma \subseteq \Omega$ whose interiors are contained in Ω, then f is holomorphic on Ω.

Proof: We will find a holomorphic F with $F' = f$. This will prove that f is holomorphic. Since holomorphicity is local, we can assume that $\Omega = D$ is a disc. Now choose $z_0 \in D$ we define

$$F(z) = \int_\Gamma f(z)dz$$

where Γ is any path from z_0 to z because D is simply connected, this F is well defined by hypothesis. Compute

$$\begin{aligned}
F'(z) = \lim_{x\to z} \frac{F(z)-F(x)}{z-x} &= \lim_{x\to z} \frac{1}{z-x}\left(\int_{z_0}^z f(y)dy - \int_{z_0}^x f(y)dy\right) \\
&= \lim_{x\to z} \frac{1}{z-x}\int_x^z f(y)dy \\
&= \lim_{x\to z}\left(\int_x^z \frac{f(y)-f(z)}{z-x}dy + \int_x^z \frac{f(z)}{z-x}dy\right) \\
&= \lim_{x\to z} f(x) + \int_x^z \frac{f(y)-f(x)}{z-x}dy \\
&= f(z) + 0
\end{aligned}$$

because the

$$\left|\int_x^z \frac{f(y)-f(x)}{z-x}dy\right| \le |z-x|\left|\frac{f(m)-f(x)}{z-x}\right| = |f(m)-f(x)|$$

where $m =$ the max value of f on (x,z), completes the proof.

Lemma 3.4.2 — Symmetry Principle.
Let D be a domain symmetric across $\mathbb{R}$, let D^+, D^-, I be as indicated. Let f^+ be holomorphic on D^+, f^- be holomorphic on D, both extend continuously to I and $f^+(z) = f^-(z)$ for $z \in I$, then

$$f(z) = \begin{cases} f^+(z) & z \in D^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in D^- \end{cases}$$

is holomorphic on D.

Proof: Note that if f is continuous on D, then

$$\left|\int_T f(z) - \int_{T_\varepsilon} f(z)\right| \le \varepsilon \cdot \left(\max_{z\in T}\left|f'(z)\right|\right) \cdot \textbf{length(T)} \to 0$$

as $\varepsilon \to 0$

Proposition 3.4.3 — Schwarz Reflection Principle.
Let D^+, D^-, I as be the ones defined in **Lemma 3.1.22**. Let f^+ be holomorphic on D^+ and extend continuously to I, then there exists f such that $f(z) = f^+(z)$ on D^+ and f is holomorphic on D

Proof: Let $f^+(z) = \overline{f(\bar{z})}$, by **A2** we have f^- is holomorphic on D^-, then apply the **Symmetry Principle**

Lemma 3.4.4 — Schwarz's Lemma.
Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and f be holomorphic on D, $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Then $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z)| = |z|$ for some $0 \neq z \in D$, then f is rotation $f(z) = \lambda z$ for some constant $|\lambda| \leq 1$.

Proof: Let

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & \mathbf{z=0} \end{cases}$$

Note that g is holomorphic on D, since

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$$

Now consider g on $|z| < r < 1$, then

$$|g(z)| \leq \max_{|w|=r} |g(w)| \leq \max_{|w|=r} \frac{|f(w)|}{|w|} \leq \frac{1}{r} \to 1$$

as $r \to 1$. so we have $|g(z)| \leq 1$ on D, then for $z \neq 0$

$$\frac{|f(z)|}{|z|} \leq 1 \quad \Longrightarrow \quad |f(z)| \leq |z|$$

for $z = 0$ we have $|g(0)| = |f'(0)| \leq 1$. If $|f(z)| = |z|$ at some $z \in D$, then $|g(z)| = 1$, so by **maximum modulus theorem** g is constant on D. Let $g(z) = \lambda$ and $|z| = 1$, then we have $f(z) = \lambda z$ as desired.

■ **Remark 3.3** If f is holomorphic on domain D, then f is infinitely differentialbe on D

■ **Remark 3.4** In $\mathbb{R}$, an infinitely differentiable function has a Taylor series representation.

3.5 Series

Definition 3.5.1 — Convergent Series.
A series $\sum_{n=1}^{\infty} z_n$ is **convergent** if $\lim_{n \to \infty} \sum_{i=1}^{n} z_i$ converges.

Definition 3.5.2 — Cauchy Series.
A series $\sum_{n=1}^{\infty} z_n$ is **Cauchy** if $\lim_{k\to\infty} \sum_{n=k}^{\infty} z_n = 0$.

Definition 3.5.3 — Uniformly Convergent.
A sequence $\{f_n\}$ is **uniformly convergent** on a set if $\forall \varepsilon > 0$, $\exists N > 0$, $\forall z \in S$, $\exists L$, $\forall n > 0$

$$|f_n(z) - L| < \varepsilon$$

Lemma 3.5.1 If $f_n \to f$ uniformly on S, then

$$\int_S f_n \to \int_S f$$

Definition 3.5.4 — Uniformly Convergent Series.
A series is **uniformly convergent** if its sequence of partial sum is **uniformly convergent**.

Definition 3.5.5 — Absolutely Convergent Series.
A series is **absolutely convergent** if the series $\sum_{n=0}^{\infty} z_n$ converges.

Definition 3.5.6
Let's define

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the open disk of radius r centered at z_0. Let $\overline{D_r(z_0)} = D_r(z_0) \cup C_r(z_0)$ be its closure.

Definition 3.5.7
Let $\{x_n\} \subseteq \mathbb{R}$, then

$$\limsup x_n = \lim_{n\to\infty} \sup_{k>n} x_k$$

Proposition 3.5.2 — Ratio Test.
If $\limsup \left|\frac{z_{n+1}}{z_n}\right| < 1$, then $\sum_{n=0}^{\infty} z_n$ converges absolutely. If $\limsup \left|\frac{z_{n+1}}{z_n}\right| > 1$, then $\sum_{n=0}^{\infty} z_n$ diverges.

Proposition 3.5.3 — Root Test.
If $\limsup |z_n|^{\frac{1}{n}} < 1$, then then $\sum_{n=0}^{\infty} z_n$ converges absolutely. If $\limsup |z_n|^{\frac{1}{n}} > 1$, then $\sum_{n=0}^{\infty} z_n$ diverges

Proposition 3.5.4 — Comparison Test.
If $\sum_{n=0}^{\infty} x_n$ converges with $x_n \in \mathbb{R}$ and $|z_n| \leq x_n$ for all n, then $\sum_{n=0}^{\infty} z_n$ converges absolutely.

Proposition 3.5.5 — Weierstrass M Test.
Let $\{f_n\}_{n=1}^{\infty}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in S$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on S

Proof: Let $g_n(z) = \sum_{k=0}^{n} f_k(z)$, then g_n is uniformly Cauchy on S

Definition 3.5.8 — Power Series.
A **Power Series** about z_0 is a series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $z_n \in \mathbb{C}$

Theorem 3.5.6
If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at point z with $|z - z_0| = R$, then it converges absolutely on $D_R(z_0)$ and converges uniformly on any closed subdisk of $D_r(z_0)$

Proof: Let $w \in D_R(z_0)$ and $|w - z_0| < r < R$, then

$$|a_n(w - z_0)^n| = \underbrace{|a_n(z - z_0)^n|}_{\to 0 \quad \text{so is bounded} \leq M} \cdot \underbrace{\left| \frac{a_n(w - z_0)^n}{a_n(z - z_0)^n} \right|}_{\leq \frac{r}{R}} \leq M \cdot \underbrace{\left(\frac{r}{R} \right)^n}_{\frac{r}{R} < 1}$$

Now we can see that $M\left(\frac{r}{R}\right)^n$ is a convergent geometric series so $\sum_{n=0}^{\infty} a_n (w - z_0)^n$ converges absolutely by comparison. Apply the **Weierstrass M-Test** to the above to get uniformly convergence on $\overline{D}_r(z_0)$

Theorem 3.5.7 — Taylor's Theorem.
Let f be holomorphic on $D_r(z_0)$, then for all $z \in D_R(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Proof: Choose $z \in D_R(z_0)$ and let $|z - z_0| < r < R$, by Cauchy Integration formula

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z} dw$$

for all $w \in C_r(z)$. Then

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)-(z-z_0)} = \frac{f(w)}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty} f(w) \cdot \left(\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right)$$

Now we have

$$\left| f(w) \cdot \left(\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right)\right| \leq \max_{w \in C_r(z_0)} |f(w)| \cdot \frac{|z-z_0|^n}{r^{n+1}} = \frac{1}{r} \cdot \max_{w \in C_r(z_0)} |f(w)| \left(\frac{z-z_0}{r}\right)^n$$

Since $\left|\frac{z-z_)}{r}\right| < 1$, then by weierstrass M-test, this series converges uniformly on $C_r(z_0)$, thus we may integrate term by term. Then

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=0}^{\infty} f(w) \frac{(z-z_0)^n}{(w-z_0)^{n+1}} dw = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw (z-z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)(z_0)}}{n!}(z-z_0)^n$$

■ **Remark 3.5** The term "analytic" means expressible as n power series many text will use "analytic" in place of "holomorphic".

■ **Example 3.15**

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots\ldots$$

with $R = \infty$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots.$$

with $R = \infty$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots.$$

with $R = \infty$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots.$$

with $R = 1$ ■

■ **Example 3.16** Taylor series for e^{2z} about 0, we have

$$e^{(2z)} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = 1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \ldots$$

Converges for $|2z| < \infty$, implies that $R = \infty$. ■

■ **Example 3.17** Let's look at $\frac{1}{\frac{1}{2}z^2+1}$, let $w = -\frac{1}{2}z^2$, then

$$\frac{1}{\frac{1}{2}z^2+1} = \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}z^2\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}z^{2n}$$

and $\left|\frac{1}{2}z^2\right| < 1$ implies $|z| < \sqrt{2}$, so $R = \sqrt{2}$ ■

■ **Example 3.18** Taylor series for $\cos(z) + i\sin(z)$ about 0:

$$\cos(z)+i\sin(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} + i\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} = 1+iz-\frac{z^2}{2!}-i\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots. = \sum_{n=0}^{\infty}\frac{i^n z^n}{n!} = e^{iz}$$

■

■ **Example 3.19** Recall for $r > 0$

$$\int_{C_r(z_0)}(z-z_0)^n = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = -1 \end{cases}$$

Let f be analytic on $D_R(z_0)$, then

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

uniform convergence on $\overline{D_r(z_0)}$ and the convergence is uniform on closed subdisks.
so for all $r < R$ we have

$$\int_{C_r(z_0)} f(z)dz = \int_{C_r(z_0)}\sum_{n=0}^{\infty}a_n(z-z_0)^n dz = \sum_{n=0}^{\infty}a_n\cdot\int_{C_r(z_0)}(z-z_0)^n dz = 0$$

■

■ **Example 3.20** Find a series representation for $\frac{e^z}{z^2}$ about $z = 0$.

$$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^n}(1+z+\ldots) = \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}$$

Now consider for $r > 0$

$$\int_{C_r(0)}\frac{e^z}{z^2} = \int_{C_r(0)}\sum_{n=0}^{\infty}\frac{z^n}{(n+2)!} = \sum_{n=0}^{\infty}\int_{C_r(0)}\frac{z^n}{(n+2)!} = 2\pi i\cdot\frac{1}{(-1+2)!} = 2\pi i$$

■

Lemma 3.5.8
If $f_n \to f$ is uniformly on S, then

$$\int_S f_n \to \int_S f$$

Theorem 3.5.9
Let f be holomorphic on a domain $\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, then on that annulus, f has a **Laurent Series** (generalized Cauchy series)

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

which converges on the annulus and converges uniformly on closed subannuli.

Definition 3.5.9 — Isolated Singnlarity.
An **isolated singnlarity** of a function f is a point z_0 where f is not holomorphic, but where f is holomorphic on some punnctured disk $0 < |z - z_0| < r$

■ **Remark 3.6** If $f(z)$ has zero at z_0, then $\frac{1}{f(z)}$ has singularity at z_0.

Definition 3.5.10 — Zero of Order m.
An analytic function f has a **zero of order m** at z_0 if $\frac{f(z)}{(z-z_0)^m}$ is analytical at z_0 but $\frac{f(z)}{(z-z_0)^{m+1}}$ is not.

Equivalently if $f(z) = \sum_{n=0}^{\infty}$ the order is the smallest n such that $a_n \neq 0$

Definition 3.5.11 — Singularity.
A **singularity** of f is a point where f is not analytic but is a limit point of the points where f is analytic.

Definition 3.5.12
Let z_0 be an isolated singularity of f, let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be the Laurrent series of f at z_0

If $a_{-m} \neq 0$ but $a_n = 0$ for all $n > m$, we call z_0 a **pole order of m** $\iff$ $(z - z_0)^m f(z)$ is analytic at z_0 but $(z - z_0)^{m+1} f(z)$ is not

If $a_{-n} = 0$ for all $n > 0$, we call this a **removable singularity**. In this case, we have

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim\limits_{z \to z_0} f(z) & z = z_0 \end{cases}$$

is analytic at z_0.

If $a_{-n} \neq 0$ for infinitely many $n > 0$, we call this **essential singularity**.

■ **Example 3.21 Removable singularity** examples:

$$f(z) = \frac{z}{z} \quad f(z) = \frac{\sin(z)}{z} = \frac{1}{z} \cdot \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

■

■ **Example 3.22** Let a function

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n \quad \textbf{with} \ \ a_m \neq 0$$

on a punctured disk $0 < |z - z_0| < r$. Let Γ be a simple, closed positively oriented contour in the annulus with z_0 inside the loop, then

$$\int_\Gamma f(z)dz = \int_\Gamma \sum_{n=-m}^{\infty} a_n(z-z_0)^n dz = \sum_{n=-m}^{\infty} \int_\Gamma a_n(z-z_0)^n = 2\pi i \cdot a_{-1}$$

■

Definition 3.5.13 — Residue.
Given f, z_0, Γ as before, we define the **residue** of f at z_0 to be

$$\frac{1}{2\pi i} \int_\Gamma f(z)dz = a_{-1} = \mathbf{res}_{z_0}(f)$$

Definition 3.5.14 — Meromorphic.
A function f is called **meromorphic** on a domain D if it's holomorphic on all of D except for a set of isolated poles.

Theorem 3.5.10 — Residue Theorem.
Let f be meromorphic on a simply connected domain D and let Γ be a simple, closed, positively oriented contour lying in D. Let $z_1, ..., z_k$ be the poles of f inside Γ, then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{i=1}^{k} \mathbf{res}_{z_i}(f)$$

■ **Example 3.23** Consider $f(z) = \frac{1}{z^2+z} = \frac{1}{z(z+1)}$, so for $0 < |z| < 1$, then

$$\frac{1}{z(z+1)} = \frac{1}{z} \cdot \left(\frac{1}{1-(-z)}\right) = \frac{1}{z} \sum_{n=0}^{\infty} (-z)^n = \frac{1}{z} - 1 + z - z^2 ....$$

so it has order 1 and residue 1
for $0 < |z+1| < 1$, then

$$\frac{1}{z(z+1)} = \frac{1}{z+1} \cdot \frac{1}{z+1-1} = \frac{1}{z+1} \cdot \left(\frac{1}{1-(z+1)}\right) = -\frac{1}{z+1} \sum_{n=0}^{n} (z+1)^n = \frac{-1}{z+1} - 1 - (z+1) - (z+1)^2 - ..$$

so is has simple pole and residue -1.
for $|z| > 1$, then

$$\frac{1}{z(z+1)} = \frac{1}{z} \cdot \frac{1}{z+1} \cdot \frac{\frac{1}{z}}{\frac{1}{z}} = \frac{1}{z^2} + \frac{1}{1+\frac{1}{z}} = \frac{1}{z^2} \cdot \sum_{n=0}^{n} \left(\frac{1}{z}\right)^n = ... + \frac{1}{z^4} + + \frac{1}{z^3} + \frac{1}{z^2}$$

■

■ **Remark 3.7** Recall f has pole of order m at $z_0 \iff (z-z_0)^m f(z)$ has removeable singularity at z_0, $(z-z_0)^{m-1} f(z)$ has a pole at z_0. That is

$$(z-z_0)^m \left(\frac{a_{-m}}{(z-z_0)^m} + \dots \right) = a_{-m} + a_{m-1}(z-z_0) + \dots.$$

f has a zero of order m at $z_0 \iff \frac{f(z)}{(z-z_0)^m}$ has a removeable singgularity at z_0, $\frac{f(z)}{(z-z_0)^{m+1}}$ has a pole at z_0.

$$\frac{1}{(z-z_0)^m}(a_m(z-z_0)^m + \dots)) = a_m + a_{m+1}(z-z_0) + ..$$

Let f and g be analytic at z_0, let f have a zero of order m at z_0 and let g have a zerp pf order n at z_0, then

$$\frac{f(z)}{g(z)} = \frac{\frac{f(z)}{(z-z_0)^m}(z-z_0)^m}{\frac{g(z)}{(z-z_0)^n}(z-z_0)^n} = (z-z_0)^{m-n} h(z) \quad \textbf{where } h(z) \textbf{ is analytic at } z_0$$

and we see that

$$\frac{f(z)}{g(z)} \quad \textbf{has} \quad \begin{cases} \textbf{a zero of order } m-n \textbf{ at } z_0 & \textbf{if } m > n \\ \textbf{a pole of order } n-m \textbf{ at } z_0 & \textbf{if } m < n \\ \textbf{a removable singularity} & \textbf{if } m = n \end{cases}$$

■ **Example 3.24** $\frac{1}{z(z+1)}$ has simple poles at $z=0$ and $z=-1$ ■

■ **Example 3.25** $\frac{z+3}{z^3(z+1)^2(z-2)}$ has order 3 pole at 0, order 2 pole at -1 and simple pole at 2. ■

■ **Example 3.26**

$$\frac{\cos(z)-1}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) = -\frac{1}{2!} + \frac{z^2}{4!} - \dots$$

so it has removeable singularity at $z=0$. ■

■ **Example 3.27** Let f has a simple pole at z_0 and a Laurent series:

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$$

is same punctured disk about z_0, then

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \dots$$

so that

$$\lim_{z\to z_0}((z-z_0)f(z)) = a_{-1} = \text{res}_{z_0}(f)$$

so if f has a simple pole at z_0, then $\text{res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$ ■

■ **Example 3.28** Let $f(z) = \frac{1}{z(z+1)}$, then $\text{res}_0(f) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} z \cdot \frac{1}{z(z+1)} = \frac{1}{0+1} = 1$. Similarly, we have $\text{res}_{-1}(f) = \lim_{z\to -1}(z+1)\frac{1}{z(z+1)} = \frac{1}{-1} = 1$ ■

■ **Example 3.29** Now let f have a pole of order m at z_0, then

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{-m+1}} + \ldots.$$

so that

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \ldots$$

so

$$\frac{d}{dz}((z-z_0)^m f(z)) = a_{-m+1} + 2a_{-m+2}(z-z_0) + \ldots + (m-1)a_{-1}(z-z_0)^{m-2} + .$$

and

$$\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)) = a_{-1}(m-1)! + \ldots$$

then

$$\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)) = a_{-1}(m-1)!$$

and

$$a_{-1} = \text{res}_{z_0}(f) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z))$$

■

■ **Example 3.30** Consider $f(z) = \frac{e^z+1}{z^3}$, then $e^0 + 1 = z \neq 0$ so f has an order 3 pole at 0, then

$$\text{res}_0(f) = \frac{1}{(3-1)!} \lim_{z\to z_0} \frac{d^2}{dz^2}\left(z^3 \cdot \frac{e^z+1}{z^3}\right) = \frac{1}{2}$$

■

■ **Example 3.31** Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series for f in some annulus, so

$$a_{-1} = \frac{1}{2\pi i}\oint_{\Gamma} f(z)dz$$

where Γ is simple, closed, positively oriented contour looping around the inner circle of annulus.
Now we see

$$(z-z_0)^{-m-1} f(z) = \ldots. + \frac{a_m}{z-z_0} + \ldots. \quad \Longrightarrow \quad a_m = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{m+1}}dz$$

Note: for a Taylor series, this is equivalent to $\frac{f^{(n)}(z_0)}{n!}$ by Cauchy's Integration Theorem. ■

3.6 Integration II

Proposition 3.6.1 Let $f(z) = \frac{g(z)}{h(z)}$ where g, h are analytic at z_0. Let $g(z_0) \neq 0$ and $h(z_0) = 0$, $h'(z_0) \neq 0$. That is f has a simple pole at z_0, then

$$\text{res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)} = g(z_0)\lim_{z\to z_0}\frac{z-z_0}{h(z)} = g(z_0)\lim_{z\to z_0}\frac{z-z_0}{h(z)-h(z_0)} = \frac{g(z_0)}{h'(z_0)}$$

■ **Example 3.32** Find residues of all poles of $f(z) = \frac{1}{z^3-1}$, note that $z^3 - 1 = 0 \iff z \in \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}$, then f has 3 simple poles. Then residue at simple pole z is $\frac{1}{3z^2}$. This gives us that

$$\text{res}_1(f) = \frac{1}{3} \qquad \text{res}_{e^{\frac{2\pi i}{3}}}(f) = \frac{1}{3}e^{\frac{2\pi i}{3}} \qquad \text{res}_{e^{\frac{4\pi i}{3}}}(f) = \frac{1}{3}e^{\frac{4\pi i}{3}}$$

■

■ **Example 3.33** Consider the following:

$$\int_0^\infty \frac{1}{x^4+1}dx \quad \textbf{and} \quad I = \int_0^\infty \frac{1}{x^4+1}dx$$

Note that

$$2I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$$

Let Γ_R be the line segment running from $-R$ to R in $\mathbb{R}$, then

$$2I = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{z^4+1}dz$$

Let C_R be the upper semicircle running from R to $-R$, note $\Gamma_R + C_R$ is simple closed positive oriented tour. so we can sue residue theorem. Consider

$$\left|\int_{C_R} \frac{1}{z^4+1}dz\right| \leq \left|\int_{C_R} \frac{1}{R^4}dz\right| \leq |\pi i R| \cdot \frac{1}{R^4} \leq \frac{\pi}{R^3} \to 0 \quad \textbf{as} \quad R \to \infty$$

Next we locate the poles of $\frac{1}{z^4+1}$ and find their residues. Note that $z^4 + 1 = 0 \iff z \in \left\{e^{\frac{k\pi i}{4}} : k = 1,3,5,7\right\}$. Then

$$\text{res}_{z_0}(f) = \frac{1}{4z^3} \quad \textbf{where} \quad z_0 \in \left\{e^{\frac{k\pi i}{4}} : k = 1,3,5,7\right\}$$

Then we have

$$\oint_{C_R+\Gamma_R} \frac{1}{z^4+1}dz = \frac{\pi}{\sqrt{2}}$$

so

$$2I = \lim_{R\to\infty} \int_{\Gamma_R} = \lim_{R\to\infty} \left(\int_{\Gamma_R+C_R} f(z)dz - \int_{C_R} f(z)dz\right) = \frac{\pi}{\sqrt{2}} - 0 = \frac{\pi}{\sqrt{2}}$$

■

Definition 3.6.1
Extended complex plane $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$

Definition 3.6.2
Define the behavior of $f(z)$ at ∞ to behavior of $f(\frac{1}{z})$ at 0.

■ **Example 3.34** For example, let $f(z) = z^2+1$, so that $f(\frac{1}{z}) = \frac{1}{z^2}+1 = \frac{1}{z^2}+\frac{0}{z}+1$, so it has order 2 and residue 0 and

$$\lim_{R\to\infty}\int_{C_R(0)} z^2+1dz = 0$$

also

$$\int_{-C_\infty(0)} f(z)dz = -2\pi i res_\infty(f)$$

■

■ **Example 3.35** Let $f(z) = \frac{z+1}{z-i}$, so that $f(\frac{1}{z}) = \frac{\frac{1}{z}+1}{\frac{1}{z}-i} = \frac{1+z}{1-iz}$ at $z=0$ and $f(\frac{1}{z}) = 1$ so f is analytic at ∞. ■

■ **Example 3.36** Let $f(z) = \sin(z)$ so $f(\frac{1}{z}) = \sin\frac{1}{z}$ does not converge as $z\to 0$, so $\sin(z)$ has essential singularity at ∞

At an isolated singularity z_0

$$\lim_{z\to z_0} f(z) = c\in\mathbb{C} \quad\Longrightarrow\quad f \textbf{ analytic at } z_0 \textbf{ (or removeable singularity)}$$

$$\lim_{z\to z_0} |f(z)| = \infty \quad\Longrightarrow\quad f \textbf{ has a pole at } z_0$$

$$\lim_{z\to z_0} f(z) \textbf{ does not exist in } \hat{\mathbb{C}} \quad\Longrightarrow\quad f \textbf{ has an essential singularity at } z_0$$

■

■ **Example 3.37** Consider the following

$$\int_0^\infty \frac{1}{x^3+1}dx$$

Let $f(z) = \frac{1}{z^3+1}$ so f has poles at $z = -1, e^{\frac{i\pi}{3}}, e^{\frac{5i\pi}{3}}$. We define

$$I = \int_0^\infty \frac{1}{x^3+1}dx = \lim_{R\to\infty}\int_{\Gamma_1} f(z)dz$$

and we can see that

$$\left|\int_{C_R}\frac{1}{z^3+1}dz\right| \sim R\cdot\frac{1}{R^3} = R^{-2}\to 0 \ \textbf{ as } \ R\to\infty$$

Let $\Gamma_2 : r_2(t) = t\cdot e^{\frac{2\pi i}{3}}$ for $t\in[0,R]$,

$$\int_{\Gamma_2}\frac{1}{z^3+1}dz = \int_0^R \frac{1}{\left(t\cdot e^{\frac{2\pi i}{3}}\right)^3+1}\cdot e^{\frac{2\pi i}{3}}dt = \int_0^R \frac{e^{\frac{2\pi i}{3}}}{t^3+1}dt = e^{\frac{2\pi i}{3}}\int_0^\infty f$$

Now

$$\int_{\Gamma_1+C_R-\Gamma_2} f = 2\pi i res_{e^{\frac{i\pi}{3}}}(f) = \frac{2\pi i}{3}e^{-\frac{2\pi i}{3}}$$

This gives us that

$$\frac{2\pi i}{3}e^{-\frac{2\pi i}{3}} = I + 0 - e^{\frac{2\pi i}{3}}I \quad\Longrightarrow\quad I = \frac{2\sqrt{3}\pi}{9}$$

■

Definition 3.6.3 — Cauchy Principal Value.
Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, define the **Cauchy Principal Value** of $\int_{-\infty}^{\infty} f(x)dx$ is

$$\textbf{p.v.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

Note that $\int_{-\infty}^{\infty} f(x)dx$ exists, then

$$\textbf{p.v.} \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

■ **Example 3.38** Find **p.v.** for

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}dx$$

Let $f(z) = \frac{\cos(z)}{1+z^2}$, then

$$\left|\int_{C_R} \frac{\cos(z)}{1+z^2}dz\right| = \left|\int_{C_R} \frac{\frac{1}{2}(e^{iz}+e^{-iz})}{1+z^2}dz\right|$$

but consider e^{-iz} at z is iR, as $R \to \infty$ $e^{-i(iR)} = e^R \to \infty$
Consider

$$I = \textbf{p.v.} \int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}dz$$

then

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}dz\right| \sim \frac{1}{R^2} \cdot R = \frac{1}{R} \to \infty$$

and

$$\int_{C_R+\Gamma} f(z)dz = 2\pi i \text{res}_i(f) = 2\pi i \left[\frac{e^{iz}}{2z}\right]_{z=i} = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

so $I = \frac{\pi}{e} - 0 = \frac{\pi}{e}$

Now we consider

$$I_2 = \textbf{p.v.} \int_{-\infty}^{\infty} \frac{e^{-iz}}{1+z^2}dz$$

then

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}dz\right| \sim \frac{1}{R^2}R \sim \frac{1}{R} \to 0$$

then similarly we have

$$\int_{C_R+\Gamma} f = -2\pi i \text{res}_{-i}(f) = -2\pi i \left[\frac{e^{-iz}}{2z}\right]_{z=-i} = \frac{\pi}{e}$$

so $I_2 = \frac{\pi}{e} - 0 = \frac{\pi}{e}$.
Then we have

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}dx = \frac{1}{2}I + \frac{1}{2}I_2 = \frac{\pi}{e}$$

■

■ **Example 3.39** Consider

$$\int_0^{2\pi} \sin^2\theta d\theta$$

Let $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$, then $\sin(\theta) = \frac{1}{2i}(z + \frac{1}{z})$, then

$$\begin{aligned}
\int_0^{2\pi} \sin^2\theta d\theta = \int_{C_1(0)} \left(\frac{1}{2i}(z+\frac{1}{z})\right)^2 \frac{d\theta}{dz} dz = \int_{C_1(0)} \left(\frac{1}{2i}(z+\frac{1}{z})\right)^2 \cdot \frac{1}{iz} dz \\
= -\frac{1}{4i}\int_{C_1(0)} z + \frac{2}{z} + \frac{1}{z^3} dz \\
= -\frac{1}{4i}\text{res}_0\left(z + \frac{2}{z} + \frac{1}{z^3}\right) = -\frac{1}{4i}(2\pi i)(-2) = \pi
\end{aligned}$$

■

■ **Example 3.40** Let f be continuous on $[a,b]$ except at c with $a < c < b$, then

$$\textbf{p.v.} \int_a^b f(x)dz = \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx\right)$$

■

■ **Example 3.41** Consider

$$\textbf{p.v.} \int_0^{2\pi} \frac{\cos^2\theta}{1 - 3\sin\theta} d\theta$$

Let $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, $\frac{d\theta}{dz} = \frac{1}{iz}$. Now we can rewrite it as

$$\int_{C_1(0)} \frac{\left(\frac{z+\frac{1}{z}}{2}\right)^2}{1 - \frac{3}{2i}(z - \frac{1}{z})} \frac{1}{iz} dz = \frac{1}{4i} \frac{z^4 + 2z^2 + 1}{z^2(-\frac{3}{2i}z^2 + z + \frac{3}{2i})} dz$$

Note that

$$-\frac{3}{2i}z^2 + z + \frac{3}{2i} = 0 \qquad \Longrightarrow \qquad z = \frac{-i \pm 2\sqrt{2}}{-3}$$

■

Proposition 3.6.2 Let $p(z)$, $q(z)$ be polynomial with $\deg(p) \le \deg(q) - 2$, then for any arc C_R of $C_R(0)$,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{p(z)}{q(z)} dz\right| = 0$$

This is because

$$\left|\int_{C_R} \frac{p(z)}{q(z)}\right| \sim R \cdot \frac{R^{\deg(p)}}{R^{\deg(q)}} = R \cdot R^{-2} = \frac{1}{R} \to \infty \quad \textbf{as} \;\; R \to \infty$$

Lemma 3.6.3
Let $a > 0$ a d $\deg(q) \geq 1 + \deg(p)$, let C_R be the upper half of $C_R(0)$, then

$$\lim_{R\to\infty} e^{iaz}\frac{p(z)}{q(z)}dz = 0$$

Proof: Parameterize C_R by Re^{it} with $t \in [0,\pi]$, now

$$\int_{C_R} e^{iaz}\frac{p(z)}{q(z)}dz = \int_0^{\pi} e^{iaRe^{it}} \cdot \frac{p(Re^{it})}{q(Re^{it})} \cdot Rie^{it}dt$$

Note that

$$\left|e^{iaRe^{it}}\right| = \left|e^{iaR(\cos(t)+i\sin(t))}\right| = e^{-aR\sin(t)}$$

Then for large enough R, exists $K \in R$ such that

$$\left|\frac{p(Re^{it})}{q(Re^{it})}\right| \leq \frac{K}{R}$$

so that

$$\left|\int_0^{\pi} e^{iaRe^{it}} \cdot \frac{p(Re^{it})}{q(Re^{it})} \cdot Rie^{it}dt\right| \leq \int_0^{\pi} e^{-aR\sin(t)}\frac{K}{R}Rdt = K\int_0^{\pi} e^{-aR\sin(t)}dt = 2K\int_0^{\frac{\pi}{2}} e^{-aR\sin(t)}dt$$

Since $\sin(t) \geq \frac{2t}{\pi}$ on $[0,\frac{\pi}{2}]$, then $e^{-R\sin(t)} \leq e^{-aR\frac{2t}{\pi}}$, so

$$K\int_0^{\pi} e^{-aR\sin(t)}dt = 2K\int_0^{\frac{\pi}{2}} e^{-aR\sin(t)}dt \leq K\int_0^{\pi} e^{-aR\sin(t)}dt = 2K\int_0^{\frac{\pi}{2}} e^{-aR\frac{2t}{\pi}}dt$$

$$= 2K \cdot \left(-\frac{\pi}{2aR}\right)(e^{-aR}-1) \to \frac{\pi K}{aR} \to 0 \textbf{ as } R \to \infty$$

Remark 3.8

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)}dz \quad \textbf{Need } \deg(q) \geq 2 + \deg(p)$$

$$\int_{-\infty}^{\infty} \cos(z)\frac{p(z)}{q(z)}dz \quad \textbf{Need } \deg(q) \geq 1 + \deg(p)$$

Lemma 3.6.4 — Jordan's Lemma.
Let f be meromorphic with a simple pole at z_0, and Γ_r be parametrized by $r(t) = z_0 + re^{i\theta}$ with $\theta_1 < \theta < \theta_2$, then

$$\lim r \to 0^+ \int_{\Gamma_r} f(z)dz = i(\theta_2 - \theta_1)res_{z_0}(f)$$

Proof:

$$f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n = \frac{a_{-1}}{z-z_0} + g(z)$$

where g is analytic so g is continuous then $\exists R$ such that for $0 < r \leq R$, $\exists M > 0$ s.t. $|g(z)| \leq M$, so that

$$\left|\int_{\Gamma_r} g(z)dz\right| \leq M \cdot \text{length}(\Gamma_r) = M \cdot (\theta_2 - \theta_1)r \to 0 \quad \textbf{as} \quad r \to 0^+$$

Then

$$\int_{\Gamma_r} f(z)dz = \int_{\Gamma_r} \frac{a_{-1}}{z - z_0}dz + 0 = a_{-1}\int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}} rie^{i\theta} d\theta = a_{-1}\int_{\theta_1}^{\theta_2} id\theta i \cdot res_z(f)$$

■ **Example 3.42** Consider

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx$$

Let $f(z) = \frac{z^{\frac{1}{3}}}{1+z^2}$ with branch cut along the positive real axis, then

$$\left|\int_{C_R} \frac{z^{\frac{1}{3}}}{1+z^2}dz\right| \sim \frac{R^{\frac{1}{3}}}{R^2} \sim R^{-\frac{2}{3}} \to 0 \ \textbf{ as } \ R \to \infty$$

Similarly,

$$\left|\int_{C_R} \frac{z^{\frac{1}{3}}}{1+z^2}dz\right| \sim \frac{r^{\frac{1}{3}}}{1} r \sim r^{\frac{4}{3}} \to 0 \ \textbf{ as } \ r \to 0^+$$

Let $\int_\Gamma f \to \int_0^\infty f(z)dz = I$, then

$$\int_{\Gamma_2} f(z)dz = \int_{\Gamma_2} \frac{z^{\frac{1}{3}}}{1+z^2}dz = \int_{\Gamma_1} \frac{(ze^{2\pi i})^{\frac{1}{3}}}{1+z^2}dz = I \cdot e^{\frac{2\pi i}{3}}$$

Let f has simple pole at $z = \pm i$ with residues, so that $res_z(f) = \frac{z^{\frac{1}{3}}}{2z}$, then

$$res_i(f) = -i\frac{\sqrt{3}}{4} \qquad res_{-i}(f) = -\frac{1}{2}$$

Then we have

$$\oint_{C_R + C_r + \Gamma_1 - \Gamma_2} f = 2\pi i\left(\frac{1}{4} - i\frac{\sqrt{3}}{4} - \frac{1}{2}\right) = 0 + 0 + I - Ie^{\frac{2\pi i}{3}} = I\left(1 - e^{\frac{2\pi i}{3}}\right) \quad \Longrightarrow \quad I = \frac{\pi i e^{-\frac{2\pi i}{3}}}{1 - e^{\frac{2\pi i}{3}}} = \frac{\pi}{\sqrt{3}}$$

■

■ **Example 3.43**

$$\textbf{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx = \lim_{R\to\infty, r\to 0^+} \left(\int_{-R}^{-r} \frac{e^{ix}}{x}dx + \int_r^R \frac{e^{ix}}{x}dx\right)$$

then by **Jordan's Lemma**

$$\int_{C_r} \frac{e^{iz}}{z}dz = i(0 - \pi)\text{res}_0(f) = -\pi i$$

Let $\Gamma : [-R, R]$, then

$$\lim_{R\to\infty, r\to 0^+} \int_\Gamma f(z)dz = \oint_{C_R + C_r + \Gamma} f - \int_{C_R} f - \int_{C_r} f = 0 - 0 - (-\pi i) = \pi i$$

■

Remark 3.9
Two techniques, either:
1. convert everything to rectangular, clear denominator.
2. convert to a trig function

Example 3.44 Consider

$$I=\int_0^\infty \frac{1}{1+x^3}dx$$

we let $f(z)=\frac{\log(z)}{1+z^3}$, branch cut along the positive real axis., then

$$\int_{\Gamma_1}\frac{\log(z)}{1+z^3}dz=\int_r^R\frac{\ln(x)}{1+x^3}dx$$

and

$$\int_{\Gamma_2}\frac{\log(z)}{1+z^3}dz=\int_r^R\frac{\log\left(xe^{2\pi i}\right)}{1+x^3}dx=\int_r^R\frac{\ln(x)+2\pi i}{1+x^3}dx$$

then we have

$$\int_{\Gamma_1} f-\int_{\Gamma_2} f=\int_r^R\frac{\ln(x)}{1+x^3}dx-\int_r^R\frac{\ln(x)+2\pi i}{1+x^3}=2\pi i\cdot I$$

■

Remark 3.10 Let $f\neq 0$ be meromorphic on D and let Γ be a simple, positively oriented closed contour with Γ and its interior is in D. Consider $\frac{f'}{f}$ is meromorphic and its poles can only lie at poles and zeros of f. Let z_0 be an order-m zeros of f, then

$$f(z)=(z-z_0)^m g(z)\quad g(z_0)\neq 0\quad g \textbf{ is analytic}$$

Now we have

$$f'(z)=m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)$$

so that

$$\frac{f'(z)}{f(z)}=\frac{m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)}{(z-z_0)^m g(z)}=\frac{m}{z-z_0}+\frac{g'(z)}{g(z)}$$

Let z_0 be an order-m pole of f, then

$$f(z)=\frac{h(z)}{(z-z_0)^n}\quad h(z_0)\neq 0\quad h \textbf{ is analytic}$$

so that

$$f'=\frac{-m(z-z_0)^{m-1}h(z)+(z-z_0)^m h'(z)}{(z-z_0)^{2m}}=-\frac{m}{z-z_0}+\frac{h'(z)}{h(z)}$$

Theorem 3.6.5 — The Argument Principle. Let f be meromorphic and inside a simple, close, positively oriented contour Γ. Let $N_0(f)$ be the number of zeros in Γ and $N_p(f)$ be the number of poles in Γ (both couted with multiplicity), then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz=N_0(f)-N_p(f)$$

Definition 3.6.4 — Curling Number.
Let Γ be a closed contour and let $z_0 \neq \Gamma$. The **curling number** of Γ about z_0, denoted $n(\Gamma, z_0)$ is the unique integer n such that Γ is homeomorphic to

$$\underbrace{C_1(z_0) + C_1(z_0) + \ldots + C_1(z_0)}_{\textbf{n in total}}$$

in $\mathbb{C} \setminus \{z_0\}$

Lemma 3.6.6 For $z_0 \in \Gamma$,

$$\oint_\Gamma \frac{1}{z - z_0} dz = 2\pi i \cdot n(\Gamma, z_0)$$

Proposition 3.6.7 Let $f(\Gamma)$ be which $\Gamma : \gamma(t) : [a,b] \to \Gamma$, then

$$\int_\Gamma \frac{1}{z - z_0} dz = \int_a^b \frac{1}{\gamma(t) - z_0} \gamma'(t) dt$$

so that

$$\int_\Gamma \frac{1}{f(z) - z_0} dz = \int_a^b \frac{1}{f(\gamma(t)) - z_0} f'(\gamma(t))\gamma'(t) dt = \int_{f(\Gamma)} \frac{f'(z)}{f(z) - z_0} dz$$

This gives us that

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z) - z_0} dz = n(f(\Gamma), z_0)$$

Moreover, for $z_0 = 0$ we have that

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z) - z_0} dz = n(f(\Gamma), z_0)$$

■ **Example 3.45** Note that

$$\frac{d}{dz} log(f(z)) = \frac{f'(z)}{f(z)}$$

then

$$\oint_{\Gamma_{i\theta}} \frac{f'(z)}{f(z)} dz = [log(f(z))]_{z_0}^{z_1} = 2\pi i \cdot n(f(\Gamma), 0)$$

Let $f = re^{i\theta}$ we have

$$log(f(z)) = +i\theta$$

■

Theorem 3.6.8 — The Dog-walking Theorem.
Let Γ_1, Γ_2 be parametrized by $\gamma_1, \gamma_2 : [a,b] \to \mathbb{C}$ and $\forall t \in [a,b]$ with $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$. Then $n(\Gamma_1, 0) = n(\Gamma_2, 0)$

Proof: NOte that $\gamma_1, \gamma_2 \neq 0$, consider $\Gamma : \gamma(t) = \frac{\gamma_2(t)}{\gamma_1(t)}$, then

$$|1-\gamma(t)| = \left|1 - \frac{\gamma_2(t)}{\gamma_1(t)}\right| = \left|\frac{\gamma_1(t)-\gamma_2(t)}{\gamma_1(t)}\right| < 1$$

so Γ lies in $D_1(1)$ so $n(\Gamma,0)=0$. Let $\gamma_1 = r_1 e^{i\theta_1}$ and $\gamma_2 = r_1 e^{i\theta_2}$ where $r_1, r_2, \theta_1, \theta_2$ are functions of t, then

$$\gamma = \frac{\gamma_2}{\gamma_1} = \frac{r_2}{r_1 e^{i(\theta_1-\theta_2)}}$$

and

$$n(\Gamma_1,0) = \theta_1(b) - \theta_1(a) \quad \textbf{and} \quad n(\Gamma_2,0) = \theta_2(b) - \theta_2(a)$$

so that

$$0 = n(\Gamma,0) = \theta_2(b) - \theta_2(a) - (\theta_1(b) - \theta_1(a)) = n(\Gamma_2,0) - n(\Gamma_1,0)$$

which is $n(\Gamma_2,0) = n(\Gamma_1,0)$

Theorem 3.6.9 — The Generalized Dog-walking Theorem.
Let Γ_1, Γ_2 be parametrized by $\gamma_1, \gamma_2 : [a,b] \to \mathbb{C}$ and $\forall t \in [a,b]$ with

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| + |\gamma_2(t)|$$

then $n(\Gamma_2,0) = n(\Gamma_1,0)$

Proof: Let $\gamma(t) = \frac{\gamma_1(t)}{\gamma_2(t)}$, assume for contradiction that exist $c > 0$ abd $t \in [a,b]$ such that $\gamma(t) = -c$. Then $\gamma_1(t) = -c\gamma_2(t)$, so that

$$|\gamma_1(t) - \gamma_2(t)| = |(-c-1)\gamma_2(t)| = (c+1)|\gamma_2(t)|$$

but $|\gamma_2(t)| + |\gamma_1(t)| = |\gamma_2(t)| + |-c\gamma_2(t)| = (1+c)|\gamma_2(t)|$, which contradicts the **Dog-walking Theorem**, so there is no such c exists. Then $\Gamma : \gamma(t)$ lies in the $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, so $n(\Gamma,0) = 0$, that is $n(\Gamma_2,0) = n(\Gamma_1,0)$.

Theorem 3.6.10 — Rouche's Theorem.
Let f, g be analytic on and inside a simple closed contour Γ. Let $|g(z)| < |f(z)|$ for all $z \in \Gamma$, then $f+g$ and f have the same number of zeros (connted with multiplicity inside)

Proof: Let $h = f+g$ then

$$|h(z) + (-f(z))| = |g(z)| < |-f(z)|$$

on Γ. Then

$$n(h(\Gamma),0) = n(f(\gamma),0) \quad \textbf{that is} \quad N_0(h) = N_0(f)$$

■ **Example 3.46** All 5 zeros of $h(z) = z^5 + 3z + 1$ lie inside $|z| < 2$. Let $f(z) = z^5$ and $g(z) = 3z+1$ on $C_2(0)$, so $|f(z)| = 32$ and $|g(z)| = 7 < |f(z)|$, so by **Rouche's Theorem** h and f have same number of zeros inside $C_2(0)$ ■

■ **Example 3.47** How many zeros does $z+3+2e^z$ have in the left half-plane $\text{Re}(z) < 0$? Let Γ_R be the contour as "D" reflect by y-axis. Let $f(z) = z+3$ and $g(z) = 2e^z$, so $|g(z)| = 2e^{\text{Re}(z)}$ so $|g(z)| \leq 2$ on Γ_R for all R and

$$|f(z)| \geq \begin{cases} |3+iy| & z = iy \\ R-3 & |z| = R \end{cases} \geq \begin{cases} 3 & z = iy \\ R-3 & |z| = R \end{cases}$$

so for all $R > 5$ we have $|f(z)| > |g(z)|$ on Γ_R, then f has the same number of zeros inside Γ_R as $z+3+2e^z$, $f(z) = z+3$ has one zero inside Γ_R namely -3, so $z+3+2e^z$ has exactly one zero in the left half-plane. ■

Definition 3.6.5
A point z is a **limit point** of a set if there exists a sequence $\{z_n\} \subseteq S$ with $z_n \neq z$ but $\lim_{n\to\infty} z_n = z$

Theorem 3.6.11 Let f be holomorphic on a domain D, let $Z \subseteq D$ be the set of zeros of D if Z has a limit point in D, f is identically zero on D

Proof: Let z_0 be the limit of $\{w_n\} \subseteq Z$ and $z_0 \neq w_n$ for all n, conside $D_\varepsilon(z_0)$ for some sufficiently small $\varepsilon > 0$, that is

$$f(z) = \sum_{n=0}^{\infty} z_n (z-z_0)^n$$

on $D_\varepsilon(z_0)$. If f is not identically 0 on $D_\varepsilon(z_0)$, then there exists a minimal $m \geq 0$ such that $a_m \neq 0$, write

$$f(z) = a_m (z-z_0)^m (1+g(z-z_0))$$

where $g(z-z_0) \to 0$ as $z \to z_0$. Let k be sufficiently large that $w_k \in D_\varepsilon(z_0)$, $w_k \in D_\varepsilon(z_0)$ for all $K \geq k$. Now $f(w_k) = 0$ but

$$0 = f(w_k) = a_m (w_k - z_0)^n (1+g(w_k - z_0))$$

and $a_m \neq 0$, $(w_k - z_0)^n \neq 0$ and $g(w_k - z_0) \to 0$ as w_{k0} as $k \to \infty$. so for large enough k, $|g(w_k - z_0)| < 1$, so $1+g(w_k - z_0) \neq 0$, which is a contradiction, so $f = 0$ on $D_\varepsilon(z_0)$. Let U be the interior of Z, we just showed that U is non-empty, U is open by definition, let $\{z_n\} \subseteq U$ converging $z_n \to z$, f is continuous so $f(z) = 0$, by earlier argument, $z \in U$. Then U is closed, so $V = D \setminus U$ is open we have $D = U \cup V$ and $U \cap V$, U, V are open and D is connected, so one of U, V is empty. U is non-empty, so $V = \emptyset$ so $U = D$, then f is 0 on D

Corollary 3.6.12 Let f, g analytic on D and $f(z) = g(z)$ on $S \subseteq D$ where S has limit point in D, then $f(z) = g(z)$ on D

Proof: apply the above theorem to $f - g$.

Corollary 3.6.13 Let f be analytic and non-constant on a domain D and let $z_0 \in D$, $f(z_0) = w_0$, then there exist $\varepsilon > 0$ such that $\overline{D_\varepsilon(z_0)} \subseteq D$ and $f(z) - w_0$ has zero in $\overline{D_\varepsilon(z_0)} \setminus \{z_0\}$

Proof: Let $f(z) - w_0$ is a non-constant analytic function, so its zero cannot have a limit point, done.

Theorem 3.6.14 — Open Mapping Theorem.
If f is holomorphic on a domain D, then f is a open map on D (map open set to to open set)

Proof: It suffices to show $f(D)$ is open. Let $z_0 \in D$ and $f(z_0) = w_0$, let $w \in \mathbb{C}$ and

$$g(z) = f(z) - w = f(z) - w_0 + w_0 - w$$

Choose $\delta > 0$ such that $D_\delta(z_0) \subseteq D$ and such that $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$ which exists by the previous corollary. Now we choose $\varepsilon > 0$ such that $|f(z) - w_0| \geq \varepsilon$ on $|z - z_0| = \delta$, so for all $w \in D_\varepsilon(w_0)$ we have $|f(z) - w_0| \geq \varepsilon > |w - w_0|$ on the circle $|z - z_0| = \delta$, so by **Rouche's Theorem** g and $f(z) - w_0$ have the same number of zeros in $D_\delta(z_0)$, namely one. Then $\exists z \in D_\delta \subseteq D$, $g(z) = 0 = f(z) - w \implies f(z) = w \implies w \in f(D)$, then $D_\varepsilon(w_0) \subseteq f(D)$

■ **Example 3.48** Let f be analytic on a domain D and $\text{Re}(f(z))$ is constant , then f is constant, $\text{Re}(f(z)) = K$ contains no open set, so f must be constant by the contrapositive of open mapping theorem. ■

Definition 3.6.6 — Gamma Function.
The **gamma function** is defined for $s > 0$ in $\mathbb{R}$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

Lemma 3.6.15 Γ extends to an analytic function on $\text{Re}(s) > 0$ and

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

still holds there

Proof: It suffices to show lemma on

$$S = \{z \in \mathbb{C} : \delta < \text{Re}(s) < M\}$$

for any $0 < \delta < M < \infty$. Let $\text{Re}(s) = \sigma$, now

$$\int_0^\infty e^{-t} t^{s-1} dt = \lim_{\varepsilon > 0} \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt$$

Consider

$$F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt$$

Note that $F_\varepsilon(s)$ is analytic with

$$F'_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} (s-1) t^{s-2} dt$$

Recall that the limit of a uniformly convergent sequence of analytic function is analytic. Consider

$$\begin{aligned}|\Gamma(s) - F_\varepsilon(s)| &= \left|\int_0^\infty e^{-t}t^{s-1}dt - \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}dt\right| \\ &= \left|\int_0^\varepsilon e^{-t}t^{s-1}dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{s-1}dt\right| \\ &\le \int_0^\varepsilon e^{-t}t^{\sigma-1}dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}dt\end{aligned}$$

Now for $\varepsilon < 1$:

$$\left|\int_0^\varepsilon e^{-t}t^{\sigma-1}dt\right| \le \varepsilon \cdot \frac{1}{\delta} \cdot \varepsilon^{\delta-1} = \frac{\varepsilon^\delta}{\delta}$$

Similarly we have

$$\left|\int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}dt\right| \le \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{M-1}dt \to 0$$

as $\frac{1}{\varepsilon} \to 0$.

Then $F_\varepsilon(s) \to \Gamma(s)$ uniformly, so Γ is analytic on S so Γ is analytic on $\mathrm{Re}(s) > 0$

Proposition 3.6.16

Let $n \in \mathbb{Z}_{\ge 0}$, then $\Gamma(n+1) = n!$

Lemma 3.6.17 For $0 < \mathrm{Re}(a) < 1$, then

$$\int_0^\infty \frac{v^{a-1}}{1+v}dv = \frac{\pi}{\sin(\pi a)}$$

Proof: Let $v = e^x$, then

$$\int_0^\infty \frac{v^{a-1}}{1+v}dv = \int_{-\infty}^\infty \frac{e^{(a-1)x}}{1+e^x}dx = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx$$

Let $f(z) = \frac{e^{az}}{1+e^z}$ and integrate over a region. That is

$$\left|\int_{\Gamma_2} f(z)dz\right| = \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}}dt\right| \le C \cdot \frac{e^{aR}}{e^R} \sim Ce^{(a-1)R} \to 0 \ \textbf{as} \ R \to \infty$$

$$\left|\int_{\Gamma_4} f(z)dz\right| \le Ce^{-aR} \to 0 \ \textbf{as} \ R \to \infty$$

$$\int_{\Gamma_1} f(z)dz = \int_{-R}^R \frac{e^{ax}}{1+e^x}dx$$

$$\int_{\Gamma_3} f(z)dz = -e^{2\pi ia}\int_{-R}^R \frac{e^{ax}}{1+e^x}dx$$

Note that f has a pole at $z = \pi i$, then

$$\begin{aligned}\lim_{z\to\pi i}(z-\pi i)f(z) = \lim_{z\to\pi i}(z-\pi i)\frac{e^{az}}{1+e^z} &= \lim_{z\to\pi i} e^{az}\left(\frac{z-\pi i}{e^z - e^{\pi i}}\right)\\ &= e^{a\pi i}\left(\lim_{z\to\pi i}\frac{e^z-e^{\pi i}}{z-\pi i}\right)^{-1}\\ &= e^{a\pi i}\cdot(e^{\pi i})^{-1}\\ &= -e^{a\pi i}\\ &= \mathrm{res}_{\pi i}(f)\end{aligned}$$

Then we have

$$\int_\Gamma f(z)dz = 2\pi i(-e^{a\pi i}) = (1-e^{2\pi i a}) = I$$

so that

$$I = -2\pi i\frac{e^{a\pi i}}{1-e^{2\pi i}} = \frac{\pi}{\sin(\pi a)}$$

Theorem 3.6.18

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Proof: IT suffices to show this on $0 < \mathrm{Re}(s) < 1$:

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{(1-s)-1}du = \int_0^\infty e^{-u}u^{-s}du$$

Let $u = vt$, $v > 0$, then

$$\Gamma(1-s) = t\int_0^\infty e^{-vt}(vt)^{-s}dv$$

This give us

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty e^{-t}t^{s-1}\left(\int_0^\infty te^{-vt}(vt)^{-s}dv\right)dt = \int_0^\infty\int_0^\infty e^{-t(v+1)}v^{-s}dvdt = \int_0^\infty \frac{v^{-s}}{v+1}dv$$

By the lemma from above we have

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{v^{-s}}{v+1}dv = \frac{\pi}{\sin(\pi s)}$$

Definition 3.6.7 — Riemann Zeta Function.
For real $s > 1$ as

$$\zeta(s) = \sum_{i=1}^\infty \frac{1}{n^s}$$

so ζ immediately has an analytic continuation to $\mathrm{Re}(s) > 1$ and the formula

$$\zeta(s) = \sum_{i=1}^\infty \frac{1}{n^s}$$

is still valid. If $s=\sigma+it$ for $\sigma,t\in\mathbb{R}$, if $\sigma>1+>1$, then

$$\left|\sum_{i=1}^{\infty}\frac{1}{n^s}\right|\leq\sum_{i=1}^{\infty}\left|\frac{1}{n^s}\right|=\sum_{i=1}^{\infty}\left|\frac{1}{e^{s\log(n)}}\right|=\sum_{i=1}^{\infty}\frac{1}{e^{\sigma\log(n)}}=\sum_{i=1}^{\infty}\frac{1}{n^{\sigma}}\leq\sum_{i=1}^{\infty}\frac{1}{n^{1+\delta}}$$

then $\zeta(s)$ is analytic on $\mathrm{Re}(s)>1$. Consider the Euler product:

$$\prod_{\textbf{p price}}\frac{1}{1-p^s}$$

for $\mathrm{Re}(s)>1$, then we see that

$$\frac{1}{1-p^{-s}}=\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+.....\right)=\sum_{i=1}^{\infty}\frac{1}{p^{ns}}$$

then

$$\prod_{s}\frac{1}{1-p^{-s}}=\left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}\right)\left(1+\frac{1}{3^s}+\frac{1}{9^s}+...\right)\left(1+\frac{1}{5^s}+\frac{1}{25^s}+...\right)$$

but we have unique factorization of positive integers:

$$\textbf{formula above}=\sum_{j_1,j_2,...}\left(\frac{1}{2^{j_1}3^{j_2}5^{j_3}...}\right)^s=\sum_{i=1}^{\infty}\frac{1}{n^s}=\zeta(s)$$

Theorem 3.6.19 $\zeta(s)-\frac{1}{s-1}$ has an analytic continuation to $\mathrm{Re}(s)>0$. Then $\zeta(s)$ is rercmorphic on $\mathrm{Re}(s)>0$ with a simple pole of residu 1 at $s=1$

Proof: Consider

$$\sum_{1\leq n\leq N}\frac{1}{n^s}-\int_1^n\frac{1}{x^s}ds$$

Let

$$_n(s)=\int_n^{n+1}\frac{1}{n^s}-\frac{1}{x^s}ds$$

By the mean value theorem

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right|\leq\frac{|s|}{n^{\sigma+1}}$$

on $n\leq x\leq n+1$. Then we have uniform convergence of $_n(s)$ on $1+\sigma>1\iff\mathrm{Re}(s)>0$. Then

$$\sum_{i=1}^{\infty}\delta_n(s)$$

is analyticiction on $\mathrm{Re}(s)>0$. Now

$$\sum_{1\leq n\leq N}\frac{1}{n^s}=\sum_{n=1}^{N}\delta_n(s)+\int_1^N\frac{1}{x^s}ds$$

Now

$$\lim_{N\to\infty}\int_1^N \frac{1}{x^s}dx = \int_1^\infty \frac{1}{x^s}dx = \left[\frac{1}{1-s}x^{1-s}\right]_0^\infty$$

on $\text{Re}(s) > 1$, this $= \frac{1}{s-1}$. Thus,

$$\sum_{i=1}^{N} {}_n(s) + \int_1^N \frac{1}{x^s}ds$$

converges uniformly and matches $\zeta(s)$ on $\text{Re}(s) > 1$, so

$$\zeta(s) - \frac{1}{s-1} = \sum_{i=1}^{\infty} {}_n(s)$$

is analytic on $\text{Re}(s) > 0$

Theorem 3.6.20 $\zeta(s)$ has no zeros on the line $\text{Re}(s) = 1$

Proof: Let $x, y \in \mathbb{R}$, $y \neq 0$ and define

$$h(x) = \zeta^3(x)\zeta^4(x+iy)\zeta(x+2yi)$$

Now

$$\zeta(s) = \prod_s \frac{1}{1-p^{-s}}$$

so

$$\ln|\zeta(s)| = \ln\prod_p \left|\frac{1}{1-p^{-s}}\right| = -\sum_p \ln|1-p^{-s}| = -\text{Re}\sum_p \log\left(1-p^{-s}\right)$$

Now

$$-\log(1-w) = \sum_{i=1}^{\infty}\frac{w^n}{n}$$

for $|w| < 1$, so

$$\ln|\zeta(s)| = \text{Re}\sum_p\sum_n \frac{1}{n}p^{-sn}$$

and then

$$\begin{aligned}\ln|h(x)| &= 3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+2iy)| \\ &= 3\,\text{Re}\sum_p\sum_p \frac{1}{n}p^{-nx} + 4\,\text{Re}\sum_p\sum_n \frac{1}{n}p^{-ns-iny} + \text{Re}\sum_p\sum_n \frac{1}{n}p^{-nx-2iny} \\ &= \sum_p\sum_n \frac{1}{n}p^{-nx}\,\text{Re}\left(3 + 4p^{-iny} + p^{-2iny}\right)\end{aligned}$$

Note that

$$p^{-iny} = e^{nyi\ln(p)} \quad \textbf{has} \quad \text{Re}(p^{iny}) = \cos(-ny\ln(p))$$

and

$$\text{Re}(p^{-2iny}) = \cos(-2ny\ln(p))$$

so that

$$\text{Re}\left(3+4p^{-iny}+p^{-2iny}\right) = 3+4\cos(-ny\ln(p))+\cos(-2ny\ln(p))$$

Let $\theta = -ny\ln(p)$, so this is

$$\text{Re}\left(3+4p^{-iny}+p^{-2iny}\right) = 3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0$$

Then we have $\ln|h(x)| \geq 0$, so $|h(x)| \geq 1$ we have

$$\frac{|h(x)|}{x-1} = |(x-1)\zeta(x)|^3 \left|\frac{\zeta(x+iy)}{x-1}\right|^4 |\zeta(x+2iy)| \geq \frac{1}{x-1}$$

As $x \to 1^+$ we have $|\zeta(x+2iy)| \to |\zeta(1+2iy)|$

$$\lim_{x\to 1} |(x-1)\zeta(x)| = 1$$

if $\zeta(1+iy) = 0$, but

$$\lim_{x\to 1^+} \frac{\zeta(x+iy)}{x-1} = \zeta'(1+iy)$$

so $\lim_{x\to 1^+} \dfrac{|h(x)|}{x-1}$ converges to some finite value. but this is $\geq \frac{1}{x-1}$ and $\lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$, this is contradiction so $\zeta(1+iy) \neq 0$