# PMATH 450 WINTER 2021

Lebesgue Integration and Fourier Analysis

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## Lecture Notes

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# Contents

# 1. Outer Measure

## 1.1 Borel Sets

**Definition 1.1.1 — $\sigma$-algebra.**
Let $X$ be a set, we call $Q \subseteq \mathcal{P}(X)$ a **$\sigma$-algebra** of the subset $X$ if

(1) $\emptyset \in Q$
(2) $A \in Q \Longrightarrow X \setminus A \in Q$
(3) $A_1, A_2, .... \in Q \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in Q$

■ **Remark 1.1** For $Q \in \mathcal{P}(X)$ is a **$\sigma$-algebra**:

1. $X \in Q$ and $X \setminus \emptyset = X \in Q$
2. $A, B \in Q \Longrightarrow A \cup B \in Q$ by using $A \cup B = A \cup B \cup \emptyset \cup \emptyset \ldots \in Q$
3. $A_1, A_2, .... \in Q \Longrightarrow \bigcap_{i=1}^{\infty} A_i \in Q$ by using $\bigcap_{i=1}^{\infty} A_i = X \setminus \left( \bigcup_{i=1}^{\infty} X \setminus A_i \right)$
4. $A, B \in Q \Longrightarrow A \cap B \in Q$

■ **Example 1.1** $\{\emptyset, X\}$ is the smallest **$\sigma$-algebra** where given a set $X$ ■

■ **Example 1.2** $Q = P(X)$ is a **$\sigma$-algebra** ■

■ **Example 1.3** $Q = \{A \subseteq \mathbb{R} : \text{A is open}\}$ is not a **$\sigma$-algebra**. We take $A = (0, 1) \in Q$ but $\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin Q$ ■

■ **Example 1.4** $Q = \{A \subseteq \mathbb{R} : \text{A is open or closed}\}$ is not a $\sigma$-**algebra**. $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\} \notin Q$ because $\mathbb{Q}$ is neither open or closed set. ■

**Proposition 1.1.1**
Let $X$ be a set, $C \subseteq \mathcal{P}(X)$, then $Q \coloneqq \bigcap \{B : \text{B is a } \sigma\text{-}\mathbf{algebra},\ C \subseteq B\}$ is a $\sigma$-**algebra**, and it's also the smallest $\sigma$-**algebra** containing $C$

**Definition 1.1.2 — Borel Set.**
The elements of $Q = \bigcap \{B : C \subseteq B, \text{B is } \sigma\text{-algebra}\}$ **(Borel $\sigma$-algebra)** are called **Borel sets** where $C = \{A \subseteq \mathbb{R} : \text{A is open}\}$

■ **Remark 1.2**
1. **Open set** $\Longrightarrow$ **Borel set**
2. **Closed set** $\Longrightarrow$ **Borel set**
3. **Countable set** $\Longrightarrow$ **Borel set** i.e. $\{X_1, ....\} = \bigcup_{i=1}^{\infty} X_i \Longrightarrow$ **Borel set**
4. $[a, b) = [a, b] \setminus \{b\} = \underbrace{[a, b]}_{\textbf{closed}} \cap \underbrace{(\mathbb{R} \setminus \{b\}}_{\textbf{open})} \Longrightarrow$ **Borel set**

## 1.2 Outer Measure 1

**Definition 1.2.1 — Measure.** (on $\mathbb{R}$)
A function $m : \mathcal{P}(\mathbb{R}) \longrightarrow [0, \infty) \cup \{\infty\}$ called a **measure** if:

(1) $m(a, b) = m([a, b]) = m((a, b]) = b - a$
(2) $m(A \cup B) \leq m(A) + m(B)$
(3) $A \cap B = \emptyset \Longrightarrow m(A \cup B) = m(A) + m(B)$

**Definition 1.2.2 — (Lebegue) Outer Measure.**
**Outer Measure** is a function $m^* : \mathcal{P}(\mathbb{R}) \Longrightarrow [0, \infty) \cup \{\infty\}$ where

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \ell(I_i) : \ A \subseteq \bigcup_{i=1}^{\infty} I_i \ \text{ where } I_i \text{ is bounded, open interval} \right\}$$

• $\ell(I_i)$ is the length of the interval $I_i$

■ **Example 1.5** For $\varepsilon > 0$, $\emptyset \subseteq (0, \varepsilon) \Longrightarrow m^*(\emptyset) \leq \ell(0, \varepsilon) = \varepsilon$ and $m^*(\emptyset) \geq 0 \Longrightarrow m^*(\emptyset) = 0$ ■

■ **Example 1.6** $m^*(A) = 0$ where $A = \{X_1, X_2, .....\}$

Proof: Note that $A \subseteq \bigcup_{i=1}^{\infty} \left(X_i - \frac{\varepsilon}{2^{i+1}}, X_i + \frac{\varepsilon}{2^{i+1}}\right)$ for $\varepsilon > 0$, then

$$m^*(A) \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2} \cdot \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\varepsilon}{2} \cdot \left\{ \frac{1}{1 - \frac{1}{2}} \right\} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, so we have $m^*(A) = 0$ as desired. ■

## 1.3 Outer Measure 2

**Proposition 1.3.1** If $A \subseteq B$, then $m^*(A) \leq m^*(B)$

**Lemma 1.3.2** If $a, b \in \mathbb{R}$ with $a \leq b$, then $m^*([a, b]) = b - a$

*Proof.* Let $\varepsilon > 0$ be given, since $[a, b] \subseteq (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$, we have $m^*([a, b]) \leq b - a + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, so by definition of outer measure we have $m^*([a, b]) \leq b - a$. Let $I_i$ $(i \in \mathbb{N})$ be bounded open interval s.t. $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Note that $[a, b]$ is compact, so $\exists n \in \mathbb{N}$ s.t. $[a, b] \subseteq \bigcup_{i=1}^{m} I_i$. Then we have

$$b - a \leq \sum_{i=1}^{n} \ell(I_i) \leq \sum_{i=1}^{\infty} \ell(I_i) \quad \Longrightarrow m^*([a, b]) \geq b - a$$

so we have $b - a \leq m^*([a, b]) \leq b - a$, this gives us $m^*([a, b]) = b - a$ ■

**Proposition 1.3.3** If $I$ is an interval, then $m^*(I) = \ell(I)$

*Proof.* When $I$ is bounded with endpoints where $a \leq b$, so for $\varepsilon > 0$,$I \subseteq [a, b] \Longrightarrow m^*(I) \leq b - a$ and $[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}] \subseteq I \Longrightarrow b - a - \varepsilon \leq m^*(I)$ By definition of outer measure we have $b - a \leq m^*(I)$. Then we have $m^*(I) + b - a = \ell(I)$ as desired

When $I$ is unbounded, $\forall n \in \mathbb{N}$, $\exists I_n \subseteq I$ such that $\ell(I_n) = n$. This gives us that $m^*(I) \geq m^*(I_n) = n$, then $m^*(I) = \infty = \ell(I)$ as desired.

Hence, we have $m^*(I) = \ell(I)$, which completes the proof. ■

## 1.4 Properties

**Proposition 1.4.1 — Outer Measure is Translation Invariant.** i.e. $m^*(x+A) = m^*(A)$

*Proof.*

$$\begin{aligned} m^*(x+A) = \inf\left\{\sum \ell(I_i) : \ x+A \subseteq \bigcup_{i=1}^{\infty} I_i\right\} &= \inf\left\{\sum \ell(I_i) : \ A \subseteq \bigcup_{i=1}^{\infty} (I_i - x)\right\} \\ &= \inf\left\{\sum \underbrace{\ell(I_i - x)}_{J_i} : \ A \subseteq \bigcup_{i=1}^{\infty} (I_i - x)\right\} \\ &= \inf\left\{\sum \ell(J_i) : \ A \subseteq \bigcup_{i=1}^{\infty} J_i\right\} \\ &= m^*(A) \end{aligned}$$

■

**Proposition 1.4.2 — Outer Measure has Countably Subadditivity.**
That means if $A_i \subseteq \mathbb{R}$, then $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$

*Proof.* **WLOG**, we assume $m^*(A_i) < \infty$. Let $\varepsilon > 0$ be given and fix $i \in \mathbb{N}$. Then there exists open bounded intervals $I_{i,j}$ s.t. $A \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ and $\sum_{i=1}^{\infty} \ell(I_{i,j}) \leq m^*(A_i) + \frac{\varepsilon}{2^i}$. We can see that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j}^{\infty} I_{i,j}$ and so

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i,j} \ell(I_{i,j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{i,j}) \leq \sum_{i=1}^{\infty}\left(m^*(A_i) + \frac{\varepsilon}{2}\right) = \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, so we have $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$ as desired. ■

**Corollary 1.4.3 — Outer Measure has Finitely Subadditivity.**
If $A_1, A_2, \ldots.., A_n \subseteq \mathcal{P}(\mathbb{R})$, then

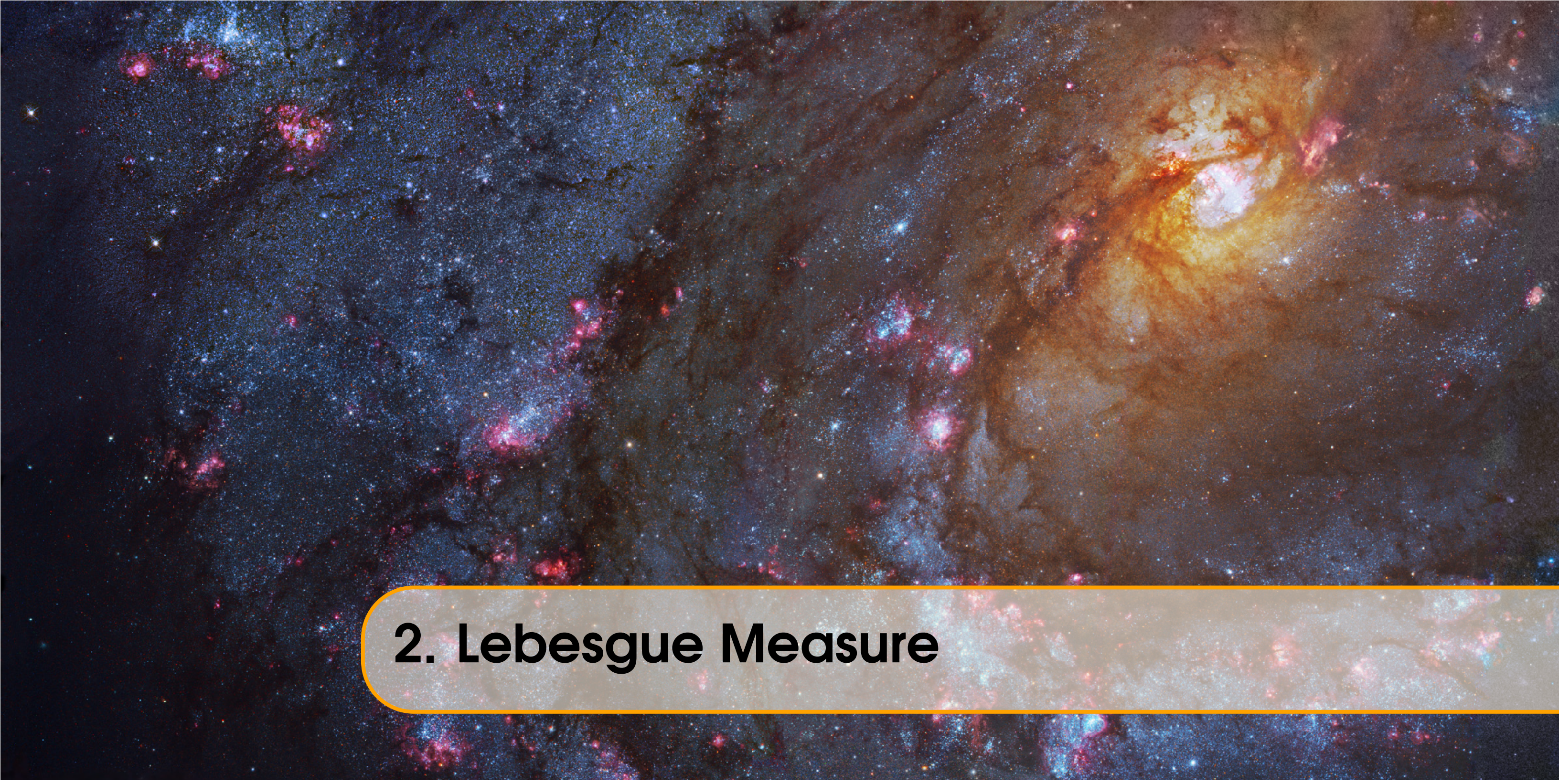
$$m^*(A_1 \cup \ldots \cup A_n) \leq m^*(A_1) + \ldots + m^*(A_n)$$

**Exercise 1.4.1**
Prove that if $A \subseteq \mathbb{R}$ has positive outer measure, then there exists a bounded subset of $A$ which also has positive outer measure.

**Solution:** For sake of contradiction, we suppose every bounded subset of $A$ has 0 outer measure. Since $A \subseteq \mathbb{R}$ has positive outer measure, so we say $m^*(A) > 0$. Now we construct a sequence of bounded subset of $A$. Consider $A_i = A \cap [n, n+1]$ for all $n \in \mathbb{Z}$, then we have $A = \bigcup_{i \in \mathbb{Z}} A_i$. Then we have

$$0 < m^*(A) = m^*\left(\bigcup_{i \in \mathbb{Z}} A_i\right) \leq \sum_{i \in \mathbb{Z}} m^*(A_i) = \sum_{i \in \mathbb{Z}} 0 = 0$$

That gives $0 < 0$, which is a **contradiction!** Hence, there exists a bounded subset of $A$ has positive outer measure, which completes the proof. ■



# 2. Lebesgue Measure

## 2.1 Measurable Sets

**Goal:** Restrict the domain of $m^*$ to only include sets s.t. whenever $A \cap B = \emptyset$ we have

$$m^*(A \cup B) = m^*(A) + m^*(B)$$

**Definition 2.1.1 — Measurable Set.**
We say a set $A \subseteq \mathbb{R}$ is **measurable** if $\forall X \subseteq \mathbb{R}$, $m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$

■ **Remark 2.1** Since $X = (X \cap A) \cup (X \setminus A)$, so we always have $m^*(X) \leq m^*(X \cap A) + m^*(X \setminus A)$

■ **Remark 2.2** If $A \subseteq \mathbb{R}$ is **measurable** and $B \subseteq \mathbb{R}$ with $A \cap B = \emptyset$, then

$$m^*(\underbrace{A \cup B}_{X}) = m^*(X \cap A) + m^*(X \setminus A) = m^*(A) + m^*(B)$$

**Goal:** Show a lot of sets are measurable

**Proposition 2.1.1**
If $m^*(A) = 0$, then $A$ is **measurable**

**Proof:** Let $X \subseteq \mathbb{R}$, since $X \cap A \subseteq A$, we have $0 \leq m^*(X \cap A) \leq m^*(A) = 0$. Then we have that $m^*(X \cap A) = 0$, so

$$m^*(X \cap A) + m^*(X \setminus A) = m^*(X \setminus A) \leq m^*(X)$$

**Proposition 2.1.2**

$A_1, A_2, ...., A_n$ are measurable, then $\bigcup_{i=1}^{n} A_i$ is measurable.

**Proof:** It suffices to prove the result when $n = 2$. Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(X \setminus A) \\ &= m^*(X \cap A) + m^*((X \setminus A) \cap B) + m^*((X \setminus A) \setminus B) \\ &= m^*(X \setminus A) + m^*((X \setminus A) \cap B) + m^*(X \setminus (A \cup B)) \\ &\geq m^*((X \cap A) \cup ((X \setminus A) \cap B)) + m^*(X \setminus (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B)) \end{aligned}$$

Note that $X = (X \cap (A \cup B)) \cup (X \setminus (A \cup B))$, then

$$m^*(X) \leq m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B))$$

Therefore, we have $\forall X \subseteq \mathbb{R}$, $m^*(X) = m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B))$ as desired.

**Proposition 2.1.3**

$A_1, ....A_n$ are measurable and $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $A = A_1 \cup \ldots \cup A_n$, if $X \subseteq \mathbb{R}$, then

$$m^*(X \cap A) = \sum_{i=1}^{n} m^*(X \cap A_i)$$

**Proof:** It suffices to prove the result when $n = 2$. Let $A, B \subseteq \mathbb{R}$ be measurable set with $A \cap B = \emptyset$. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} m^*(X \cap (A \cup B)) &= m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \setminus A) \\ &= m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

**Corollary 2.1.4 — Finite Additivity.**

Let $A_1, ...., A_n$ be measurable sets and $A_i \cap A_j \neq \emptyset$ for $i \neq j$, then

$$m^*(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} m^*(A_i)$$

## 2.2 Countable Additivity

**Lemma 2.2.1**
Let $A_i \subseteq \mathbb{R}$ be measurable sets for $i \in \mathbb{N}$, if $A_i \cap A_j \neq \emptyset$ for $i \neq j$, then

$$A := \bigcup_{i=1}^{\infty} A_i$$

is measurable.

**Proof:** Let $B_n := A_1 \cup \ldots \cup A_n$, so for $X \subseteq \mathbb{R}$ we have

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \setminus B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \setminus A) \\ &\stackrel{\textbf{prop}}{=} \sum_{i=1}^{n} m^*(X \cap A_i) + m^*(X \setminus A) \end{aligned}$$

By taking $n \to \infty$, we have

$$\begin{aligned} m^*(X) &\geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \setminus A) \\ &\geq m^*\left(\bigcup_{i=1}^{\infty} (X \cap A_i)\right) + m^*(X \setminus A) \\ &= m^*(X \cap A) + m^*(X \setminus A) \end{aligned}$$

as desired.

**Proposition 2.2.2**
If $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \setminus A$ is measurable.

**Proof:** Let $X \subseteq \mathbb{R}$, so

$$\begin{aligned} m^*(X \cap (\mathbb{R} \setminus A)) + m^*(X \setminus (\mathbb{R} \setminus A)) &= m^*(X \setminus A) + m^*(X \cap A) \\ &= m^*(X) \end{aligned}$$

**Proposition 2.2.3**
Let $A_i \subseteq \mathbb{R}$ me measurable for $i \in \mathbb{N}$, then $A = \bigcup_{i=1}^{\infty} A_i$ is measurable.

**Proof:** Let $B_n = A_n \setminus (A_1 \cup \ldots A_n)$ for $n \geq 2$, so we have

$$B_n = \underbrace{A_n}_{\textbf{measurable}} \cap \left( \underbrace{\mathbb{R} \setminus (A_1 \cup \ldots A_{n-1})}_{\textbf{measurable}} \right)$$

Then we have $B_n$ is measurable and for $i \neq j$, $B_i \cap B_j = \emptyset$. This gives us that $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ is measurable as desired.

**Corollary 2.2.4**
The collection $\mathscr{L}$ of (Lebesgue) measurable sets is a $\sigma$-algebra of sets in $\mathbb{R}$

**Proposition 2.2.5 — Countable Additivity.**
Let $A_i \subseteq \mathbb{R}$ be measurable for $i \in \mathbb{N}$, if $A_i \cap A_j \neq \emptyset$ for $i \neq j$, then

$$m^* \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} m^*(A_i)$$

**Proof:** Obviously we have $m^* \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} m^*(A_i)$, and note that

$$m^* \left( \bigcup_{i=1}^{\infty} A_i \right) \geq m^* \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} m^*(A_i)$$

By taking $n \to \infty$ we have $m^* \left( \bigcup_{i=1}^{\infty} A_i \right) \geq \sum_{i=1}^{\infty} m^*(A_i)$, which completes the proof.

## 2.3 Borel Implies Measurable

**Goal 1:** Show Borel sets are measurable.

**Proposition 2.3.1**
If $a \in \mathbb{R}$, then $(a, \infty)$ is measurable.

**Proof:** Let $X \subseteq \mathbb{R}$, we want to show that $m^*(X \cap (a, \infty)) + m^*(X \setminus (a, \infty)) \leq m^*(X)$

**Case 1**: $a \notin X$, we will show $m^*(\underbrace{X \cap (a,\infty)}_{X_1}) + m^*(\underbrace{X \cap (-\infty,a)}_{X_2}) \leq m^*(X)$

Let $(I_i)$ be a sequence of bounded open intervals s.t. $X \subseteq \bigcup_{i=1}^{\infty} I_i$. Define $I_i' = I_i \cap (a,\infty)$ and $I_i'' = I_i \cap (-\infty,a)$. Note that

$$X_1 \subseteq \bigcup_{i=1}^{\infty} I_i' \quad \text{and} \quad X_2 \subseteq \bigcup_{i=1}^{\infty} I_i''$$

so we have

$$m^*(X_1) \leq \sum_{i=1}^{\infty} \ell(I_i') \quad \text{and} \quad m^*(X_2) \leq \sum_{i=1}^{\infty} \ell(I_i'')$$

Then we see that

$$m^*(X_1) + m^*(X_2) \leq \sum_{i=1}^{\infty} \ell(I_i') + \sum_{i=1}^{\infty} \ell(I_i'') = \sum_{i=1}^{\infty} \left[\ell(I_i') + \ell(I_i'')\right] = \sum_{i=1}^{\infty} \ell(I_i)$$

By the definition of inf, we have

$$m^*(X_1) + m^*(X_2) \leq m^*(X)$$

**Case 2:** $a \in X$, left it as exercise. Hint: $X' = X \setminus \{a\}$

**Theorem 2.3.2 — Every Borel Set is measurable.**

**Proof:** omitted

**Definition 2.3.1 — Lebesgue Measure.**
A function $m : \mathscr{L} \to [0,\infty) \cup \{\infty\}$ defined by $m(A) = m^*(A)$ is called **Lebesgue Measure**

## 2.4 Properties

**Proposition 2.4.1 — Excision Property.**
If $A \subseteq B$ and $A$ is measurable with $m(A) < \infty$. Then

$$m^*(B \setminus A) = m^*(B) - m(A)$$

**Proof:**

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \setminus A) \\ &= \underbrace{m(A)}_{<\infty} + m^*(B \setminus A) \quad \textbf{since } m^*(B \cap A) = m^*(A) = m(A) \end{aligned}$$

**Theorem 2.4.2 — Continuity of Measure.**

1. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ are measurable, then

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$$

2. If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ are measurable and $m(B_1) < \infty$, then

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$$

**Proof for 1:** Since $m(A_k) \leq m\left(\bigcup_{i=1}^{\infty} A_i\right)$ for all $k \in \mathbb{N}$, we have $\lim_{n\to\infty} m(A_n) \leq m\left(\bigcup_{i=1}^{\infty} A_i\right)$. If $\exists k \in \mathbb{N}$ such that $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done. Then we may assume each $m(A_k) < \infty$. For each $k \in \mathbb{N}$, let $D_k = A_k \setminus A_{k-1}$ and $A_0 = \emptyset$. Note that $D_k$'s are measurable and they are pairwise disjoint. We also have $\bigcup_{i=0}^{\infty} D_i = \bigcup_{i=0}^{\infty} A_i$, then

$$\begin{aligned} m\left(\bigcup_{i=0}^{\infty} A_i\right) &= m(\bigcup_{i=0}^{\infty} D_i) \\ &= \sum_{i=0}^{\infty} m(D_i) \quad \textbf{by Prop 2.2.5} \\ &= \sum_{i=1}^{\infty} (m(A_i) - m(A_{i-1})) \quad \textbf{by Prop 2.4.1} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (m(A_i) - m(A_{i-1})) \\ &= \lim_{n\to\infty} m(A_n) - \underbrace{m(A_0)}_{=0} \quad \textbf{since } A_0 = \emptyset \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

as desired.

**Proof for 2:** For $k \in \mathbb{N}$, we define $D_k = B_1 \setminus B_k$. Note that $D_k$'s are measurable and

$D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$. Then by **1** we have

$$m\left(\bigcup_{i=1}^{\infty} D_i\right) = \lim_{n\to\infty} m(D_n)$$

and we see that

$$\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} B_1 \setminus B_i = B_1 \setminus \left(\bigcap_{i=1}^{\infty} B_i\right)$$

and so

$$\begin{aligned}\lim_{n\to\infty} m(D_n) = m\left(\bigcup_{i=1}^{\infty} D_i\right) &= m\left(B_1 \setminus \left(\bigcap_{i=1}^{\infty} B_i\right)\right) \\ &= m(B_1) - m\left(\bigcap_{i=1}^{\infty} B_i\right)\end{aligned}$$

However, we note that

$$\lim_{n\to\infty} m(D_n) = \lim_{n\to\infty} m(B_1) - m(B_n) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

This gives us that

$$m(B_1) - m\left(\bigcap_{i=1}^{\infty} B_i\right) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

That is

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$$

which completes the proof.

■ **Example 2.1**

Let $B_i = (i, \infty)$ then we have

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = m(\emptyset) = 0 \quad \text{and} \quad \lim_{n\to\infty} m(B_n) = \infty$$

**Why this does not fit Theorem 2.4.2?** Because $m(B_1) = \infty$ ■

**Exercise 2.4.1**

Let $A \subseteq \mathbb{R}$ has finite outer measure, prove that $A$ is measurable if and only if

$$b - a = m^*((a, b) \cap A) + m^*((a, b) \setminus A)$$

for any open bounded interval $(a, b)$

**Solution:** $\Longrightarrow$ Assume that $A$ is measurable, then for any $X \subseteq \mathbb{R}$ we have

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

Now we can just take $X$ be an arbitrary open bounded interval $(a, b)$, so we have

$$m^*((a, b)) = m^*((a, b) \cap A) + m^*((a, b) \setminus A)$$

Note that $m^*((a, b)) = \ell((a, b)) = b - a$, then we have

$$b - a = m^*((a, b) \cap A) + m^*((a, b) \setminus A)$$

as desired.

$\Longleftarrow$ Assume that

$$b - a = m^*((a, b) \cap A) + m^*((a, b) \setminus A)$$

for any open bounded interval $(a, b)$.
Since $m^*(A) < \infty$, so for any $\varepsilon > 0$ and by the definition of outer measure we have

$$\sum_{i=1}^{\infty} \ell(I_i) < m^*(A) + \varepsilon$$

where $A \subseteq \bigcup_{i=1}^{\infty} I_i$. Since each $I_i$ is open bounded interval so it's a Borel set. We also note that $m^*(I_i) = \ell(I_i)$ for each $i$, then we have

$$m^*(I_i) = m^*(I_i \cap A) + m^*(I_i \setminus A)$$

Consider to sum each $i$ for the equation above, we get

$$\sum_{i=1}^{\infty} m^*(I_i) = \sum_{i=1}^{\infty} m^*(I_i \cap A) + \sum_{i=1}^{\infty} m^*(I_i \setminus A)$$

this gives us that

$$\sum_{i=1}^{\infty} m^*(I_i) \geq m^*\left(\bigcup_{i=1}^{\infty} I_i \cap A\right) + m^*\left(\bigcup_{i=1}^{\infty}(I_i \setminus A)\right)$$

Notice that

$$\bigcup_{i=1}^{\infty} I_i \cap A = A \quad \textbf{and} \quad \bigcup_{i=1}^{\infty}(I_i \setminus A) = \left(\bigcup_{i=1}^{\infty} I_i\right) \setminus A$$

Then we have that

$$m^*(A) + m^*\left(\left(\bigcup_{i=1}^{\infty} I_i\right) \setminus A\right) \leq \sum_{i=1}^{\infty} m^*(I_i) = \sum_{i=1}^{\infty} \ell(I_i) < m^*(A) + \varepsilon$$

This gives us that

$$m^*\left(\left(\bigcup_{i=1}^{\infty} I_i\right) \setminus A\right) < \varepsilon$$

Note that $\bigcup_{i=1}^{\infty} I_i$ is an open set and contains $A$, then by **A1Q5b** the set $A$ is measurable, which completes the proof. ■

## 2.5 Non-Measurable Set

**Lemma 2.5.1**
Let $A \subseteq \mathbb{R}$ be bounded and measurable, $\Lambda \subseteq \mathbb{R}$ be bounded and countably infinite. If $\lambda + A$ with $\lambda \in \Lambda$ are pairwise disjoint, then $m(A) = 0$

**Proof:** Note that $\bigcup_{\lambda}(\lambda + A)$ is bounded and measurable. Then we have $m\left(\bigcup_{\lambda}(\lambda + A)\right) < \infty$, so that

$$m\left(\bigcup_{\lambda}(\lambda + A)\right) = \sum_{\lambda} m(\lambda + A) = \sum_{\lambda} m(A) < \infty$$

Then $m(A) = 0$

**Construction**
We start with $\emptyset \neq A \subseteq \mathbb{R}$, consider

$$a \sim b \iff a - b \in \mathbb{Q}$$

Then this $\sim$ is an equivalence relation.
Let $C_A$ denote a single choice of equivalence class representatives for $A$ relative to $\sim$.

■ **Remark 2.3** The sets $\lambda + C_A$ with $\lambda \in \mathbb{Q}$ are pairwise disjoint. Because

$$x \in (\lambda_1 + C + A) \cap (\lambda_2 + C_A)$$

implies $x = \lambda_1 + a = \lambda_2 + b$ where $a, b \in C_A$, then $a - b = \lambda_2 - \lambda_1 \in \mathbb{Q}$. This gives us that

$$a \sim b \implies a = b \implies \lambda_1 = \lambda_2$$

**Theorem 2.5.2 — Vitali Theorem.**
Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable set.

**Proof:** By **Quiz 1**, we may assume $A$ is bounded. Say $A \subseteq [-N, N]$ for some $N \in \mathbb{N}$.

**Claim:** $C_A$ is non-measurable.

Assume $C_A$ is measurable, let $\Lambda \subseteq \mathbb{Q}$ be bounded and countable. By the **Lemma and Remark** we have $m(C_A) = 0$. Let $a \in A$, then $a \sim b$ for some $b \in C_A$. In particular, $a - b = \lambda \in \mathbb{Q}$. Moreover, $\lambda \in [-2N, 2N]$. Taking $\Lambda_0 = \mathbb{Q} \cap [-2N, 2N]$ we have that

$$A \subseteq \bigcup_{\lambda \in \Lambda_0} \underbrace{(\lambda + C_A)}_{=0}$$

This leads to a **contradiction!**

**Corollary 2.5.3**
There exists $A, B \subseteq \mathbb{R}$ s.t.

$$A \cap B = \emptyset \quad \textbf{and} \quad m^*(A \cup B) < m^*(A) + m^*(B)$$

**Proof:** Let $C$ be non-measurable set, then there exists $X \subseteq \mathbb{R}$ s.t.

$$m^*(X) < m^*(X \cap C) + m^*(X \setminus C)$$

Take $A = X \cap C$ and $B = X \setminus C$, then we are done.

## 2.6 Cantor-Lebesgue Function

**Proposition 2.6.1 — The Cantor set is Borel and has measure zero.**

**Proof:** $C$ is closed so it's Borel. Note that $C = \bigcap_{i=1}^{\infty} C_i$ and $C_i$ are measurable with

$C_1 \supseteq C_2 \supseteq C_3 \supseteq$ ..... and $m(C_1) < \infty$. By continuity of measure, we have

$$m(C) = \lim_{n\to\infty} m(C_i) = \lim_{n\to\infty} \frac{2^i}{3^i} = 0$$

**Construction of Cantor-Lebesgue Function**

1. For $i \in \mathbb{N}$, let $\mathcal{U}_i$ be union of open intervals deleted in the process of constructing $C_1, C_2, ....C_i$ i.e. $\mathcal{U}_i = [0,1] \setminus C_i$
2. $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ i.e. $\mathcal{U} = [0,1] \setminus C$
3. Say $\mathcal{U}_i = I_{i,1} \cup I_{i,2} \cup \ldots \cup I_{i,2^{i-1}}$, we define

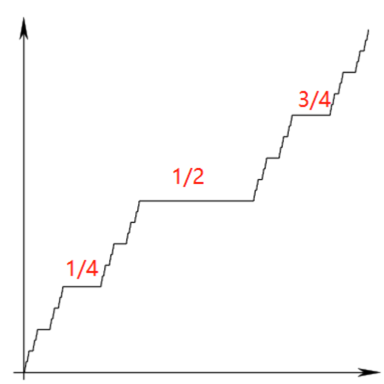
$$\varphi : \mathcal{U}_i \to [0,1] \quad \textbf{by } \varphi\,|_{I_{i,j}} = \frac{j}{2^i}$$

e.g) $\mathcal{U}_1 = \underbrace{\left(\frac{1}{3}, \frac{2}{3}\right)}_{\mapsto \frac{1}{2}}$ and $\mathcal{U}_2 = \underbrace{\left(\frac{1}{9}, \frac{2}{9}\right)}_{\mapsto \frac{1}{4}} \cup \underbrace{\left(\frac{1}{3}, \frac{2}{3}\right)}_{\mapsto \frac{2}{4}} \cup \underbrace{\left(\frac{7}{9}, \frac{8}{9}\right)}_{\mapsto \frac{3}{4}}$

4. Define $\varphi : [0,1] \to [0,1]$ by for $0 \neq x \in C$

$$\varphi(x) = \sup\{\varphi(t) : \ t \in \mathcal{U} \cap [0,x)\}$$

and $\varphi(0) = 0$. This is the Cantor-Lebesgue Function:




■ **Remark 2.4** Things to know about $\varphi$:
1. $\varphi$ is increasing
2. $\varphi$ is continuous.
· $\varphi$ is continuous on $\mathcal{U}$
· $x \in C$ with $x \neq 0, 1$. For large $i$, $\exists a_i \in I_{i,j}$ and $b_i \in I_{i,j+1}$ s.t.

$$a_i < x < b_i$$

but $\varphi(b_i) - \varphi(a_i) = \frac{j+1}{2^i} - \frac{j}{2^i} = \frac{1}{2^i} \to \infty$. Then there is no jump up! The point for $x \in \{0,1\}$'s proof is similar, so it's continuous.
· $\varphi : \mathcal{U} \to [0,1]$ is differentiable and $\varphi' = 0$
· $\varphi$ is onto, $\varphi(0) = 0$, $\varphi(1) = 1$, **IVT**

## 2.7 Non-Borel Set

**A non-Borel Set**
Let $\varphi$ be the Cantor-Lebesgue Function, consider $\psi : [0,1] \to [0,2]$ defined by

$$\psi(x) = x + \varphi(x)$$

Then $\psi$ is strictly increasing, continuous and onto. This implies $\psi$ is invertible.

**Proposition 2.7.1**
1. $\psi(C)$ is measurable and has **positive** measure.
2. $\psi$ maps a particular (measurable) subset of $C$ to a non-measurable set.

**Proof (for 1):** By **A1**, $\psi^{-1}$ is continuous, then $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed. Then $\psi(C)$ measurable. Note that $[0,1] = C \cup \mathcal{U}$ and $C \cap \mathcal{U} = \emptyset$, so $[0,2] = \psi(C) \cup \psi(\mathcal{U})$ with $\psi(C) \cap \psi(\mathcal{U}) = \emptyset$. Then

$$2 = m(\psi(C)) + m(\psi(\mathcal{U}))$$

it's suffices to show that

$$m(\psi(\mathcal{U})) = 1$$

We say $\mathcal{U} = \bigcup_{i=1}^{\infty} I_i$ a disjoint union of open intervals, then $\psi(\mathbb{U}) = \bigcup_{i=1}^{\infty} \psi(I_i)$ so that $m(()\psi(\mathcal{U})) = \sum_{i=1}^{\infty} m(\psi(I_i))$. **No that** $\forall i \in \mathbb{N}$, $\exists r \in \mathbb{R}$ s.t. $\varphi(x) = r$ for all $x \in I_i$. In particular, $\psi(x) = x + r$ for all $x \in I_i$ and so $\psi(I_i) = r + I_i$. Then

$$m(\psi(\mathcal{U})) = \sum_{i=1}^{\infty} m(I_i) = m\left(\bigcup_{i=1}^{\infty} I_i\right) = m(\mathcal{U})$$

Since $[0,1] = \mathcal{U} \cup C$ we have that

$$1 = m(\mathbb{U}) + \underbrace{m(C)}_{=0} = m(\mathcal{U})$$

Hence, $m(\psi(\mathcal{U})) = m(\mathcal{U}) = 1 > 0$

**Proof (for 2):** By **Vitali**, $\psi(C)$ contains a subset $A \subseteq \psi(C)$ which is non-measurable. Let $B = \psi^{-1}(A) \subseteq C$, then $\psi(B) = A$ is non-measurable as resquired.

**Theorem 2.7.2**
The Cantor set contains an element of $\mathscr{L} \setminus \mathcal{B}$

**Proof:** Take $B \subseteq C$, so $B$ is measurable, then $\psi(B)$ is not measurable. By **A1**, if $B$ is Borel, then $\psi(B)$ is Borel this leads to a **contradiction**. Hence $B$ is not Borel.

**Exercise 2.7.1** Let $A \subseteq \mathbb{R}$ be a non-measurable set with finite outer measure. Prove that there does not exists a measurable set $B \subseteq A$ such that $m(B) = m^*(A)$

**Solution:** Let $A \subseteq \mathbb{R}$ is non-measurable and $m^*(A) < \infty$
For sake of contradiction, we suppose there exists a measurable set $B \subseteq A$ such that $m(B) = m^*(A)$
Since $m^*(A) < \infty$ so we have $m^*(B) \leq m^*(A) < \infty$, note that $B$ is measurable. Then we have

$$m^*(A \setminus B) = m^*(A) - m(B) = 0$$

This gives us that $A \setminus B$ is measurable, that is $A \setminus B \in \mathscr{L}$. Since $B \subseteq A$ and $B \in \mathscr{L}$, then

$$\underbrace{(A \setminus B)}_{\in \mathscr{L}} \cup \underbrace{B}_{\in \mathscr{L}} = A \in \mathscr{L}$$

That means $A$ is measurable, it's a **contradiction**!
Hence, there does not exists $B \subseteq A$ is measurable s.t. $m(B) = m^*(A)$, which completes the proof. ■

# 3. Measurable Functions

## 3.1 Measurable Functions

**Definition 3.1.1 — Measurable Function.**
$A \subseteq \mathbb{R}$ is measurable, we say $f : A \to \mathbb{R}$ is **measurable** if and only if for all open $\mathcal{U} \subseteq \mathbb{R}$, $f^{-1}(\mathcal{U})$ is **measurable**

**Proposition 3.1.1**
If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous, then $f$ is measurable.

**Proposition 3.1.2**
$A \subseteq \mathbb{R}$ is measurable, and $\mathcal{X}_A : \mathbb{R} \to \mathbb{R}$ where

$$\mathcal{X}_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then $\mathcal{X}_A$ is measurable.

**Proposition 3.1.3**
Let $A \subseteq \mathbb{R}$ be measurable, $f : A \to \mathbb{R}$, the following are **equivalent**:
1. $f$ is measurable
2. $\forall a \in \mathbb{R}$, $f^{-1}(a, \infty)$ is measurable.
3. $\forall a < b$ with $a, b \in \mathbb{R}$, $f^{-1}(a, b)$ is measurable.

**Proof:**

**1 $\implies$ 2:** Trivial

**2 $\implies$ 3:** Let $b \in \mathbb{R}$ so that $f^{-1}(b,\infty)$ is measurable, then $\mathbb{R} \setminus f^{-1}(b,\infty) = f^{-1}(\mathbb{R} \setminus (b,\infty)) = f^{-1}((-\infty,b])$ is measurable as well. We see that $(-\infty,b) = \bigcup_{i=1}^{\infty}\left(-\infty, b-\frac{1}{i}\right)$ and so

$$f^{-1}(-\infty,b) = \bigcup_{i=1}^{\infty} f^{-1}\left(-\infty, b-\frac{1}{i}\right)$$

is measurable. Finally, for $a<b$, we can write

$$(a,b) = (a,\infty) \cup (-\infty,b) \implies f^{-1}(a,b) = f^{-1}(a,\infty) \cap f^{-1}(-\infty,b)$$

is measurable.

**3 $\implies$ 1:** Trivial

**Proposition 3.1.4**

Let $A \subseteq \mathbb{R}$ be measurable and $f,g : A \to \mathbb{R}$ are measurable.
1. For all $a,b \in \mathbb{R}$, $af+bg$ is measurable.
2. The function $fg$ is measurable.

**Proof for 1:** Let $a \in \mathbb{R}$, for $\alpha \in \mathbb{R}$ $(af)^{-1}(\alpha,\infty) = \{x \in A : af(x) > \alpha\}$
If $a>0$,

$$(af)^{-1}(\alpha,\infty) = \left\{x \in A : f(x) > \frac{\alpha}{a}\right\} = f^{-1}\left(\frac{\alpha}{a},\infty\right)$$

is measurable.
If $a<0$, $(af)^{-1}(\alpha,\infty) = f^{-1}\left(-\infty,\frac{\alpha}{a}\right)$ is measurable.
If $a=0$, $af$ continuous $\implies$ measurable.
We now show that $f+g$ is measurable. For $\alpha \in \mathbb{R}$

$$\begin{aligned}
(f+g)^{-1}(\alpha,\infty) &= \{x \in A : f(x)+g(x) > \alpha\} \\
&= \{x \in A : f(x) > \alpha - g(x)\} \\
&= \{x \in A : \exists q \in \mathbb{Q},\ f(x) > q > \alpha - g(x)\} \\
&= \bigcup_{q\in\mathbb{Q}} (\{x \in A : f(x) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\
&= \bigcup_{q\in\mathbb{Q}} \Big(\underbrace{f^{-1}(q,\infty)}_{\textbf{measurable}} \cap \underbrace{g^{-1}(\alpha-q,\infty)}_{\textbf{measurable}}\Big)
\end{aligned}$$

is measurable. Then we have $f+g$ is measurable.
Since $af$ and $f+g$ are measurable, so we have $af+bg$ is measurable.

**Proof for 2:** By the quiz, $|f|$ is measurable. For $\alpha \in \mathbb{R}$:

$$\begin{aligned}(f^2)^{-1}(\alpha,\infty) &= \left\{x \in A : f^2(x) > \alpha\right\} \\ &= \begin{cases} \text{A} & \alpha < 0 \\ \{x \in A : |f|(x) > \sqrt{\alpha}\} & \alpha \geq 0 \end{cases} \\ &= \begin{cases} A & \alpha < 0 \\ |f|^{-1}\left(\sqrt{a},\infty\right) & \alpha \geq 0 \end{cases}\end{aligned}$$

is measurable, then $f^2$ is measurable. Since $(f+g)^2 = f^2 + 2fg + g^2$ is measurable, so we have $2fg$ is measurable. By **1**, the function $fg$ is measurable.

■ **Example 3.1** $\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x + \underbrace{\varphi(x)}_{\textbf{C-L}}$. $\exists A \subseteq [0,1]$ s.t. $A$ is measurable but $\psi(A)$ is not measurable. Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function s.t. $\psi^{-1}$ is continuous.

Consider $\mathcal{X}_A \circ \psi^{-1}$, then

$$\left(\mathcal{X}_A \circ \psi^{-1}\right)^{-1}\left(\frac{1}{2},\frac{3}{2}\right) = \psi^{-1}\left(\mathcal{X}_A^{-1}\left(\frac{1}{2},\frac{3}{2}\right)\right) = \psi(A)$$

which is not measurable. Then $\mathcal{X}_A \circ \psi^{-1}$ is not measurable. ■

**Proposition 3.1.5**
Let $A \subseteq \mathbb{R}$ be measurable set, if $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f \circ g$ is measurable.

**Proof:** Let $\mathcal{U} \subseteq \mathbb{R}$ be open, $(f \circ g)^{-1}(\mathcal{U}) = g^{-1}(\underbrace{f^{-1}(\mathcal{U})}_{\textbf{open}})$ is measurable.

**Definition 3.1.2**
Let $A \subseteq \mathbb{R}$, we say a property $P(x)$ $(x \in A)$ is true **almost everywhere (ae)** if

$$m(\{x \in A : P(x) \ \ \textbf{false}\}) = 0$$

**Proposition 3.1.6**
Let $f : A \to \mathbb{R}$ be measurable, if $g : A \to \mathbb{R}$ is a function and $f = g$ **ae**, then $g$ is measurable.

**Proof:** Consider

$$B = \{x \in A : f(x) \neq g(x)\}$$

so we have $m(B) = 0$. Let $\alpha \in \mathbb{R}$, so

$$\begin{aligned} g^{-1}(\alpha, \infty) &= \{x \in A : g(x) > \alpha\} \\ &= \{x \in A \setminus B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \{x \in A \setminus B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \left( \underbrace{f^{-1}(\alpha, \infty)}_{\textbf{measurable}} \cap \underbrace{(A \setminus B)}_{\textbf{measurable}} \right) \cup \underbrace{\{x \in B : g(x) > \alpha\}}_{\textbf{measure 0}} \end{aligned}$$

is measurable.

**Proposition 3.1.7**
Let $A$ be measurable, $B \subseteq A$ measurable and a function $f : A \to \mathbb{R}$ is measurable if and only if $f\mid_B$ and $f\mid_{A\setminus B}$ are measurable.

**Proof:** $\Longrightarrow$ suppose $f : A \to \mathbb{R}$ is measurable, let $\alpha \in R$, then

$$\begin{aligned} (f\mid_B)^{-1}(\alpha, \infty) &= \{x \in B : f(x) > \alpha\} \\ &= f^{-1}(\alpha, \infty) \cap B \end{aligned}$$

is measurable, then $f\mid_B$ is measurable. The proof for $f\mid_{A\setminus B}$ is similar.
$\Longleftarrow$ Suppose $f\mid_B$ and $f\mid_{A\setminus B}$ are measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned} f^{-1}(\alpha, \infty) &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A \setminus B : f(x) > \alpha\} \\ &= \underbrace{(f\mid_B)^{-1}(\alpha, \infty)}_{\textbf{measurable}} \cup \underbrace{(f\mid_{A\setminus B})^{-1}[\alpha, \infty)}_{\textbf{measurable}} \end{aligned}$$

is measurable, and so $f$ is measurable.

**Proposition 3.1.8**
Let $f_n$ be a sequence of measurable functions where $f_n : A \to \mathbb{R}$. If $f_n \to f$ pointwise **ae**, then $f$ is measurable.

**Proof:** Let $B = \{x \in A : f_n(x) \nrightarrow f(x)\}$, so that $m(B) = 0$. Now for $\alpha \in \mathbb{R}$,

$$(f\mid_B)^{-1}(\alpha, \infty) = \underbrace{f^{-1}(\alpha, \infty) \cap B}_{\textbf{measure 0}}$$

is measurable.
If suffices to show that $f\mid_{A\setminus B}$ is measurable. By replacing $f$ by $f\mid_{A\setminus B}$, we may assume

$f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$, since $f_n \to f$ pointwise, we see that for $x \in A$

$$f(x) > \alpha \qquad \Longleftrightarrow \qquad \exists n, N \in \mathbb{N},\ \forall i \geq N,\ f_i(x) > \alpha + \frac{1}{n}$$

Then we see that

$$f^{-1}(\alpha, \infty) = \bigcup_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{i=N}^{\infty} \underbrace{f_i^{-1}\left(\alpha + \frac{1}{n}, \infty\right)}_{\textbf{measurable}}$$

is measurable. Therefore, we have $f$ is measurable.

## 3.2 Simple Approximation

**Definition 3.2.1 — Simple.**
A function $\varphi : A \to \mathbb{R}$ is called **simple** if $\varphi$ is measurable and $\varphi(A)$ is finite.

■ **Remark 3.1 — Canonical Representation.**
Let $\varphi : A \to \mathbb{R}$ be measurable, $\varphi(A) = \{c_1, c_2, ...c_k\}$ where $c_i$s are distinct and $A_i = \varphi^{-1}(\{c_i\})$ is measurable. We can see that $A$ is a disjoint union of $A_i$ i.e. $A = \bigcup_{i=1}^{k} A_i$ and $\varphi = \sum_{i=1}^{k} c_i \mathcal{X}_{A_i}$

**Goal:** Show measurable functions can be approximated by simple functions

**Lemma 3.2.1**
Let $f : A \to \mathbb{R}$ be measurable and bounded, for all $\varepsilon > 0$ there exists **simple** $\varphi_\varepsilon$, $\psi_\varepsilon : A \to \mathbb{R}$ such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \qquad \textbf{and} \qquad 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$$

**Proof:** Since $f(A)$ is bounded, so $f(A) \subseteq [a, b]$. Now for any $\varepsilon > 0$, we consider $a = y_0 < y_1 < ...... < y_n = b$ where $y_{i+1} - y_i < \varepsilon$. We define $I_k = [y_{k-1}, y_k)$, $A_k = \underbrace{f^{-1}(I_k)}_{\textbf{measurable}}$. Let $\varphi_\varepsilon : A \to \mathbb{R}$ and $\psi_\varepsilon : A \to \mathbb{R}$ where

$$\varphi_\varepsilon : \sum_{k=1}^{n} y_{k-1} \mathcal{X}_{A_k} \qquad \textbf{and} \qquad \psi_\varepsilon \sum_{k=1}^{n} y_k \mathcal{X}_{A_k}$$

so $\varphi_\varepsilon$ and $\psi_\varepsilon$ are simple. Let $x \in A$, since $f(x) \in [a, b]$, so $\exists k \in \{0, 1, .., 0\}$ such that $f(x) \in I_k$. i.e. $y_{k-1} \leq f(x) < y_k$, $x \in A_k$. Moreover, $\varphi_\varepsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\varepsilon(x)$ and so

$$\varphi_\varepsilon \leq f < \psi_\varepsilon$$

Now for the same $x$, we see that $0 \leq \psi_\varepsilon(x) - \varphi_\varepsilon(x) = y_k - y_{k-1} < \varepsilon$, which completes the proof.

**Theorem 3.2.2 — Simple Approximation.**
Let $A \subseteq \mathbb{R}$ be measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence $(\varphi_n)$ of simple functions on $A$ such that $\varphi_n \to f$ pointwise and $\forall n$, $|\varphi_n| \leq |f|$

**Proof:** $\Longleftarrow$ Done
$\Longrightarrow$ Suppose $f : A \to \mathbb{R}$ is measurable.
**Case 1:** $f \geq 0$, for each $n \in \mathbb{N}$ we define

$$A_n = \{x \in A : f(x) \leq n\}$$

so that $A_n$ is measurable and $f\,|_{A_n}$ is measurable and bounded. By the **Lemma 3.2.1**, there exists simple function $(\varphi_n)$ and $(\psi_n)$ such that $\varphi_n \leq f \leq \psi_n$ on $A_n$ and $0 \leq \psi_n - \varphi_n < \frac{1}{n}$. Fix $n \in \mathbb{N}$, extend $\varphi_n : A \to \mathbb{R}$ by setting $\varphi_n(x) = n$ if $x \notin A_n$, so $0 \leq \varphi_n \leq f$. For each $n \in \mathbb{N}$, $\varphi_n : A \to \mathbb{R}$ is simple.
**Claim:** $\varphi_n \to f$ pointwise.
Let $x \in A$ and $N \in \mathbb{N}$ such that $f(x) \leq N$ i.e. $x \in A_N$. For $n \geq N$, $x \in A_n$ and so $0 \leq f(x) - \varphi_n \leq \psi_n(x) - \varphi_n(x) < \frac{1}{n}$.

**Case 2:** $f : A \to \mathbb{R}$ measurable.
We define

$$B = \{x \in A : f(x) \geq 0\} \quad \textbf{and} \quad C = \{x \in A : f(x) < 0\}$$

are measurable. Now define $g, h : A \to \mathbb{R}$

$$g = \mathcal{X}_B f \quad \textbf{and} \quad h = -\mathcal{X}_C f$$

so that $g, h$ are measurable and non-negative. By **Case 1**, there exists sequences $(\varphi_n)$ and $(\psi_n)$ of simple functions such that $\varphi_n \to g$ pointwise and $\psi_n \to h$ pointwise with $0 \leq \varphi_n \leq g$ and $0 \leq \psi_n \leq h$. Then we have

$$\underbrace{\varphi_n - \psi_n}_{\textbf{simple}} \to g - h = f \quad \textbf{pointwise}$$

and

$$|\varphi_n - \psi_n| \leq |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \leq g + h = |f|$$

which completes the proof.

# 4. Littlewood Principles

## 4.1 Littlewood Principle I

**Up to certain finiteness conditions:**
1. Measurable sets are "almost" finite, disjoint union of bounded open intervals.
2. Measurable functions are "almost" continuous.
3. Pointwise limits of measurable functions are "almost" uniform limits.

**Theorem 4.1.1**
Let A be measurable set and $m(A) < \infty$. For all $\varepsilon > 0$ there exists finitely many open bounded, disjoint intervals $I_1, I_2, ....., I_n$ such that

$$m(A \Delta \mathcal{U}) < \varepsilon$$

where $\mathcal{U} = I_1 \cup I_2 \cup ...... \cup I_n$
Note: $m(A\Delta\mathcal{U}) = m(A \setminus \mathcal{U}) + m(\mathcal{U} \setminus A)$

**Proof:** Let $\varepsilon > 0$ be given, we may find an open set $\mathcal{U}$ such that

$$m(\mathcal{U} \setminus A) < \frac{\varepsilon}{2}$$

By **PMATH 351**, there exists bounded open, disjoint intervals $I_i$ ($i \in \mathbb{N}$) such that

$$\mathcal{U} = \bigcup_{i=1}^{\infty} I_i$$

Note that $\sum_{i=1}^{\infty} I_i = m(\mathcal{U}) < \infty$. In particular, $\exists N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\varepsilon}{2}$$

Take $V = I_1 \cup .... \cup I_N$ we see that $m(A \setminus V) \leq m(\mathcal{U} \setminus V)$ and $m(V \setminus A) \leq m(\mathcal{U} \setminus) < \frac{\varepsilon}{2}$.Therefore, we have $m(A \Delta \mathcal{U}) < \varepsilon$ as desired.

## 4.2 Littlewood Principle III

**Goal:** Prove that pointwise limits of measurable functions are almost uniform limits.

**Lemma 4.2.1**
Let $A$ be a measurable set with $m(A) < \infty$ and $f_n : A \to \mathbb{R}$ be a sequence of measurable functions. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

1. $|f_n(x) - f(x)| < \alpha$ for all $x \in B$, $n \geq N$
2. $m(A \setminus B) < \beta$

**Proof:** Let $\alpha, \beta > 0$ be given, for $n \in \mathbb{N}$ define

$$A_n = \{x \in A : |f_k(x) - f(x)| < \alpha, \ \ \forall\ k \geq n\} = \bigcap_{k=n}^{\infty} \underbrace{|f_k - f|^{-1}(-\infty, \alpha)}_{\in \mathscr{L}}$$

Then every $A_n$ is measurable.

Since $f_n \to f$ pointwise, $A = \bigcup_{n=1}^{\infty} A_n$ and $(A_n)$ is ascending, by **continuity of measure**

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty$$

We may find $N \in \mathbb{N}$ such that for all $n \geq N$

$$m(A) - m(A_n) < \beta$$

we can just pick $B = A_N$, then the proof is completed.

**Theorem 4.2.2 — Littlewood 3 - Egoroff's Theorem.**
Let $A$ be a measurable set with $m(A) < \infty$ and $f_n : A \to \mathbb{R}$ be a sequence of measurable functions. If $f_n \to f$ pointwise, then for all $\varepsilon > 0$ there exists a closed set $C \subseteq A$ such that

1. $f_n \to f$ uniformly on $C$
2. $m(A \setminus C) < \varepsilon$

**Proof:** Let $\varepsilon > 0$ be given, by the **Lemma 4.2.1** for every $n \in \mathbb{N}$, there exists a measurable set $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ s.t.

1. For all $x \in A_n$ and $k \geq N(n)$, $|f_k(x) - f(x)| < \frac{1}{n}$
2. $m(A \setminus A_n) < \dfrac{\varepsilon}{2^{n+1}}$

We take $B = \bigcap_{n=1}^{\infty} A_n$ (measurable). For $n \in \mathbb{N}$ s,t, $\frac{1}{n} < \varepsilon$, $k \geq N(n)$ and $x \in B$

$$|f_k(x) - f(x)| < \frac{1}{n}$$

then $f_n \to f$ uniformly on $B$
Moreover we have

$$m(A \setminus B) = m\left(A \setminus \bigcap_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} A \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(A \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$$

By **A1**, there exists a closed set $C$ s.t. $C \subseteq B$ and $m(B \setminus C) < \frac{\varepsilon}{2}$
Since $C \subseteq B$, $f_k \to f$ uniformly on $C$ and $m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which completes the proof.

■ **Example 4.1 — Warning.**
Let $f_n : \mathbb{R} \to \mathbb{R}$ with $f_n(x) = \frac{x}{n}$, $f_n \to 0$ pointwise but $f_n \nrightarrow 0$ uniformly on any measurable sets $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \setminus B) < 1$

Need: $m(A) < \infty$ ■

## 4.3 Littlewood Principle II

**Goal:** Prove that measurable functions are "almost" continuous. (i.e. Littlewood's $2^{nd}$ Principle/ Lusin's Theorem)

**Lemma 4.3.1**
Let $f : A \to \mathbb{R}$ be a simple function. For all $\varepsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a

closed set $C \subsetneq A$ such that $f = g$ on $C$ and $m(A \setminus C) < \varepsilon$

**Proof:** Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x \in A : f(x) = a_i\}$ is measurable. By **A1** we have there exists $C_i \subseteq A_i$ is closed such that $m(A_i \setminus C_i) <$. Note that $A = \bigcup_{i=1}^{n} A_i$ and $C = \bigcup_{i=1}^{n} C_i$ are disjoint union. We can see that for all $x \in C_i$, $f(x) = a_i$, by **A1** we have $f$ is continuous on $C$ and we can extend $f\mid_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$, also we have

$$m(A \setminus C) = m\left(\bigcup_{i=1}^{n}(A_i \setminus C_i)\right) = \sum_{i=1}^{n} m(A_i \setminus C_i) < \varepsilon$$

as desired.

**Theorem 4.3.2 — Littlewodd 2 - Lusin's Theorem.**
Let $f : A \to \mathbb{R}$ be a measurable function. For all $\varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subsetneq A$ such that $f = g$ on $C$ and $m(A \setminus C) < \varepsilon$.

**Proof:** Let $\varepsilon > 0$ be given

**Case 1:** $m(A) < \infty$
Let $f : A \to \mathbb{R}$ be measurable, by the **SAT (simple approximation theorem)** there exists the simple function $f_n$ such that $f_n \to f$ pointwise. By the lemma, there exists the continuous function $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subsetneq A$ such that $f_n = g_n$ on $C_n$ and $m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}$. By **Egorff**, there exists a closed set $C_0 \subsetneq A$ such that $f_n \to f$ uniformly on $C_0$ and $m(A \setminus C_0) < \frac{\varepsilon}{2}$.
Let $C = \bigcap_{i=0}^{\infty} C_i$, so $g_n = f_n \to f$ uniformly on $C \subsetneq C_0$ so $f$ is continuous on $C$. By **A1** we may extend $f\mid_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$ and

$$\begin{aligned} m(A \setminus C) = m\left(A \setminus \bigcap_{i=0}^{\infty} C_i\right) = m\left(\bigcup_{i=1}^{\infty}\right)(A \setminus C_i) &\leq \sum_{i=0}^{\infty} m(A \setminus C_i) \\ &= m(A \setminus C_0) + \sum_{i=1}^{\infty} m(A \setminus C_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which completes the proof of **Case 1**

**Case 2:** $m(A) = \infty$
For $n \in \mathbb{N}$, we define

$$A_n := \{a \in A : |a| \in [n-1, n)\}$$

so that $A = \bigcup_{n=1}^{\infty} A_n$, by **case 1** there exists continuous function $g_n : \mathbb{R} \to \mathbb{R}$ and closed set $C_n \subseteq A_n$ such that $f = g_n$ on $C_n$ and $m(A_n \setminus C_n) <$ Consider $C = \bigcup_{i=1}^{\infty} C_i$ which is a disjoint union, so we have

$$m(A \setminus C) = m\Big(\bigcup (A_i \setminus C_i)\Big) = \sum_{i=1}^{\infty} m(A_i \setminus C_i) < \varepsilon$$

and let $g : C \to \mathbb{R}$ and $x \in C$ so that $x \in C_n$ for exactly one $n \in \mathbb{N}$. Define $g(x) = \underbrace{g_n(x)}_{c.t.s} = f(x)$. By **A1** we can extend $g$ to a continuous function on $\mathbb{R}$, which completes the proof.

# 5. Integration

## 5.1 Integration I

1. Simple functions, $\varphi : A \to \mathbb{R}$, $m(A) < \infty$
2. $f : A \to \mathbb{R}$ is bounded and measurable with $m(A) < \infty$, $\varphi_\varepsilon \leq f \leq \psi_\varepsilon$
3. $f : A \to \mathbb{R}$ measurable, $f \geq 0$

$$\sup\left\{\int_A h : \; h \in \mathbf{2}, \; 0 \leq h \leq f\right\}$$

4. $f : A \to \mathbb{R}$ be measurable function, $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

**Step 1:** $\varphi : A \to \mathbb{R}$ be simple function and $m(A) < \infty$.

**Definition 5.1.1 — Lebesgue Integral.**
Let $m(A) < \infty$, $\varphi : A \to \mathbb{R}$ be simple function with canonical representation:

$$\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$$

The **Lebesgue Integral** of $\varphi$ over $A$ is

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(A_i)$$

**Lemma 5.1.1**

Let $m(A) < \infty$ where $A$ is measurable, if $B_1, B_2, ...., B_n \subseteq A$ are measurable and disjoint, and $\varphi : A \to \mathbb{R}$ is defined by

$$\varphi = \sum_{i=1}^{n} b_i \mathcal{X}_{B_i}$$

then

$$\int_A \varphi = \sum_{i=1}^{n} b_i m(B_i)$$

**F**or $n = 2$: If $b_1 \neq b_2$, then $\varphi = b_1\mathcal{X}_{B_1} + b_2\mathcal{X}_{B_2}$ is the canonical representation, if $b_1 = b_2$ then

$$b_1\mathcal{X}_{B_1} + b_1\mathcal{X}_{B_2} = b_1\{\mathcal{X}_{B_1} + \mathcal{X}_{B_2}\} = \underbrace{b_1 \cdot \mathcal{X}_{B_1\cup B_2}}_{\textbf{con rep}}$$

then we have

$$\int_A \varphi = b_1 m(B_1 \cup B_2) = b_1 \cdot (m(B_1) + m(B_2)) = b_1 m(B_1) + b_2 m(B_2)$$

**Proposition 5.1.2**

Let $\varphi, \psi : A \to \mathbb{R}$ be simple function with $m(A) < \infty$, for all $\alpha, \beta \in \mathbb{R}$ we have

$$\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$$

**Proof:** Let

$$\varphi(A) = \{a_1, ..., a_n\} \quad \textbf{and} \quad \psi(A) = \{a_1, ..., a_n\} \quad \textbf{are distinct}$$

Define

$$C_{ij} = \{x \in A : \varphi(x) = a_i, \quad \psi(x) = b_j\} = \varphi^{-1}(\{a_i\}) \cap \psi^{-1}(\{b_j\}) \quad \textbf{measurable}$$

Then we have

$$\alpha\varphi + \beta\psi = \sum_{i,j} (\alpha a_i + \beta b_j)\mathcal{X}_{C_{ij}}$$

where $C_{ij}$ is pairwise disjoint, so by the **Lemma 5.1.1** we have

$$\begin{aligned}
\int_A \alpha\varphi + \beta\psi &= \sum_{i,j} (\alpha a_i + \beta b_j) \cdot m(C_{ij}) \\
&= \sum_{i,j} \alpha a_i m(C_{ij}) + \sum_{i,j} \beta b_i m(C_{ij}) \\
&= \sum_i \alpha a_i \left( \sum_j m(C_{ij}) \right) + \sum_j \beta b_i \left( \sum_i m(C_{ij}) \right) \\
&= \alpha \sum_i \alpha(m(\{x \in A : \varphi(x) = a_i\})) + \beta \sum_j \alpha(m(\{x \in A : \psi(x) = b_i\})) \\
&= \alpha \int_A \varphi + \beta \int_A \psi
\end{aligned}$$

**Proposition 5.1.3**

Let $\varepsilon, \psi : A \to \mathbb{R}$ be simple function and $m(A) < \infty$, if $\varphi \leq \psi$, then

$$\int_A \varphi \leq \int_A \psi$$

## 5.2 Integration II

**Step 2:** $f : A \to \mathbb{R}$ be bounded and measurable with $m(A) < \infty$

**Definition 5.2.1 — Upper/Lower Lebesgue Integral.**

$$\underline{\int_A} f = \sup \left\{ \int_A \varphi : \varphi \leq f \quad \textbf{is simple} \right\} \quad \textbf{and} \quad \overline{\int_A} f = \inf \left\{ \int_A \psi : f \leq \psi \quad \textbf{is simple} \right\}$$

**Proposition 5.2.1**

Let $m(A) < \infty$ and $f : A \to \mathbb{R}$ be bounded and measurable, then

$$\underline{\int_A} f = \overline{\int_A} f$$

**Proof:** For all $n \in \mathbb{N}$, $\exists$ simple function $\varphi_n, \psi_n : A \to \mathbb{R}$ such that

$$\varphi_n \leq f \leq \psi_n \quad \textbf{and} \quad \psi_n - \varphi_n < \frac{1}{n}$$

We see that

$$0 \leq \overline{\int_A} f - \underline{\int_A} f \leq \int_A \psi_n - \int_A \varphi_n = \int_A (\psi_n - \varphi_n) \leq \int_A \frac{1}{n} = \frac{1}{n} \cdot m(A) \to 0$$

**Definition 5.2.2 — Lebesgue Integral.**
Let $m(A) < \infty$ and $f : A \to \mathbb{R}$ be bounded measurable functions, we define the **(Lebesgue Integral) of** $f$ **over** $A$ by

$$\int_A f = \underline{\int_A} f = \overline{\int_A} f$$

**Proposition 5.2.2**
Let $f.g : A \to \mathbb{R}$ be bounded measurable and $m(A) < \infty$. For any $\alpha, \beta \in \mathbb{R}$

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

**Proof:** Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be simple function where $\varphi_1 \leq f \leq \psi_1$ and $\varphi_2 \leq g \leq \psi_2$, so

$$\begin{aligned}
\int_A +f + g = \overline{\int_A} f + g \leq \int_A (\psi_1 + \psi_2) \\
&= \int_A \psi_1 + \int_A \psi_2 \\
&\leq \inf \left\{ \int_A \psi_1 + \int_A \psi_2 : f \leq \psi_1, g \leq \psi_2 \right\} \\
&= \inf \left\{ \int_A \psi_1 : f \leq \psi_1 \ \ \textbf{simple} \right\} + \inf \left\{ \int_A \psi_2 : g \leq \psi_2 \ \ \textbf{simple} \right\} \\
&= \int_A f + \int_A g
\end{aligned}$$

$$\begin{aligned}
\int_A f + g = \underline{\int_A} f + g \geq \int_A \varphi_1 + \varphi_2 \\
&= \int_A \varphi_1 + \int_A \varphi_2
\end{aligned}$$

Similarly, by taking sup we have $\int_A f + g \geq \int_A f + \int_A g$, so we have the addition

$$\int_A f + g = \int_A f + \int_A g$$

**Scalar multiple is similar**, then the results follows.

**Proposition 5.2.3**
Let $f, g : A \to \mathbb{R}$ be bounded and measurable, $m(A) < \infty$. If $f \leq g$ then

$$\int_A f \leq \int_A g$$

**Proof:**

$$\int_A (g - f) \geq \int_A 0 = 0 \quad \Longrightarrow \quad \int_A g - \int_A f \geq 0 \quad \Longrightarrow \quad \int_A g \geq \int_A f$$

## 5.3 Bounded Convergence Theorem

**Proposition 5.3.1**
Let $f : A \to \mathbb{R}$ be bounded and measurable, let $B \subseteq A$ be measurable and $m(A) < \infty$, then

$$\int_B f = \int_A (f \cdot \mathcal{X}_B)$$

**Proof:** If $f = \mathcal{X}_C$ and $C \subseteq A$ be measurable, then

$$\int_A \mathcal{X}_C \mathcal{X}_B = \int_A \mathcal{X}_{B \cap C} = m(B \cap C) = \int_B \mathcal{X}_{C|B}$$

If $f$ is simple, let $f = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$, then

$$\int_A f \mathcal{X}_B = \sum a_i \int_A \mathcal{X}_{A_i} \mathcal{X}_B = \sum a_i \int_B \mathcal{X}_{A_i} = \int_B \left( \sum a_i \mathcal{X}_{A_i} \right) = \int_B f$$

Now $f : A \to \mathbb{R}$ bounded and measurable, let $f \leq \psi$ be simple, so

$$\int_A f \mathcal{X}_B \leq \int_A \psi \mathcal{X}_B = \int_B \psi$$

By taking the inf over all such $\psi$, we have that

$$\int_A f \mathcal{X}_B \leq \overline{\int_B} f = \int_B f$$

Taking $\varphi \leq f$, $\varphi$ is simple, we obtain

$$\underline{\int_B} f = \int_B f \leq \int_A f \mathcal{X}_b$$

as desired.

**Proposition 5.3.2**
Let $f : A \to \mathbb{R}$ be bounded measurable and $m(A) < \infty$. If $B, C \subseteq A$ are measurable and disjoint, then

$$\int_{B\cup C} f = \int_B f + \int_C f$$

**Proof:**

$$\int_{B\cup C} f = \int_A f\mathcal{X}_{B\cup C} = \int_A f \cdot (\mathcal{X}_B + \mathcal{X}_C) = \int_A f\mathcal{X}_B + \int_A f\mathcal{X}_C = \int_B f + \int_C f$$

**Proposition 5.3.3**
Let $f : A \to \mathbb{R}$ be bounded and measurable with $m(A) < \infty$, then

$$\left|\int_A f\right| \leq \int_A |f|$$

**Proof:**

$$-|f| \leq f \leq |f| \quad \Longrightarrow \quad -\int_A |f| \leq \int_A f \leq \int_A |f|$$

Take the absolve value we have

$$\left|\int_A f\right| \leq \int_A |f|$$

as desired.

**Proposition 5.3.4**
Let $(f_n)$ be bounded measurable sequence and $f_n : A \to \mathbb{R}$ with $m(A) < \infty$. If $f_n \to f$ uniformly then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

**Proof:** Let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ such that

$$|f_n - f| < \frac{\varepsilon}{m(A) + 1}$$

for $n \geq N$, then for $n \geq N$ we have

$$\left|\int_A f_n - \int_A f\right| = \left|\int_A (f_n - f)\right| \leq \int_A |f_n - f| \leq m(A) \cdot \frac{\varepsilon}{m(A) + 1} < \varepsilon$$

■ **Example 5.1**
Let $f_n : [0,1] \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0 & 0 \leq n \leq \frac{1}{n} \\ n & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \end{cases}$$

We can see $f_n \to 0$ and

$$\int_{[0,1]} f_n = 1 \quad \textbf{and} \quad \int_{[0,1]} 0 = 1$$

■

**Theorem 5.3.5 — Bounded Convergence Theorem.**
Let $(f_n)$ be a sequence of measurable functions and $f_n : A \to \mathbb{R}$ with $m(A) < \infty$. If $\exists M > 0$ such that $|f_n| \leq M$ for all $n$ and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

**Proof:** Let $\varepsilon > 0$ be given, by **Egoroff's Theorem**, there exists measurable set $B \subseteq A$ and $N \in \mathbb{N}$ s.t. for $n \geq N$

$$|f_n - f| < \frac{\varepsilon}{2 \cdot (m(B) + 1)} \quad \textbf{and} \quad m(A \setminus B) < \frac{\varepsilon}{4M}$$

For $n \geq N$ we have

$$\begin{aligned} \left| \int_A f_n - \int_A f \right| \leq \int_A |f_n - f| &= \int_B |f_n - f| + \int_{A\setminus B} |f_n - f| \\ &\leq \int_B |f_n - f| + \int_{A\setminus B} (|f_n| + |f|) \\ &\leq \int_B |f_n - f| + 2 \cdot M \cdot m(A \setminus B) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

## 5.4 Integration III

**Definition 5.4.1**
1. We say $f$ has finite **support** if

$$A_0 := \{x \in A : f(x) \neq 0\}$$

has finite measure.

2. We say $f$ is **BF function** if $f$ is bounded and has finite support.

3. If $f : A \to \mathbb{R}$ is **BF** then

$$\int_A f := \int_{A_0} f$$

**Definition 5.4.2**

Let $f : A \to \mathbb{R}$ be measurable and $f \geq 0$, we define

$$\int_A f := \sup\left\{\int_A h : 0 \leq h \leq f \quad \mathbf{BF}\right\}$$

**Proposition 5.4.1**

Let $f, g : A \to \mathbb{R}$ be measurable function and $f, g \geq 0$, then

1. $\forall \alpha, \beta \in \mathbb{R}$

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

2. If $f \leq g$, then $\int_A f \leq \int_A g$
3. If $B, C \subseteq A$ are measurable and $B \cap C = \emptyset$, then

$$\int_{B \cup C} f = \int_B f + \int_C f$$

**Proposition 5.4.2 — Chebychev's Inequality.**

$f : A \to \mathbb{R}$ be non-negative measurable function, then for all $\varepsilon > 0$

$$m(\{x \in A : f(x) \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_A f$$

**Proof:** Let $\varepsilon > 0$ be given and let

$$A_\varepsilon = \{x \in A : f(x) \geq \varepsilon\}$$

such that $m(A_\varepsilon) < \varepsilon$ and $\underbrace{\varphi}_{\mathbf{BF}} = \varepsilon \cdot \mathcal{X}_{A_\varepsilon} \leq f$, so $\varepsilon m(A_\varepsilon) = \int_A \varphi \leq \int_A f$

If $m(A_\varepsilon) = \infty$, for $n \in \mathbb{N}$ define $A_{\varepsilon,n} := A_\varepsilon \cap [-n, n]$. By the continuity of measure

$$\infty = m(A_\varepsilon) = \lim_{n\to\infty} m(A_{\varepsilon,n})$$

For $n \in \mathbb{N}$, $\varphi_n = \varepsilon \mathcal{X}_{A_{\varepsilon,n}}$ (BF) we see that $\varphi_n \leq f$. Therefore, we have

$$\infty = m(A_\varepsilon) = \lim_{n\to\infty} m(A_{\varepsilon,n}) = \lim_{n\to\infty} \frac{1}{\varepsilon} \int_A \varphi_n \leq \int_A f$$

**Proposition 5.4.3**
Let $f : A \to \mathbb{R}$ with $f \geq 0$, then

$$\int_A f = 0 \iff f = 0 \ \textbf{ae}$$

**Proof:** $\implies \int_A f = 0$.

$$m(\{x \in A : f(x) \neq 0\}) \leq \sum m\left(\left\{x \in A : f(x) \geq \frac{1}{n}\right\}\right) \leq \sum n \cdot \underbrace{\int_A f}_{=0} = 0$$

$\impliedby$ Suppose $B = \{x \in A : f(x) \neq 0\}$ has measure 0, so

$$\int_A f = \int_B f + \underbrace{\int_{A \setminus B} f}_{=0} = \int_B f = 0$$

## 5.5 Fatou's Lemma and MCT

**Theorem 5.5.1 — Fatou's Lemma.**
Let $(f_n)$ be a measurable, non-negative sequence of functions and $f_n : A \to \mathbb{R}$. If $f_n \to f$ pointwise then

$$\int_A f \leq \liminf \int_A f_n$$

**Proof:** Let $0 \leq h \leq f$ be a **BF** function, we say $A_0 = \{x \in A : h(x) \neq 0\}$. It's suffices to show

$$\int_A h \leq \liminf \int_A f_n$$

Since for each $n \in \mathbb{N}$ we let

$$h_n = \min\{h, f_n\} \quad \textbf{measurable}$$

Note:
1. $0 \leq h_n \leq h \leq M$ for some $M > 0$ for all $n \in \mathbb{N}$.
2. For $x \in A_0$ and $n \in \mathbb{N}$, (a) $h_n(x) = h(x)$ or (b) $h_n(x) = f_n(x) \leq h(x)$ and

$$0 \leq h(x) - h_n(x) = h(x) - f_n(x) \leq f(x) - f_n(x) \to 0$$

Then $h_n \to h$ pointwise on $A_0$. By **BCT**

$$\lim_{n\to\infty} \int_{A_0} h_n = \int_{A_0} h \implies \lim_{n\to\infty} \int_A h_n = \int_A h$$

Since $h_n \leq f_n$ on $A$, so

$$\int_A h = \lim_{n\to\infty} \int_A h_n = \lim_{n\to\infty} \inf \int_A h_n \leq \lim_{n\to\infty} \inf \int_A f_n$$

■ **Example 5.2**

Let $A = (0, 1]$ and $f_n = n \cdot \mathcal{X}_{(0,\frac{1}{n})}$, so $f_n \to 0$ pointwise. We also have

$$\int_A 0 = 0 \quad \int_A f_n = n \cdot m\left(0, \frac{1}{n}\right) = 1 \quad \lim_{n\to\infty} \inf \int_A f_n = 1$$

■

**Theorem 5.5.2 — MCT.**

Let $(f_n)$ be a non-negative measurable function and $f_n : A \to \mathbb{R}$. If $(f_n)$ is increasing and $f_n \to f$ pointwise then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

**Proof:**

$$\int_A f \underbrace{\leq}_{\textbf{FL}} \lim_{n\to\infty} \inf \int_A f_n \leq \lim_{n\to\infty} \sup \int_A f_n \leq \int_A f$$

■ **Remark 5.1**

1. If $\varphi : A \to \mathbb{R}$ is simple and $m(A) < \infty$ then

$$\int_A \varphi < \infty$$

2. If $f : A \to \mathbb{R}$ is bounded and measurable, also $m(A) < \infty$, then

$$\int_A f < \infty$$

**Definition 5.5.1**

If $f : A \to \mathbb{R}$ is measurable and $f \geq 0$, then we say $f$ is **integrable** iff

$$\int_A f < \infty$$

## 5.6 Integration IV

The general integral

**Definition 5.6.1**
Let $f : A \to \mathbb{R}$ be measurable function

$$f^+(x) = \max\{f(x), 0\} \quad \textbf{positive part}$$
$$f^-(x) = \max\{-f(x), 0\} \quad \textbf{negative part}$$

Note:

$$f^+ + f^- = |f| \qquad f^+ - f^- = f \qquad f^+, f^- \quad \textbf{are measurable}$$

**Proposition 5.6.1**
Let $f : A \to \mathbb{R}$ be measurable function, then $f^+, f^-$ are **integrable** if and only if $|f|$ is **integrable**

**Proof:** $\Longrightarrow$:

$$|f| = f^+ + f^- \quad \Longrightarrow \quad \int_A |f| = \underbrace{\int_A f^+}_{<\infty} + \underbrace{\int_A f^-}_{<\infty}$$

$\Longleftarrow$:

$$\int_A f^+ \leq \int_A |f| < \infty \qquad \int_A f^- \leq \int_A |f| < \infty \quad \Longrightarrow \quad f^+, f^- \quad \textbf{are integrable}$$

**Definition 5.6.2 — Integrable Function.**
Let $f : A \to \mathbb{R}$ be measurable, we say $f$ is **integrable** if and only if $|f|$ is **integrable** if and only if $f^+, f^-$ are **integrable**, and we define

$$\int_A f = \int_A f^+ - \int_A f^-$$

**Proposition 5.6.2 — Comparison Test.**
Let $f : A \to \mathbb{R}$ be measurable, $g : A \to \mathbb{R}$ be non-negative and integrable. If $|f| \leq g$ then $f$ is integrable and

$$\left|\int_A f\right| \leq \int_A |f|$$

**Proof:**

$$\int_A |f| \leq \int_A g < \infty \quad \Longrightarrow \quad f \textbf{ is integrable}$$

$$\left|\int_A f\right| = \left|\int_A f^+ - \int_A f^-\right| \leq \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A |f|$$

**Proposition 5.6.3**
Let $f, g : A \to \mathbb{R}$ be integrable

1. $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable and $\displaystyle\int_a \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$
2. If $f \leq g$, then $\displaystyle\int_A f \leq \int_A g$
3. If $B, C \subseteq A$ are measurable with $B \cap C = \emptyset$, then $\displaystyle\int_{B \cup C} = \int_B f + \int_C f$

**Theorem 5.6.4 — Lebesgue Dominated Convergence Theorem.**
Let $(f_n)$ be a sequence of measurable function with $f_n : A \to \mathbb{R}$ and $f_n \to f$ pointwise. If there exists an integrable $g : A \to \mathbb{R}$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then $f$ is integrable and

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

**Proof:** Since we can see that $|f_n| \to |f|$ pointwise and $|f_n| \leq g$, and so $|f| \leq g$. By comparision, $f$ is integrable. Next, observe that $g - f \geq 0$, by **Fatou's Lemma**

$$\int_A g - \int_A f = \int_A (g - f) \leq \lim_{n\to\infty} \inf \int_A (g - f_n) = \int_A g - \lim_{n\to\infty} \sup \int_A f_n$$

Then, cancel the $g$ we have

$$\lim_{n\to\infty} \sup \int_A f_n \leq \int_A f$$

Also

$$\int_A g + \int_A f = \int_A (g + f) \leq \lim_{n\to\infty} \inf \int_A (g + f_n) = \int_A f + \lim_{n\to\infty} \inf \int_A f_n$$

Then, cancel the $g$ again we have

$$\int_A f \leq \lim_{n\to\infty} \inf \int_A f_n$$

so we have

$$\int_A f = \lim_{n\to\infty} \inf \int_A f_n = \lim_{n\to\infty} \sup \int_A f_n = \lim_{n\to\infty} \int_A f_n$$

which completes the proof.

## 5.7 Riemann Integration

**Definition 5.7.1 — Riemann Sum.**
Let $f : [a, b] \to \mathbb{R}$ be bounded function

1. A **partition** of $[a, b]$ is a finite set $P = \{x_0, x_1, ...., x_n\} \subseteq \mathbb{R}$ such that

$$a = x_0 < x_1 < x_2 < .... < x_n = b$$

2. Relative to $P$, we define the **lower Darboux sum:**

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \textbf{where} \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

3. Similarly, we define the **upper Darboux sum:**

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \textbf{where} \quad M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

**Definition 5.7.2**
Let $f : [a, b] \to \mathbb{R}$ be bounded function

1. **Lower Riemann Integral:**

$$R\underline{\int_a^b} f = \sup\{L(f, P) : \textbf{P is a partition}\}$$

1. **Upper Riemann Integral:**

$$R\overline{\int_a^b} f = \inf\{U(f, P) : \textbf{P is a partition}\}$$

3. We say $f$ is **Riemann Intetrable if and only if**

$$R\overline{\int_a^b} f = R\underline{\int_a^b} f$$

**Definition 5.7.3 — Step Function.**
Let $I_i, ...., I_n$ be pointwise disjoint intervals such that

$$[a, b] = \bigcup_{i=1}^{n} I_i$$

A **Step function** is a function of the form

$$f = \sum_{i=1}^{n} a_i \mathcal{X}_{I_i}$$

for some $a_i \in \mathbb{R}$

■ **Remark 5.2** Let $f : [a,b] \to \mathbb{R}$ be a bounded fuction and

$$a = x_0 < x_1 < ...... < x_n = b$$

and $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, ..., n-1$ and $I_n = [x_{n-1}, x_n]$. Then

$$L(f,P) = \sum_{i=1}^{n} m_i \ell(I_i) = R\int_a^b \varphi$$

where $\varphi(x) = m_i$ on $I_i$ ($\varphi \leq f$) and

$$U(f,P) = \sum_{i=1}^{n} M_i \ell(I_i) = R\int_a^b \psi$$

where $\psi(x) = M_i$ on $I_i$ ($f \leq \psi$) and

■ **Remark 5.3** Let $f : [a,b] \to \mathbb{R}$ be a bounded fuction, then

$$R\underline{\int_a^b} f = \sup\{L(f,P) : \textbf{P is a partition}\} = \sup\left\{R\int_a^b \varphi : \varphi \leq f \ \ \textbf{is a step function}\right\}$$

and

$$R\overline{\int_a^b} f = \inf\{U(f,P) : \textbf{P is a partition}\} = \inf\left\{R\int_a^b \psi : f \leq \psi \ \ \textbf{is a step function}\right\}$$

## 5.8 Riemann Integral VS Lebesgue Integral

**Definition 5.8.1**
Let $f : [a,b] \to \mathbb{R}$ be bounded function and let $x \in [a,b]$ and $\delta > 0$

1.
$$m_\delta(x) = \inf\{f(x) : x \in (x-\delta, x+\delta) \cap [a,b]\}$$

2.
$$M_\delta(x) = \sup\{f(x) : x \in (x-\delta, x+\delta) \cap [a,b]\}$$

3. **Lower boundary of** $f$:
$$m(x) = \lim_{\delta \to 0} m_\delta(x)$$

4. **Upper boundary of $f$:**

$$M(x) = \lim_{\delta \to 0} M_\delta(x)$$

5. **Oscillation of f:**

$$\omega(x) = M(x) - m(x)$$

■ **Remark 5.4** Let $f : [a, b] \to \mathbb{R}$ be bounded function, the following are equivalent:

1. $f$ is continuous at $x \in [a, b]$
2. $M(x) = m(x)$
3. $\omega(x) = 0$

**Lemma 5.8.1**

Let $f : [a, b] \to \mathbb{R}$ be bounded function, then

1. $m$ is measurable
2. If $\varphi : [a, b] \to \mathbb{R}$ is a step function with $\varphi \leq f$, then

$$\varphi(x) \leq m(x)$$

at all points of continuity of $\varphi$

3. $R\underline{\int_a^b} f = \int_{[a,b]} m$

**Proof 1:** Let $\alpha \in \mathbb{R}$ and $c \in [a, b]$ s.t. $m(c) > \alpha$. Choose any $m(c) > \beta > \alpha$, by the definition of $m$, there exists $\varepsilon > 0$ such that $m_\varepsilon > \beta$. However, this means that $f(x) > \beta$ for any $x \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$. Take $x \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$ so that there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap [a, b] \subseteq (c - \varepsilon, c + \varepsilon) \cap [a, b]$. It follows that $m_\delta(x) \geq \beta$ and so $m(x) \geq m_\delta(x) \geq \beta > \alpha$ as well. Therefore, $\{c \in [a, b] : m(c) > \alpha\}$ is relatively open in $[a, b]$ (i.e. is the intersection of an open set and $[a, b]$) and so is measurable.

**Proof 2:** Suppose $\varphi \leq f$ is a step function and let $x$ be a point of continuity of $\varphi$. Since $x$ is not an endpoint of a middle step, we see that there exists $\delta > 0$ and $z \in \mathbb{R}$ such that $\varphi(y) = z$ for all $y \in (x - \delta, x + \delta) \cap [a, b]$. Therefore, for all $y \in (x - \delta, x + \delta) \cap [a, b]$, we have $f(y) \geq \varphi(y) = z$. Hence, $m(x) \geq m_\delta(x) \geq z = \varphi(x)$ as required.

**Proof 3:** We begin by observing that if $\varphi \leq f$ is a step function then, by **(2)** $\varphi \leq m$ **a.e.** Therefore

$$R\underline{\int_a^b} f = \sup\left\{R\int_a^b \varphi : \varphi \leq f \ \ \textbf{step}\right\} = \sup\left\{\int_{[a,b]} \varphi : \varphi \leq f \ \ \textbf{step}\right\} \leq \int_{[a,b]} m$$

by monotonicity **a.e.**

Now for each $n \in \mathbb{N}$, let $P_n = \{a = x_0 < x_1 < .... < x_{2^n} = b\}$, where each $x_i - x_{i-1} = \frac{b-a}{2^n}$.

Then let $I_{n,1} = [a, x_1]$ and $I_{n,k} = (x_{k-1}, x_k]$ for $2 \leq k \leq n$. Define a step function $\varphi_n \leq f$ by setting $\varphi_n(x) = \inf\{f(x) : x \in I_{n,k}\}$ for all $x \in I_{n,k}$. Let $P = \bigcup_{i=1}^{\infty} P_i$ and note that $P$ has measure 0 (countable)

Fix $x \in [a, b] \setminus P$. For all $n \in \mathbb{N}$, let $I_n(x)$ denote the interval $I_{n,k}$ (as above) which contains $x$. Let $\delta > 0$ be given and let $N \in \mathbb{N}$ be such that $I_n(x) \subseteq (x - \delta, x + \delta)$ for all $n \geq N$. By **(2)**, for $n \geq N$ we have

$$m(x) \geq \varphi_n(x) \geq m_\delta(x)$$

as $\delta \to 0$ (and so $N \to \infty$) we see that

$$\lim_{n\to\infty} \varphi_n(x) = m(x)$$

In particular, we have that $\varphi_n \to m$ pointwise **a.e.**. Let $\alpha \in \mathbb{R}$ such that $|f| \leq \alpha$. Then $|\varphi_n| \leq \alpha$ for every $n$, where constant function $\alpha$ is integrable over $[a, b]$ and so we have by **LDCT** that

$$\lim_{n\to\infty} \int_{[a,b]} \varphi_n = \int_{[a,b]} m$$

Since the Riemann and Lebesgue integrals clearly agree for step functions:

$$\lim_{n\to\infty} R\int_a^b]\varphi_n = \int_{[a,b]} m$$

Therefore,

$$\int_{[a,b]} m = \lim_{n\to\infty} R\int_a^b \varphi_n \leq \sup\left\{R\int_a^b \varphi : \varphi \leq f \ \ \textbf{step}\right\} = R\underline{\int_a^b} f$$

**Lemma 5.8.2**

Let $f : [a, b] \to \mathbb{R}$ be bounded function, then

1. $M$ is measurable
2. If $\psi : [a, b] \to \mathbb{R}$ is a step function with $f \leq \psi$, then

$$M(x) \leq \psi(x)$$

at all points of continuity of $\psi$

3. $R\overline{\int_a^b} f = \int_{[a,b]} M$

**Proof:** Similar as the last lemma.

**Theorem 5.8.3 — Lebesgue.**
Let $f : [a, b] \to \mathbb{R}$ be bounded function, then $f$ is **Riemann Integrable** if and only if $f$ is continuous **a.e.**. In that case:

$$R\int_a^b f = \int_{[a,b]} f$$

**Proof:** Note that

$$R\underline{\int_a^b} f = \int_{[a,b]} m \leq \int_{[a,b]} M = R\overline{\int_a^b} f$$

so $f$ is **Riemann integrable**. Then

$$\int_{[a,b]} m = \int_{[a,b]} M \iff \int_{[a,b]} (M - m) = 0 \iff M = m \ \textbf{a.e.}$$

$$\iff \omega = 0 \ \textbf{a.e.}$$

$$\iff f \ \textbf{is continuous a.e.}$$

If $f$ is continuous **a.e.**, then $f$ is measurable and

$$R\underline{\int_a^b} f \leq \int_{[a,b]} m \leq \int_{[a,b]} f \leq \int_{[a,b]} M = R\overline{\int_a^b} f$$

Then we have

$$R\int_a^b f = \int_{[a,b]} f$$

as desired.

■ **Example 5.3** Let $f : [0, 1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

so $f$ is discontinuous on $[0, 1]$. Then $f$ is **not Riemann Integrable**
However, $f = 0$ **a.e.** on $[0, 1]$ and so

$$\int_{[0,1]} f = \int_{[0,1]} 0 = 0$$

so $f$ is **Lebesgue Integrable** ■

■ **Example 5.4** Let $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, ......\}$ and $f_n = \mathcal{X}_{\{q_1,q_2,...,q_n\}}$ and $f_n \to f$ pointwise. Then $f_n$ is increasing but $f_n \leq 1$, so

$$\underbrace{R\int_{[0,1]} f_n}_{=0} \nrightarrow \underbrace{R\int_{[0,1]} f}_{\textbf{DNE}}$$

■

# 6. $L^p$ Spaces

## 6.1 $L^p$ Spaces

Goal: Create Banach Spaces whose norm is given by Lebesgue Integration.
Recall
1. For $1 \leq p < \infty$, $(C([a.b]), \|\cdot\|_p)$ is a normed vector space, where

$$\|f\|_p^p = \int_a^b |f|^p$$

2. For $p = \infty$, $(C([a.b]), \|\cdot\|_\infty)$:

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$$

is a Banach space.

Problem: Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$, then

$$\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$$

is not a norm on the vector space of integrable function $f : A \to \mathbb{R}$. Because $\displaystyle\int_A |f|^p = 0 \iff f = 0$ **a.e.**

**Definition 6.1.1**
Let $A \subseteq \mathbb{R}$ be measurable.

1. $M(A) = \{f : A \to \mathbb{R}$ **measurable**$\}$ (vector space). $f \sim g$ if and only if $f = g$ **a.e.**. The $[f]$ is the equivalence class

2. $M(A)/\sim = \{[f] : f \in M(A)\}$ (vector space) and

$$\alpha[f] + \beta[g] = [\alpha f + \beta g]$$

■ **Remark 6.1** If $f \sim g$ and $f$ is integrable, then $g$ is integrable and $\int_A f = \int_A g$

**Definition 6.1.2 — $L^p$ Space.**
Let $A \subseteq \mathbb{R}$ be measurable set and $1 \leq p < \infty$, the $L^p$ space is defined by

$$L^p(A) = \left\{[f] \in M(A)/\sim : \int_A |f|^p < \infty\right\}$$

■ **Remark 6.2** Suppose $[f], [g] \in L^p(A)$, then $\int_A |f|^p, \int_A |g|^p < \infty$

1.

$$|f + g|^p \leq (|f| + |g|)^p \leq (2 \max\{|f|, |g|\})^p \leq 2^p(|f|^p + |g|^p)$$

Then $|f + g|^p$ is integrable by comparison.

2. $L^p(A)$ is a subspace of $M(A)/\sim$

**Definition 6.1.3 — $L^\infty$ Space.**
Let $A \subseteq \mathbb{R}$ be measurable set, then $L^\infty(A)$ is defined by

$$L^\infty(A) = \{[f] \in M(A)/\sim : \ f \textbf{ is bounded a.e.}\}$$

■ **Remark 6.3** 1. $[f], [g] \in L^\infty(A)$, we have $|f| \leq M$ and $|g| \leq N$, so we can find $B, C \subseteq A$ s.t. $m(B) = m(C) = 0$. For $x \notin B \cup C$, we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$$

2. $L^\infty(A)$ is a subspace of $M(A)/\sim$

■ **Remark 6.4** For all $n \in \mathbb{N}$,

$$|f| \leq \|[f]\|_\infty + \frac{1}{n} \textbf{ off } m(A_n) = 0$$

and

$$B = \bigcup_{i=1}^{\infty} A_n \to \textbf{ measure 0}$$

so $|f| \leq \|[f]\|_\infty$ off $B$.

**Proposition 6.1.1**
Let $A \subseteq \mathbb{R}$ be measurable set, then

$$\|[f]\|_\infty = \inf \{M \geq 0 : |f| \leq M \textbf{ a.e.}\}$$

is a norm on $L^\infty(A)$

**Proof:** 1. $\|[f]\|_\infty = 0 \implies |f| \leq \|[f]\|_\infty$ **a.e.** so $[f] = [0]$ in $L^\infty(A)$

2. $|f| \leq \|[f]\|_\infty$ off $B$ and $|g| \leq \|[g]\|_\infty$ off $C$, off $B \cup C \to$ measure 0, then

$$|f+g| \leq |f| + |g| \leq \|[f]\|_\infty + \|[g]\|_\infty$$

By the definition of inf, we have

$$\|[f+g]\|_\infty = \|[f] + [g]\|_\infty \leq \|[f]\|_\infty + \|[g]\|_\infty$$

**Abusive Notation**

$$f \equiv [f] \in L^p(A)$$

and $f = g$ in $L^p(A)$ means $f = g$ **a.e.**

**Definition 6.1.4 — Holder Conjugates.**
For $p \in (1, \infty)$ we define $q = \frac{p}{p-1}$ to be the **Holder conjugates** of $p$

Note:
1. $q = \frac{p}{p-1} \iff p = \frac{q}{q-1}$
2. $\frac{1}{p} + \frac{1}{q} = 1$ 3. We also define 1 and $\infty$ to be **Holder conjugates**

**Proposition 6.1.2 — Young's Inequality.**
Let $p, q \in (1, \infty)$ be **Holder conjugates** , for all $a, b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Proof:** We define $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ where $x \in (0, \infty)$. Then we have $f'(x) = x^{p-1} - 1$ and $f''(x) = (p-1)x^{p-2}$. When $f'(x) = 0$, we can get the critical point of $f(x)$ at $x = 1$. Since the Holder conjugates $p, q \in (1, \infty)$, then $f''(x) = (p-1)x^{p-2} > 0$ for all $x \in (0, \infty)$. Therefore, we can know $f(x)$ has global minimum at $x = 1$. Since We have $\frac{1}{p} + \frac{1}{q} = 1$, so $f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$, then $f(x) \geq 0$ on $x \in (0, \infty)$. Now we take $x = \frac{a}{b^{\frac{q}{p}}}$, then

$$\begin{aligned} f\left(\frac{a}{b^{\frac{q}{p}}}\right) = \frac{1}{p} \cdot \left(\frac{a}{b^{\frac{q}{p}}}\right)^p + \frac{1}{q} - \frac{a}{b^{\frac{q}{p}}} \geq 0 &\Longrightarrow \frac{1}{p} \cdot \frac{a^p}{b^q} + \frac{1}{q} - \frac{a}{b^{\frac{q}{p}}} \geq 0 \\ &\Longrightarrow \frac{a^p}{p} + \frac{b^q}{q} \geq ab^{q-\frac{q}{p}} \end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, then we have $q - \frac{q}{p} = q \cdot \left(1 - \frac{1}{p}\right) = q \cdot \frac{1}{q} = 1$ Therefore, by $\frac{a^p}{p} + \frac{b^q}{q} \geq ab^{q-\frac{q}{p}}$ and $q - \frac{q}{p} = 1$, we have $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$ as desired.

**Proposition 6.1.3**
Let $A \subseteq \mathbb{R}$ be measurable set and $1 \leq p < \infty$ and $q$ is the **Holder conjugate** of $p$. If $f \in L^p(A)$ and $g \in L^q(A)$, then $fg \in L^1(A)$ and

$$\int_A |fg| \leq \|f\|_p \|g\|_q$$

**Proof:** If $p = 1$ and $q = \infty$,

$$|fg| \leq |f||g| \leq |f|\|g\|_\infty \ \textbf{a.e.}$$

then $fg \in L^1(A)$.
If $1 < p < \infty$ and $q$ is the **Holder conjugate** of $p$, so

$$|fg| = |f||g| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} \quad \textbf{by Young's Inequality}$$

so $fg$ is integrable by comparison, then $fg \in L^1(A)$. Also we have

$$\int_A |fg| \leq \frac{1}{p}\int_A |f|^p + \frac{1}{q}\int_A |g|^q = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

Now we have two cases, Case 1: $\|f\|_p = \|g\|_q = 1$, so

$$\int_A |fg| \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q$$

Case 2: $\frac{f}{\|f\|_p}, \frac{g}{\|g\|_q}$ by case 1 we have

$$\frac{1}{\|f\|_p \|g\|_q} \int_A |fg| \leq 1$$

**Lemma 6.1.4**
Let $p, q$ be **Holder conjugate** and $f \in L^p(A)$, if $f \neq 0$

$$f^* = \|f\|_p^{1-p} \operatorname{sign}(f) |f|^{p-1}$$

is in $L^q(A)$ and

$$\int_A f f^* = \|f\|_p, \quad \|f^*\|_q = 1$$

**Proof:** If $p = 1$ and $q = \infty$, we have

$$f^* = \operatorname{sign}(f) \in L^\infty(A)$$

and

$$\int_A f f^* = \int_A |f| = \|f\|_1$$

2. If $1 < p < \infty$ and $q$ is the **Holder conjugate** of $p$,

$$\int_A f f^* = \|f\|_p^{1-p} \int_A |f|^p = \|f\|_p^{1-p} \|f\|_p^p = \|f\|_p$$

and

$$\|f^*\|_q^q = \|f\|_p^{(1-p)q} \int_A |f|^{(p-1)q} = \|f\|_p^{-p} \int_A |f|^p = \|f\|_p^{-p} \|f\|^p = 1$$

**Theorem 6.1.5 — Minkowski's Inequality.**
Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$. If $f, g \in L^p(A)$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

**Proof:** If $p = 1$, the result is trivial. Now we look at $1 < p < \infty$, we can see that

$$\begin{aligned} \|f + g\|_p = \int_A (f+g)(f+g)^* &= \int_A f(f+g)^* + \int_A g(f+g)^* \\ &\leq \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

## 6.2 Completeness

**Theorem 6.2.1 — Riesz-Fisher.**
For every measurable set $A \subseteq \mathbb{R}$ and $1 \le p \le \infty$, $L^p(A)$ is a **Banach Space**

**Proof:** If $p = \infty$, it's trivial. Now we look at $1 \le p < \infty$. Let $(f_n) \subseteq L^p(A)$ be strongly Cauchy. Then there exists $(\varepsilon_n) \subseteq \mathbb{R}$ such that

$$\|f_{n+1} - f_n\|_p \le \varepsilon_n^2 \quad \textbf{and} \quad \sum \varepsilon_n < \infty$$

Since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly Cauchy, then it converges. Now for each $n \in \mathbb{N}$, we define

$$A_n := \{x \in A : |f_{n+1}(x) - f_n(x)| \ge \varepsilon\} = \{x \in A : |f_{n+1}(x) - f_n(x)|^p \ge \varepsilon^p\}$$

By **Chebychev's Inequality**

$$m(A_n) \le \frac{1}{\varepsilon_n^p} \int_A |f_{n+1} - f_n|^p \le \frac{1}{\varepsilon_n^p} \cdot \varepsilon_n^{2p} = \varepsilon_n^p$$

Then we have

$$\sum m(A_n) \le \sum \varepsilon_n^p \le \left(\sum \varepsilon_n^p\right) < \infty$$

so $m\left(\lim_{n\to\infty} \sup A_n\right) = 0$. Now we fix $x \notin \lim_{n\to\infty} \sup A_n$, let

$$N = \max\{n : x \in A_n\}$$

and for $n > N$,

$$|f_{n+1}(x) - f_n(x)| < \varepsilon_n^2 \quad \textbf{and} \quad \sum \varepsilon_i < \infty$$

so $(f_n(x))$ is Cauchy. Then $f_n \to f$ pointwise **a.e.**. For $k \in \mathbb{N}$, we have

$$\|f_{n+k} - f_n\|_p \le \sum_{i=n}^{\infty} \varepsilon_i^2$$

so $|f_{n+k} - f_n|^p \to |f_n - f|^p$ pointwise **a.e.** as $k \to \infty$. By **Fatou's Lemma** we have

$$\int_A |f_n - f|^p \le \lim_{k\to\infty} \inf \int_A |f_{n+k} - f_n|^p = \lim_{k\to\infty} \inf \|f_{n+k} - f_n\|_p^p \le \left[\sum_{i=n}^{\infty} \varepsilon_i^2\right]^p \to 0$$

## 6.3 Separability

■ **Example 6.1** Let $p = \infty$, suppose $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$. For every $x \in [0,1]$ we may find

$$\left\|\mathcal{X}_{0,x} - f_{\theta(x)}\right\|_\infty < \frac{1}{2}$$

For $x \neq y$ in $[0,1]$,

$$\left\|\mathcal{X}_{[0,x]-\mathcal{X}_{0,y}}\right\|_{\infty} = 1$$

so $\theta : [0,1] \to \mathbb{N}$ is injective, which is a contradiction

**Notation:**
1. Simp($A$)=simple functions on measurable set $A$
2. Step$[a,b]$= Step functions on [a,b]
3. Step$_{\mathbb{Q}}[a,b]$=step functions on [a,b], with rational partition function values. ■

**Proposition 6.3.1**
Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$, then Simp($A$) is dense in $L^p(A)$

**Proof:** Let $f \in L^p(A)$ so $f$ is measurable. Then $\exists(\varphi_n)$ simple function so that $\varphi_n \to f$ pointwise and $|\varphi_n| \leq |f|$, then $|\varphi_n|^p \leq |f|^p$. By comparison we have $(\varphi_n) \subseteq L^p(A)$. Note that

$$\|\varphi_n - f\|_p^p = \int_A |\varphi_n - f|^p \qquad \textbf{and} \qquad |\varphi_n - f|^p \leq 2^p(|\varphi_n|^p + |f|^p) \leq 2^{p+1}|f|^p$$

which is integrable. By **LDCT** we have

$$\lim_{n\to\infty} \int_A |\varphi_n - f|^p \int_A 0 = 0$$

as desired. (This is also true for $p = \infty$)

**Proposition 6.3.2**
Step$[a,b]$ is dense in $L^p[a,b]$

**Proof:** Let $A \subseteq [a,b]$ be measurable, so $\mathcal{X}_A : [a,b] \to \mathbb{R}$. By **Littlewood I**, so for any $\varepsilon > 0$, there exists a collection of bounded open interval such that the disjoint union $\bigcup_{i=1}^{n} I_i = U$ and $m(U \Delta A) < \varepsilon^p$. Since $\mathcal{X}_U$ is a step function so

$$\|\mathcal{X}_U - \mathcal{X}_A\|_p^p = \int_A |\mathcal{X}_U - \mathcal{X}_A| = m(A \Delta U)$$

so we have $\|\mathcal{X}_U - \mathcal{X}_A\| < \varepsilon$ as desired.

**Corollary 6.3.3**
Let $1 \leq p < \infty$, Step$_{\mathbb{Q}}[a,b]$ is dense in $L^p[a,b]$, then $L^p[a,b]$ is separable.

**Proposition 6.3.4**

Let $1 \le p < \infty$, $L^p(\mathbb{R})$ is separable

**Proof:** Consider to define $F_n = f \in L^p(\mathbb{R})$ where

$$F_n = \begin{cases} \text{Step}_{\mathbb{Q}}[-n,n] & \textbf{if } x \in [-n,n] \\ 0 & \textbf{if } x \notin [-n,n] \end{cases}$$

So we have $F = \bigcup_{i=1}^{\infty} F_i$ is countable. Take $f \in L^p(\mathbb{R})$, fix $n \in \mathbb{N}$ so $f \mid_{[-n,n]} \in L^p[-n,n]$, we show

$$f\mathcal{X}_{[-n,n]} \to f \;\; \textbf{in} \;\; L^p(\mathbb{R})$$

Note that

$$\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p^p = \int_{\mathbb{R}} \left|f\mathcal{X}_{[-n,n]} - f\right|^p = \int_{\mathbb{R}\setminus[-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p \mathcal{X}_{\mathbb{R}\setminus[-n,n]}$$

and

$$\left||f|^p \mathcal{X}_{\mathbb{R}\setminus[-n,n]}\right| \le |f|^p \quad \textbf{integrable}$$

By **LDCT** we have

$$\lim_{n\to\infty} \left\|f\mathcal{X}_{[-n,n]} - f\right\|_p^p = \lim_{n\to\infty} \int_{\mathbb{R}} \left|f\mathcal{X}_{[-n,n]} - f\right|^p = \int_{\mathbb{R}} 0 = 0$$

so $\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p \to 0$. Then for each $n \in \mathbb{N}$, $\exists \varphi_n \in F$ such that $\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p < \frac{1}{n}$ so $\|\varphi_n - f\|_p \to 0$ as desired.

**Theorem 6.3.5**

Let $A \subseteq \mathbb{R}$ be measurable set and $1 \le p < \infty$, then $L^p(A)$ is separable.

**Proof:** Similar as above.

# 7. Hilbert Spaces

## 7.1 Hilbert Spaces

We let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

**Definition 7.1.1**
Let $V$ be a vector space over $\mathbb{F}$. An inner product on $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. $\forall v \in V$, $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0$ if and only if $v = 0$
2. For all $v, w \in V$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$ (complex conjugate)
3. For all $\alpha \in F$, $u, v, w \in V$, $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$

We call $(V, \langle \cdot, \cdot \rangle)$ an **inner product space**.

**Proposition 7.1.1**
Let $V$ be a inner product space, then $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm on $V$. We call $\|\cdot\|$ the norm induced by $\langle v, v \rangle$.

■ **Example 7.1** Let $A \subseteq \mathbb{R}$ be measurable, $V = L^2(A)$ with

$$\langle f, g \rangle = \int_A fg$$

is an inner product space. Note that

$$\sqrt{\langle f, f \rangle} = \left( \int_A |f|^2 \right)^{\frac{1}{2}} = \|f\|_2$$

■

■ **Example 7.2** Let $A \subseteq \mathbb{R}$ be measurable, $V = L^2 3(A, \mathbb{C})$ (see **A3**) with

$$\langle f, g \rangle = \int_A f\overline{g}$$

so we can see $\sqrt{\langle f, f \rangle} = \|f\|_2$ ■

**Proposition 7.1.2 — Porollelogrom Law.**
Let $V$ be a inner product space, for all $u, v \in V$

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

**Proof:**

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle \\
&= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle + \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\
&= 2(\langle u, u \rangle + \langle v, v \rangle) \\
&= 2\left(\|u\|^2 + \|v\|^2\right)
\end{aligned}$$

■ **Example 7.3** Let $1 \leq p < \infty$ and $V = L^p[0, 2]$, define

$$f = \mathcal{X}_{[0,1]} \qquad g = \mathcal{X}_{[1,2]}$$

then

$$\|f\|_p^2 = \left(\int_{[0,2]} |f|^p\right)^{\frac{2}{p}} = 1^{\frac{2}{p}} = 1 \qquad \|g\|_p^2 = 1^{\frac{2}{p}} = 1 \qquad \|f+g\|_p^2 = 2^{\frac{2}{p}} \qquad \|f-g\|_p^2 = 2^{\frac{2}{p}}$$

By **Porollelogrom Law**

$$2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2 \cdot (1+1) = 2$$

so $\|\|_p$ is induced by an inner product space if and only if $p = 2$. ■

■ **Remark 7.1** $\|\|_\infty$ is not induced by an inner product space.

**Definition 7.1.2 — Hilbert Space.**
A **Hilbert Space** is a complete inner product space. (i.e. A Banach space whose norm is induced by an inner product space)

■ **Example 7.4** $L^2(A)$, $L^2(A, \mathbb{C})$ are **Hilbert Spaces** ■

## 7.2 Orthogonality

**Definition 7.2.1**
Let $V$ be an inner product space, we say $v, w \in V$ are **orthogonal** if $\langle v, w\rangle = 0$

■ **Example 7.5** Let $f, g \in L^2(A, \mathbb{C})$ where $A = [-\pi, \pi]$, define $f(x) = e^{inx}$ and $g(x) = e^{imx}$ with $n \neq m$, then

$$\begin{aligned}
\langle f, g\rangle = \int_A f\bar{g} = \int_A e^{inx}e^{-imx}dx &= \int_A e^{i(n-m)x}dx \\
&= \int_A \cos((n-m)x) + i\int_A \sin((n-m)x) \\
&= R\int_{-\pi}^{\pi} \cos((n-m)x)dx + R\int_{-\pi}^{\pi} \cos((n-m)x) \\
&= \left[\frac{1}{n-m}\sin((n-m)x)\right]_{-\pi}^{\pi} + \left[-\frac{1}{n-m}\cos((n-m)x)\right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

■

**Definition 7.2.2**
$A \subseteq V$ is **orthogonal** if the elements of $A$ are pair-wise orthogonal and $\|v\| = 1$ for all $v \in A$

**Corollary 7.2.1**
Let $V$ be a inner product space and $\{v_1, ...., v_n\}$ is orthogonal, then

$$\left\|\sum_{i=1}^{n} \alpha_i v_i\right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2$$

**Theorem 7.2.2 — Pythagorean Theorem.**
Let $V$ be an inner product space, if $v_1, ...., v_n \in V$ are pairwise orthogonal, then

$$\left\|\sum_{i=1}^{n} v_i\right\|^2 = \sum_{i=1}^{n} \|v_i\|^2$$

■ **Example 7.6** Let $L = L^2(S, \mathbb{C})$ where $S = [-\pi, \pi]$, so

$$A = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : \quad n \in \mathbb{Z} \right\}$$

is pairwise orthogonal. Now we can see

$$\frac{1}{2\pi} \|e^{inx}\|_2^2 = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{inx} e^{-inx} dx = \frac{1}{2\pi} \int_{[-\pi,\pi]} 1 = 1$$

Then we have $A$ is orthogonal. ■

**Definition 7.2.3 — Orthogonal Basis.**
An **Orthogonal Basis** is a maximal orthogonal subset of $V$

**Fact:** An inner product space always has an orthogonal basis.

**Fact:** Let $H$ be **Hilbert Space**. if $W \subseteq H$ is **closed subspace**, then there exists a subspace $W^\perp \subseteq H$ s.t.

$$H = W \bigoplus W^\perp$$

and $\langle w, z \rangle = 0$ for all $w \in W$ and $z \in W^\perp$

**Theorem 7.2.3**
Let $H$ be a Hilbert Space, then $H$ has a **countable** orthogonal basis if and only if $H$ is separable.

**Proof:** $\Longrightarrow$ Let $B$ be a countable orthogonal basis for $H$
**Claim:** $W = Span(B)$, $\overline{W} = H$
Suppose $\overline{W} \neq H$, since $H = \overline{W} \bigoplus \overline{W}^\perp$. We may find $0 \neq x \in \overline{W}^\perp$. We may assume $\|x\| = 1$. Then $B \cup \{x\}$ is orthogonal, which is a **contradiction**, so we have $\overline{W} = H$. This gives us that $Span_{\mathbb{Q}}(B) = H$, so $H$ is separable.

$\Longleftarrow$ Suppose $H$ does not have a countable orthogonal basis. Let $B$ be orthogonal basis of $H$, so $B$ is uncountable. For $u \neq v$ in $B$ we have

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 = 2 \qquad \Longrightarrow \qquad \|u - v\| = \sqrt{2}$$

Suppose $X \subseteq H$ s.t. $\overline{X} = H$. For any $u \in B$, there exists $x_u \in X$ s.t. $\|u - x_u\| < \frac{\sqrt{2}}{2}$. For $u \neq v$ in $B$ we have $x_u \neq x_v$. Then $\varphi : B \to X$ with $\varphi(u) = x_u$ is an injection, which completes the proof.

■ **Example 7.7**

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : \ n \in \mathbb{Z} \right\}$$

is a countable orthogonal set in $L^2([-\pi, \pi], \mathbb{C})$. It's countable and orthogonal.
**Question:** Is it maximal? ■

## 7.3 Big Theorems

**Definition 7.3.1**
Let $H$ be inner product space with $\{v_1, v_2, ...., v_n\}$ orthogonal. If $v = \sum \lambda_i v_i$, then $\lambda_i = \langle v, v_i\rangle$. We call $\langle v, v_i\rangle$ the **Fourier coefficient** of $v$ with respect to $\{v_1, v_2, ...., v_n\}$.

**Definition 7.3.2**
Let $H$ be Hilbert Space and $\{v_1, v_2, ......\}$ be orthogonal. For $v \in H$, we call

$$\sum_{i=1}^{\infty} \langle v, v_i\rangle v_i$$

the **Fourier Series** of $v$ relative to $\{v_1, v_2, ......\}$ and write

$$v \sim \sum_{i=1}^{\infty} \langle v, v_i\rangle v_i$$

**Theorem 7.3.1 — Best Approximation.**
Let $H$ be Hilbert Space and $\{v_1, v_2, ....., v_n\}$ be orthogonal. For $v \in H$, $\|v - \sum \lambda_i\|$ is minimized when

$$\lambda_i = \langle v, v_i\rangle$$

Moreover,

$$\left\| v - \sum \rangle v, v_i\rangle v_i \right\|^2 = \|v\|^2 - \sum |\langle v, v_i\rangle|^2$$

**Proof:** Let $W = Span\{v_1, ...., v_n\}$ is closed, we can see $V = W \bigoplus W^{\perp}$. Also, for $x \in W$ and we let $v = w + z$ where $w \in W$ and $z \in W^{\perp}$. Then

$$\|v - x\|^2 = \|w + z - x\|^2 = \|w + x + z\|^2 = \|w - x\|^2 + \|z\|^2 \geq \|z\|^2 = \|v - w\|^2$$

which gives us that

$$\|v - x\| \geq \|v - w\|$$

Now we see that $v = \sum \lambda_i v_i + z$ for $z \in W^{\perp}$, then

$$\langle v, v_i\rangle = \lambda_i + 0 = \lambda_i$$

Note that we can also write $v = \sum \langle v, v_i\rangle v_i + z$ for $z \in W^{\perp}$, then

$$\|v\|^2 = \left\| \sum \langle v, v_i\rangle v_i \right\|^2 + \|z\|^2 = \sum |\langle v, v_i\rangle|^2 + \|z\|^2$$

Therefore, we have

$$\left\| v - \sum |\langle v, v_i\rangle v_i| \right\|^2 = \|z\|^2 = \|v\|^2 - \sum |\langle v, v_i\rangle|^2$$

which completes the proof.

**Theorem 7.3.2 — Bessel's Inequality.**
Let $H$ be Hilbert Space and $\{v_1, v_2, ....., v_n\}$ be orthogonal, if $v \in H$,

$$\sum_{i=1}^{n} |\langle v, v_i\rangle|^2 \leq \|v\|^2$$

**Theorem 7.3.3 — Parseval's Identity.**
Let $H$ be Hilbert Space and $\{v_1, v_2, .....\}$ be orthogonal. For $v \in H$,

$$\sum_{i=1}^{n} |\langle v, v_i\rangle|^2 = \|v\|^2 \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} \langle v, v_i\rangle v_i \right\| = 0$$

**Theorem 7.3.4 — Orthogonal Basis Test.**
Let $H$ be separable Hilbert space and $\{v_1, v_2, .....\}$ be orthogonal. The followings are equivalent

1. $\{v_1, v_2, .....\}$ is a basis

2. $\overline{Span\{v_1, v_2, .....\}} = H$

3. $\lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} \langle v, v_i\rangle v_i \right\| = 0$ for every $v \in H$

**Proof:**
**1** $\Longrightarrow$ **2 :** Done.
**2** $\Longrightarrow$ **1:** If $\{v_1, v_2, ...\}$ is **not** maximal, then we may find $u \in H$ with $\|u\| = 1$ such that $\langle u, v_i\rangle = 0$ for all $i \in \mathbb{N}$. Since $C = \{x \in H : \langle x, u\rangle = 0\}$ is closed, so $u \notin \overline{Span\{v_1, v_2, .....\}}$
**2** $\Longrightarrow$ **3:** Let $v \in H$ and $\varepsilon > 0$ be given, also let

$$\sum_{i=1}^{N} \alpha_i v_i \in Span\{v_1, v_2, .....\}$$

such that

$$\left\| v - \sum_{i=1}^{n} \alpha_i v_i \right\| < \varepsilon$$

This gives us that

$$\left\| v - \sum_{i=1}^{n} \langle v, v_i\rangle v_i \right\| < \varepsilon$$

Now for $n \geq N$, we have

$$\left\| v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i \right\| \leq \left\| v - \sum_{i=1}^{N} \langle v, v_i \rangle v_i \right\| + \left\| \sum_{i=N+1}^{n} \langle v, v_i \rangle v_i \right\| < \varepsilon + \sqrt{\sum_{i=N+1}^{\infty} |\langle v, v_i \rangle|^2} \to 0$$

as $N \to \infty$.
**3** $\Longrightarrow$ **2:** Similar.

## 7.4 Appendix

**Definition 7.4.1 — Direct Sum.**
Let $V$ be a vector space and let $U$ and $W$ be the subspaces of $V$. We say $V$ is the direct sum of $U$ and $W$, written $V = U \bigoplus W$, if every element of $V$ can be **uniquely** written in the form of $u + w$ where $u \in U$ and $w \in W$.

If may be easily verified that $V = U \bigoplus W$ if and only if $V = U + W = \{u + v : u \in U,\ w \in W\}$ and $U \cap W = \{0\}$. Our goal is the show if $H$ is a Hilbert space and $W$ is a closed subspace of $H$, then $H = W + \bigoplus W^{\perp}$, where

$$W^{\perp} = \{x \in H : \langle x, w \rangle = 0 \ \textbf{ for all } w \in W\}$$

It's straightforward to verify that $W^{\perp}$ is a subspace of $H$.

**Proposition 7.4.1**
Let $H$ be a Hilbert space and let $W$ be a closed subspace of $H$. For every $v \in H$, there exists a unique $w \in W$ such that

$$\inf \{\|x - v\| : x \in W\} = \|w - v\|$$

**Proof:** Let $\delta = \inf \{\|x - v\| : x \in W\}$, for $a, b \in W$ we see that

$$\|a - b - (b - v)\|^2 + \|a - v + b - v\|^2 = 2\|a - v\|^2 + 2\|b - v\|^2$$

by the **Parallelogram Law**. Notice that

$$\|a + b - 2v\|^2 = 4\left\|\frac{1}{2}(a + b) - v\right\|^2 \geq 4\delta^2$$

Therefore,

$$\|a - b\|^2 \leq 2\|a - v\|^2 + 2\|b - v\|^2 - 4\delta^2 \qquad (*)$$

By the definition of inf, there exists a sequence $(w_n) \subseteq W$ such that $\|w_n - v\| \to \delta$, but then

$$\|w_n - w_m\| \leq 2\|w_n - v\|^2 + 2\|w_m - v\|^2 - 4\delta^2 \to 0$$

so that $(w_n)$ is Cauchy. Since $H$ is a Hilbert space and $W$ is closed, $w_n \to w$ for some $w \in W$. Finally, we see that $\|w_n - v\| \to \|w - v\|$ and $\|w_n - v\| \to \delta$. From which we have that $\|w - v\| = \delta$. Uniqueness follows immediately from $(*)$.

# 8. Fourier Analysis

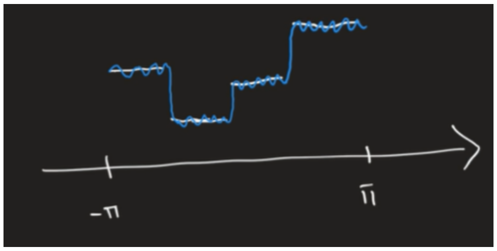
## 8.1 Fourier Series

Motivating Questions:

1. Is $\left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\}$ an orthogonal basis for $L^2([-\pi,\pi],\mathbb{C})$?
2. Is $Span\{e^{inx} : n \in \mathbb{Z}\}$ dense in $L^2([-\pi,\pi],\mathbb{C})$?
3. Is $Span\{e^{inx} : n \in \mathbb{Z}\}$ dense in $L^1([-\pi,\pi],\mathbb{C})$?

Given $f \in L^1([-\pi,\pi])$ with


Can we approximate $f$ using sinusoidal functions:



**Definition 8.1.1**
Let $T = [-\pi, \pi)$, we call $T$ the **Torus** or the **Circle**. We define $L^p(T) := L^p([-\pi, \pi], \mathbb{C})$ for $1 \leq p < \infty$ using the norm

$$\|f\|_p = \left( \frac{1}{2n} \int_T |f|^p \right)^{\frac{1}{p}}$$

and $L^p(T)$ is a separable Banach Space.

**Remark 8.1**
1. As a **group** under addition modulo $2\pi$:

$$T \cong \mathbb{R}/\mathbb{Z} \cong \{z \in C : |z| = 1\}$$

2. In this way, $T$ is locally compact abelian group.
3. There is a one-to-one correspondence between $f : T \to \mathbb{C}$ and $2\pi$-periodic functions $f : \mathbb{R} \to \mathbb{C}$

**Definition 8.1.2**
Let $f \in L^1(T)$.

1. We define the $n^{th}$ $(n \in \mathbb{Z})$ **Fourier coefficient** of $f$ by

$$\langle f, e^{inx} \rangle := \frac{1}{2\pi} \int_T f(x) e^{-inx} dx$$

2. We define the **Fourier Series** of $f$ by

$$f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

where $a_n = \langle f, e^{inx} \rangle$.
3. We let

$$S_N(f, x) = \sum_{-N}^{N} a_n e^{inx}$$

denote the $n^{th}$ partial sum of the above Fourier Series.

**Proposition 8.1.1**
Consider the **trigonometric polynomial** $f \in L^1(T)$ given by

$$f(x) = \sum_{n=-N}^{N} a_n e^{-inx}$$

for some $a_i \in \mathbb{C}$.

For each $-N, n \leq N$,

$$\langle f, e^{inx}\rangle = a_n$$

Why?

$$\frac{1}{2\pi}\int_T e^{imx}e^{-inx}dx = \delta_{m,n}$$

■ **Remark 8.2** Suppose $f \in L^1(T)$ is **real-valued**

$$f \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

For $N \in \mathbb{N}$

$$\begin{aligned} S_N(f,x) = \sum_{n=-N}^{N} a_n e^{inx} &= a_0 + \sum_{n=1}^{N}\left(a_n e^{inx} + a_{-n}e^{-inx}\right) \\ &= a_0 + \sum_{n=1}^{N}\left(\underbrace{(a_n + a_{-n})}_{b_n}\cos(nx) + \underbrace{i(a_n - a_{-n})}_{c_n}\sin(nx)\right) \\ &= a_0 + \sum_{n=1}^{N} b_n\cos(nx) + c_n\sin(nx) \end{aligned}$$

Now

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_T f(x)e^{-i0x}dx = \frac{1}{2\pi}\int_T f(x)dx \\ b_n = a_n + a_{-n} &= \frac{1}{2\pi}\int_T f(x)\left(e^{-inx} + e^{inx}\right)dx = \frac{1}{\pi}\int_T f(x)\cos(nx)dx \\ c_n = i(a_n - a_{-n}) &= \frac{i}{2\pi}\int_T f(x)\left(e^{-inx} - e^{inx}\right)dx = \frac{1}{\pi}\int_T f(x)\sin(nx)dx \end{aligned}$$

are all real-valued.

## 8.2 Fourier Coefficients

**Proposition 8.2.1**
Let $f, g \in L^1(T)$

1.
$$\langle f+g, e^{inx}\rangle = \langle f, e^{inx}\rangle + \langle g, e^{inx}\rangle$$

2. For $\alpha \in \mathbb{C}$,
$$\langle \alpha f, e^{inx}\rangle = \alpha\langle f, e^{inx}\rangle$$

3. If $\overline{f} : T \to \mathbb{C}$ is defined by $\overline{f}(x) = \overline{f(x)}$, then $\overline{f} \in L^1(T)$ and
$$\langle \overline{f}, e^{inx}\rangle = \overline{\langle f, e^{inx}\rangle}$$

**Proof (3):** Since $|f| = |\overline{f}|$ implies $\overline{f} \in L^1(T)$, then

$$\begin{aligned}\langle \overline{f}, e^{inx}\rangle = \frac{1}{2\pi}\int_T \overline{f}(x)e^{-inx}dx &= \frac{1}{2\pi}\int_T \overline{f(x)e^{inx}}dx \\ &= \frac{1}{2\pi}\int_T \mathrm{Re}\left(\overline{f(x)e^{inx}}\right) + \frac{i}{2\pi}\int_T \mathrm{Im}\left(\overline{f(x)e^{inx}}\right)dx \\ &= \frac{1}{2\pi}\int_T \mathrm{Re}\left(f(x)e^{inx}\right) - \frac{i}{2\pi}\int_T \mathrm{Im}\left(f(x)e^{inx}\right)dx \\ &= \overline{\frac{1}{2\pi}\int_T f(x)e^{inx}dx} \\ &= \overline{\langle f, e^{-inx}\rangle}\end{aligned}$$

**Proposition 8.2.2**
Let $f \in L^1(T)$ and $\alpha \in \mathbb{R}$. (By a previous remark, we may view $f : \mathbb{R} \to \mathbb{C}$ as a $2\pi$-periodic function which is integrable over $T$.) For $\alpha \in \mathbb{R}$, define $f_\alpha : \mathbb{R} \to \mathbb{C}$ by $f_\alpha(x) = f(x - \alpha)$ is integrable over $T$ and

$$\langle f_\alpha, e^{inx}\rangle = \langle f, e^{inx}\rangle e^{-inx}$$

**Proposition 8.2.3**
Let $f \in L^1(T)$, for all $n \in \mathbb{Z}$

$$\left|\langle f, e^{inx}\rangle\right| \le \|f\|_1$$

**Proof:**

$$\left|\langle f, e^{inx}\rangle\right| = \left|\frac{1}{2\pi}\int_T f(x)e^{inx}dx\right| \le \frac{1}{2\pi}\int_t \left|f(x)e^{-inx}\right|dx = \frac{1}{2\pi}\int_T |f(x)|dx = \|f\|_1$$

**Corollary 8.2.4**
Let a sequence $f_k \to f$ in $L^1(T)$, so for all $n \in \mathbb{Z}$,

$$\langle f_k, e^{inx}\rangle \to \langle f, e^{inx}\rangle$$

**Proof:**

$$\left|\langle f_k, e^{inx}\rangle - \langle f, e^{inx}\rangle\right| = \left|\langle f_k - f, e^{inx}\rangle\right| \le \|f_k - f\|_1 \to 0$$

■ **Remark 8.3** Let $Trig(T)$ denote the set of Trigonometric polynomials on $T$, by **A3** we have $\overline{Trig(T)} = L^1(T)$

**Theorem 8.2.5 — Riemann-Lebesgue Lemma.**
If $f \in L^1(T)$, then

$$\lim_{|n| \to \infty} \langle f, e^{inx} \rangle = 0$$

**Proof:** Let $\varepsilon > 0$ be given and let $P \in Trig(T)$ such that $\|f - P\|_1 < \varepsilon$. We say

$$P(x) = \sum_{k=-N}^{N} a_k e^{ikx}$$

for $n > N$ or $n < -N$ ($|n| > N$). We have that $\langle P, e^{inx} \rangle = 0$. For $|n| > N$,

$$\left| \langle f, e^{inx} \rangle \right| = \left| \langle f - P, e^{inx} \rangle \right| \leq \|f - P\|_1 < \varepsilon$$

## 8.3 Vector-Valued Integration

**Definition 8.3.1**
Let $B$ be a Banach space and let $f : [a, b] \to B$ be a function. Consider a partition $P = a = t_0 < t_1 < \ldots.. < t_n = b$ of $[a, b]$. We define a Riemann sum of $f$ over $P$ by

$$S(f, P) = \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}) \in B$$

where each $t^* \in [t_{i-1}, t_i]$

**Definition 8.3.2**
Let $B$ and $f$ be as above. We say $f$ is **Riemann integrable** if there exists $z \in B$ such that for all $\varepsilon > 0$ there exists a partition $P_\varepsilon$ of $[a, b]$ such that whenever $P$ is a refinement partition of $P_\varepsilon$ and $S(f, P)$ is a Riemann sum then

$$\|S(f, P) - z\| < \varepsilon$$

We call $z$ the integral of $f$ over $[a, b]$ and write $z = R\int_a^b f(x)dx$

**Theorem 8.3.1 — Cauchy Criterion.**
Let $B$ be a Banach space and let $f : [a, b] \to B$ be a function. Then $f$ is Riemann integrable if and only if for all $\varepsilon > 0$ there exists a partition $P_\varepsilon$ of $[a, b]$ so that whenever $P$ and $Q$ are refinements of $P_\varepsilon$ we have

$$\|S(f, P) - S(f, Q)\| < \varepsilon$$

for any Riemann sums $S(f, P)$ and $S(f, Q)$

**Proof:** $\Longrightarrow$ Suppose $f$ is Riemann integrable with $z = R\int_a^b f(x)dx$. Let $\varepsilon > 0$ be given, we may find a partition $P_{\varepsilon/2}$ such that whenever $P$ is a refinement partition of $P_{\varepsilon/2}$, then $\|S(f,P) - z\| < \frac{\varepsilon}{2}$. In particular, if $P$ and $Q$ are refinement of $P_{\varepsilon/2}$, then

$$\|S(f,P) - S(f,Q)\| \leq \|S(f,P) - z\| + \|z - S(f,Q)\| < \varepsilon$$

$\Longleftarrow$ Assume the Cauchy criterion. In particular, for each $n \in \mathbb{N}$ we may find a partition $P_n$ of $[a,b]$ which corresponds to $\varepsilon = \frac{1}{n}$, as per the Cauchy criterion. **WLOG** we may assume each $P_{n+1}$ is a refinement of $P_n$. For each $n \in \mathbb{N}$, elt $S(f,P_n)$ be a Riemann sum. Let $\varepsilon > 0$ be given, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$, we see that for $m,n \geq N$

$$\|S(f,P_m) - S(f,P_n)\| < \frac{1}{N} < \varepsilon$$

Since $B$ is a Banach space, so $S(f,P_n) \to z \in B$.

We claim that $f$ is Riemann integrable with $R\displaystyle\int_a^b f(x)dx = z$. Let $N$ and $P_N$ be as above. Moreover, there exists $M > N$ such that $\|S(f,P_M) - z\| < \frac{\varepsilon}{2}$. Now, if $P$ is any refinement partition of $P_N$, then

$$\|S(f,P) - z\| \leq \|S(f,P) - S(f,P_M)\| + \|S(f,P_M) - z\| < \varepsilon$$

This result can then be used to show the following, which we shall state and use as a fact. The proof is quite similar to the proof for $B = \mathbb{R}$

**Theorem 8.3.2**
If $B$ is a Banach space and $f : [a,b] \to B$ is continuous, then $f$ is Riemann integrable.

## 8.4 Summability Kernels

**Goal:** Given $f \in L^1(T)$, determine when $S_n(f,x) \to f(x)$ pointwise in $L^1$?

**Main tool:** Summability Kernels and convolution.

**Definition 8.4.1 — Convolution.**
Let $f,g \in L^1(T)$, the **convolution** of $f$ and $g$ is the function $f * g : T \to \mathbb{C}$ given by

$$(f * g)(x) = \frac{1}{2\pi}\int_T f(t)g(x-t)dt = \frac{1}{2\pi}\int_T f(t)g_t(x)dt$$

**Facts:**
1. Given $f, g \in L^1(T)$, $f * g \in L^1(T)$ as well
2. $\|f+g\|_1 \leq \|f\|_1 \cdot \|g\|_1$
3. This makes $L^1(T)$ a **Banach Algebra**

**Let $C(T)$ denote the set of continuous function $T \to \mathbb{C}$**

**Definition 8.4.2 — Summability Kernel.**
A **Summability Kernel** is a sequence $(K_n) \subseteq C(T)$ s.t.

1. $\frac{1}{2\pi}\int_T K_n = 1$
2. $\exists M > 0$, $\forall n \in \mathbb{N}$, $\|K_n\|_1 \leq M$
3. For all $0 < \delta < \pi$,
$$\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta} |K_n| + \int_{\delta}^{\pi} |K_n|\right) = 0$$

**Proposition 8.4.1**
Let $(B, \|\cdot\|_B)$ be a Banach Space, let $\varphi : T \to B$ be continuous function. Let $(K_n) \subseteq C(T)$ be a summability kernel, then

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(t)\varphi(t)dt = \varphi(0)$$

in the $B$-norm

**Proof:** Let $0 < \delta < \pi$, notice that

$$\begin{aligned}\frac{1}{2\pi}\int_T k_n(t)\varphi(t) - \varphi(0) &= \frac{1}{2\pi}\int_T k_n(t)(\varphi(t) - \varphi(0))dt \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)(\varphi(t)-\varphi(0))dt + \frac{1}{2\pi}\int_{-\pi}^{-\delta} k_n(t)(\varphi(t)-\varphi(0))dt \\ &+ \frac{1}{2\pi}\int_{\delta}^{\pi} k_n(t)(\varphi(t)-\varphi(0))dt\end{aligned}$$

Let the sum of the last two integrals in the above equation be labelled by $(*)$, but then

$$\left\|\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)(\varphi(t)-\varphi(0))dt\right\|_B \leq \max_{|t|\leq\delta} \|\varphi(t)-\varphi(0)\|_B \|k_n\|_T \quad (1)$$

and
$$\|*\|_B \leq \max_{t\in[-\pi,\pi]} \|\varphi(t)-\varphi(0)\|_B \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} |k_n(t)|dt + \int_{\delta}^{\pi} |k_n(t)|dt\right) \quad (2)$$

By continuity, (1) can be made as small as we like by letting $\delta \to 0$. Let $n \to \infty$ to make (2) as small as we like, so the result follows.

■ **Remark 8.4** By A3, $\varphi : T \to L^1(T)$ given by $\varphi(t) = f_t = f(x-t)$ is continuous.

**Theorem 8.4.2**
Let $f \in L^1(T)$ and $(K_n)$ be summability kernel in $L^1(T)$, then

$$\lim_{n\to\infty} K_n * f = f$$

**Proof:** Since

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(t)\varphi(t)dt = \varphi(0)$$

where $\varphi : T \to L^1$, $t \mapsto f_t$. That is

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(t)f(x-t)dt = f(x)$$

implies

$$\lim_{n\to\infty} (K_n * f)(x) = f(x)$$

as desired.

## 8.5 Dirichlet Kernel

Recall: If $(K_n)$ is a Summability Kernel and $f \in L^1(T)$, then $\lim_{n\to\infty} K_n * f = f$ in $L^1(T)$
Want: Find $(K_n)$ s.t. $K_n * f = S_n(f)$

■ **Remark 8.5** Let $f \in L^1(T)$, for $n \in \mathbb{Z}$ consider $\varphi_n(x) = e^{inx} \in L^1(T)$, then

$$\begin{aligned}
(\varphi_n * f)(x) = \frac{1}{2\pi}\int_T \varphi_n(t)f_t(x)dt &= \frac{1}{2\pi}\int_T e^{int}f(x-t)dt \\
&= \frac{1}{2\pi}e^{inx}\int_T e^{-in(x-t)}f(x-t)dt \\
&= \frac{1}{2\pi}e^{inx}\int_T e^{int}f(-t)dt \qquad \textbf{by A3} \\
&= \frac{1}{2\pi}e^{inx}\int_T e^{-int}f(t)dt \qquad \textbf{exercise} \\
&= e^{inx}\langle f, e^{inx}\rangle
\end{aligned}$$

■ **Remark 8.6** Let $f \in L^1(T)$, if $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$, then

$$\begin{aligned}(P * f)(x) = \frac{1}{2\pi}\int_T P(t)f(x-t)dt &= \sum_{k=-n}^{n} \frac{a_n}{2\pi}\int_T e^{ikt} f(x-t)dt \\ &= \sum_{k=-n}^{n} a_n(\varphi_k * f)(x) \\ &= \sum_{k=-n}^{n} a_n e^{ikx}\langle f, e^{ikx}\rangle\end{aligned}$$

**Definition 8.5.1 — Dirichlet Kernel.**
Let $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$, this is called **Dirichlet Kernel** of order $n$, so we have

$$(D_n * f)(x) = \sum_{k=-n}^{n} e^{ikx}\langle f, e^{ikx}\rangle = S_n(f, x)$$

where $S_n$ is the n-th particle sum.

■ **Remark 8.7** The $(D_n)$ is **not** a summability kernel

**Proof:** It's easy to show that

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}t\right)}$$

for all $t \neq 0$. Therefore

$$\|D_n\|_1 = \frac{1}{2\pi}\int_T \left|\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}t\right)}\right| dt \geq \frac{1}{\pi}\int_T \left|\frac{\sin\left(\frac{1}{2}t\right)}{t}\right| dt$$

Since $\left|\sin\left(\frac{t}{2}\right)\right| \leq \left|\frac{t}{2}\right|$ for all $t$, so

$$\|D_n\|_1 \geq \frac{1}{\pi}\int_{-\pi(n+\frac{1}{2})}^{\pi(n+\frac{1}{2})} \frac{|\sin t|}{|t|} dt = \frac{2}{\pi}\int_0^{\pi(n+\frac{1}{2})} \frac{|\sin(t)|}{t} dt > \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin(t)| dt = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}$$

Therefore, $\lim_{n\to\infty} \|D_n\|_1 = \infty$, which is not bounded so $D_n$ is not summability kernel

## 8.6 Fejer Kernel

Idea: Consider

$$y_n = \frac{x_1 + x_2 + \dots\dots\dots + x_n}{n}$$

exercise: If $x_n \to x$, then $y_n \to x$

**Definition 8.6.1 — Fejer Kernel.**
We say the

$$F_n(x) = \frac{D_0(x) + D_1(x) + ...... + D_n(x)}{n+1}$$

be the **Fejer Kernel** of order $n$

■ **Remark 8.8**
$F_0(x) = D_0(x) = 1$
$F_1(x) = \frac{e^{-i2x} + 2e^{-ix} + 3e^{i0x} + 2e^{ix} + e^{i2x}}{}3$
........

$$F_n = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$$

**Lemma 8.6.1**

$$F_n(t) = \begin{cases} \frac{1}{n+1}\left(\frac{\sin\left(\frac{(n+1)t}{2}\right)}{\sin\left(\frac{1}{2}t\right)}\right)^2 & \forall t \neq 0 \\ n+1 & t = 0 \end{cases}$$

**Proof:** Notice that

$$\sin^2 \frac{t}{2} = \frac{1}{2}(1 - \cos(t)) = \frac{1}{4}e^{-it} + \frac{1}{2} - \frac{1}{4}^{it}$$

and

$$\left(\frac{1}{4}e^{-it} + \frac{1}{2} - \frac{1}{4}^{it}\right) \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{n+1}\left(-\frac{1}{4}e^{-i(n+1)t} + \frac{1}{2} - \frac{1}{4}e^{i(n+1)t}\right)$$

then take the values of $t$, the results follows.

■ **Remark 8.9** $(F_n)$ is a summability kernel

**Proof:** First, we will show $\frac{1}{2\pi}\int_T F_n(t)dt = 1$. Since $\frac{1}{2\pi}\int_T ce^{ijt}dt \neq 0$ if $j \neq 0$, then

$$\frac{1}{2\pi}\int_T F_n(t)dt = \frac{1}{2\pi}\int_T 1dt = 1$$

It's obviously from **Lemma 8.6.1** that $F_n(t) \geq 0$, so $\frac{1}{2\pi}\int_T |F_n(t)|dt < M$ for some $M$. If

$t \notin (-\delta, \delta)$, then $|F_n(t)| \leq \frac{M}{n+1}$ where

$$M = \sup\left\{ \left| \frac{1}{\sin\frac{t}{2}} \right|^2 : t \in [-\pi, -\delta] \cup [\delta, \pi] \right\}$$

Hence, the third condition holds, so $F_n$ is a summability kernel.

**Definition 8.6.2 — Cesaro Mean.**

$$F_n * f = \frac{1}{n+1} \sum_{k=0}^{n} D_k * f = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f) = \frac{S_0(f) + S_1(f) + \dots\dots + S_n(f)}{n+1} := \underbrace{\sigma_n(f)}_{\textbf{n-th Cesaro Mean}}$$

**Theorem 8.6.2**
Let $f \in L^1(T)$ and $(F_n)$ be the Fejer Kernel, then

$$\lim_{n\to\infty} F_n * f = \lim_{n\to\infty} \sigma_n(f) = f$$

in $L^1(T)$

■ **Remark 8.10** If $(S_n(f))$ converges in $L^1(T)$, then $S_n(f) \to f$ in $L^1(T)$.

## 8.7 Fejer's Theorem

Idea: $L^1$ convergence is great theoretically, but pointwise convergence is practical.

**Theorem 8.7.1 — Fejer's Theorem.**
For $f \in L^1(T)$ and $t \in T$, consider

$$\omega_f(t) = \frac{1}{2} \lim_{x\to 0^+} (f(t+x) + f(t-x))$$

provided the limit exists, then

$$\sigma_n(f,t) \to \omega_f(t)$$

In particular, if $f$ is continuous at $t$, then

$$\sigma_n(f,t) \to f(t)$$

**Proof:** Assume that $\omega_f(t_0)$ exists and let $\varepsilon > 0$ be given. Since $\sigma_n(f) = F_n * f$, then

$$\begin{aligned}\sigma_n(f,t_0) - \omega_f(t_0) &= \frac{1}{2\pi}\int_T F_n(t)(f(t_0-t)-\omega_f(t_0))dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} F_n(t)(f(t_0-t)-\omega_f(t_0))dt + \frac{1}{2\pi}\int_{\delta}^{\pi} F_n(t)(f(t_0-t)-\omega_f(t_0))dt \\ &+ \frac{1}{2\pi}\int_{-\delta}^{\delta} F_n(t)(f(t_0-t)-\omega_f(t_0))dt \\ &:= (1)+(2)+(3)\end{aligned}$$

Since $F_n(t) = F_n(-t)$, so

$$(3) = \frac{1}{\pi}\int_0^{\delta} F_n(t)\Big(\frac{f(t_0-t)+f(t_0+t)}{2} - \omega_f(t_0)\Big)dt$$

By hypothesis, we may choose $\delta$ such that if $0 < t < \delta$, then

$$\left|\frac{f(t_0-t)+f(t_0+t)}{2} - \omega_f(t_0)\right| < \frac{\varepsilon}{2}$$

so that

$$|(3)| \le \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} F_n(t)dt = \varepsilon$$

WE can also choose $N$ s.t. if $n \ge N$, then

$$\sup\{F_n(t) \mid t \in (-\pi,\delta)\cup(\delta,\pi)\} < \frac{\varepsilon}{\|f_{t_0} - \omega_f(t_0)\| + 1}$$

Hence, we have

$$|(1)+(2)| \le \frac{\varepsilon}{\|f_{t_0} - \omega_f(t_0)\| + 1}\cdot\frac{1}{2\pi}\int_T |f(t_0-t) - \omega_f(t_0)|dt < \varepsilon$$

so the result follows.

In partice:
1. Fix $x \in T$
2. Prove $(S_n(f,x))$ converges
3. Then $S_n(f,x) \to \omega_f(x)$
4. If $f$ is continuous at $x$, then $S_n(f,x) \to f(x)$ i.e. $S(f,x) = f(x)$

■ **Example 8.1** Let $f \in L^1(T)$ and $f(x) = |x|$, then

$$S_n(f,x) = a_0 + \sum_{k=1}^{n}(b_k\cos(kx) + c_k\sin(kx))$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(kx)dx = \frac{2(-1)^k - 2}{k^2\pi} \qquad c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(kx)dx = 0$$

Then we have

$$S_n(f,x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^k - 1}{k^2}\cos(kx)\right) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\frac{n+1}{2}} \frac{-2}{(2k-1)^2}\cos((2k-1)x)$$

Note that $(S_n(f,x))$ converges by comparsion test with $\sum \frac{1}{(2k-1)^2}$. Since $f$ is continuous, so

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$$

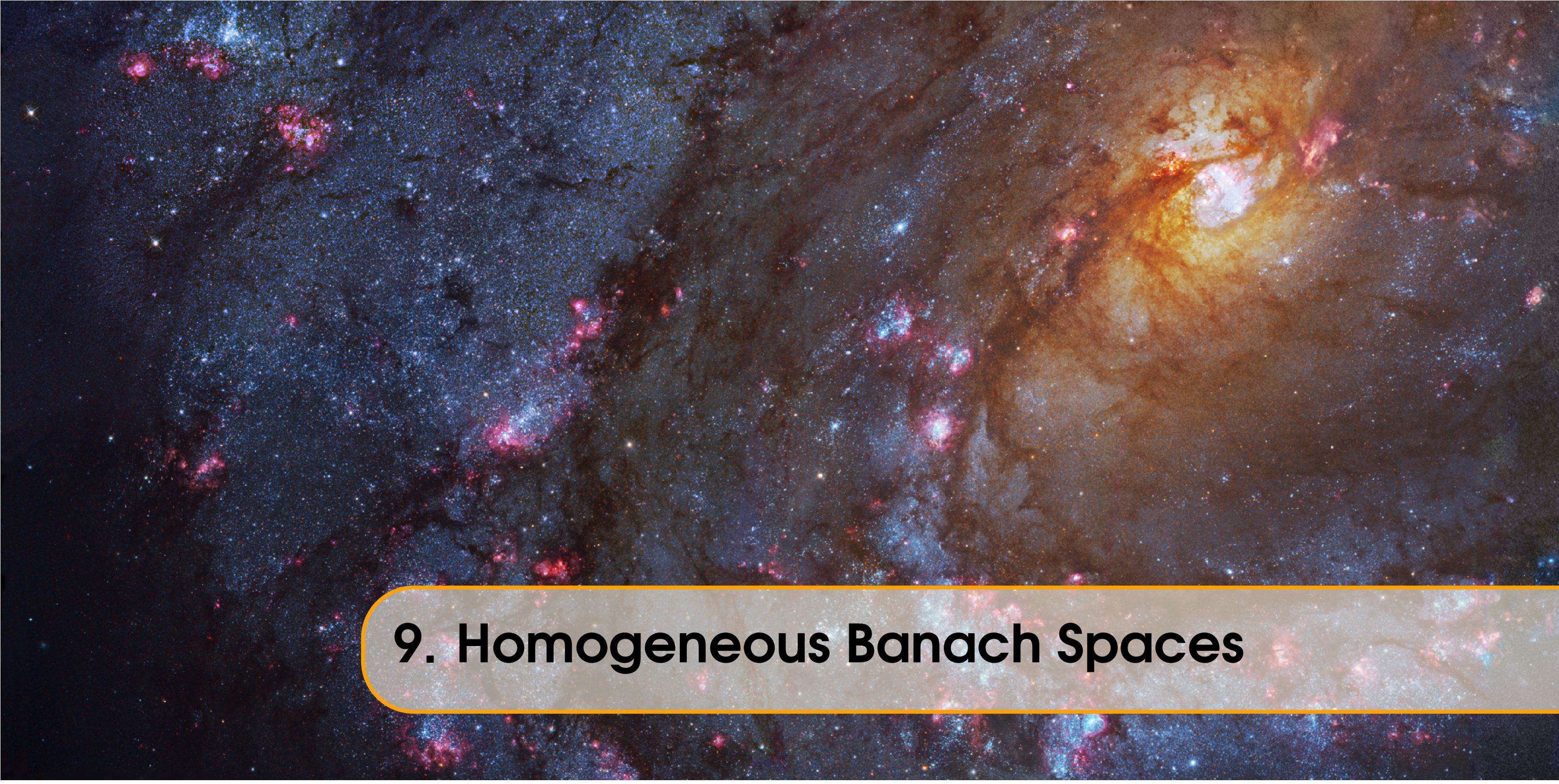
1. Taking $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

2.

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{\pi^2}{8} \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

■



# 9. Homogeneous Banach Spaces

## 9.1 Homogeneous Banach Spaces

Goal: Generalize what we have done for $L^1(T)$ to $L^p(T)$ with $p < \infty$. In particular, we look at $L^2(T)$.

**Definition 9.1.1 — Homogeneous Banach Space.**
A **homogeneous Banach space** is a Banach space $(B, \|\|_B)$ such that

1. $B$ is a subspace of $L^1(T)$
2. $\|\cdot\|_1 \leq \|\cdot\|_B$
3. $\forall f \in B$, $\forall \alpha \in T$, $\|f_\alpha\|_B = \|f\|_B$ **translation invariant**
4. $\forall f \in B$, $\forall t_0 \in T$, $\lim\limits_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$

■ **Example 9.1** $(L^p(T), \|\cdot\|_p)$ for $p < \infty$ is a homogeneous Banach space. ■

**Theorem 9.1.1**
Let $B$ be a homogeneous Banach space and $(k_n)$ be summability kernel, then for all $f \in B$

$$\lim_{n \to \infty} \|k_n * f - f\|_B = 0$$

**Proof:** First we have

$$\frac{1}{2\pi} \int_T k_n(t) f_t dt = k_n * f$$

We note that
$$\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(t)\varphi(t)dt = \varphi(0)$$
for all continuous function $\varphi : T \to B$. By previous result we have for $\varphi : T \to B$, $\varphi(t) = f_t$ is continuous (for all $f \in B$), then we have
$$\|k_n * f - f\|_B \to 0$$
as desired.

■ **Remark 9.1**
1. In the homogeneous Banach space $B$, taking $k_n = F_n$, then we have $\|\sigma_n(f) - f\|_B \to 0$ for all $f \in B$
2. Taking $B = L^p(T)$:

(a) $\|\sigma_n(f) - f\|_p \to 0$
(b) $\overline{Trig(T)} = L^p(T)$

■ **Remark 9.2** In $L^2(T)$:

1. $\overline{Trig(T)} = L^2(T)$
2. $\overline{span\{e^{inx} : n \in \mathbb{Z}\}} = L^2(T)$
3. $\left\{e^{inx} : n \in \mathbb{Z}\right\}$ is **ONB**
4. Let the above **ONB** be written as $\{v_1, v_2, ....\}$, then for all $f \in L^2(T)$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, v_i\rangle v_i = f$$
5. If $v = e^{ikx}$,
$$\langle f, v\rangle v = \left(\frac{1}{2\pi}\int_T f(x)e^{-ikx}dx\right)e^{ikx} = \langle f, e^{ikx}\rangle e^{ikx}$$
6. For all $f \in L^2(T)$, $\|S_n(f) - f\|_2 \to 0$

## 9.2 Additional Materials

**Definition 9.2.1 — Lebesgue Point.**
We say $x_0 \in \mathbb{R}$ is a **Lebesgue Point** of $f$ is
$$\lim_{h\to 0^+} \frac{1}{h} \int_{[0,h]} \left|\frac{f(x_0 - x) + f(x_0 + x)}{2} - f(x_0)\right| dx = 0$$
**Fact:** For $f$ as above, almost every $x_0 \in \mathbb{R}$ is a Lebesgue Point of $f$.

**Theorem 9.2.1**
Let $f$ the same as before, if $x_0$ is a Lebesgue Point of $f$, then

$$\sigma_n(f, x_0) \to f(x_0)$$

**Corollary 9.2.2**

$$\sigma_n(f) \to f \qquad \textbf{a.e.}$$

**Theorem 9.2.3 — Dini's Test.**
Let $f : \mathbb{R} \to \mathbb{C}$ with period $2\pi$, $\int_T |f| < \infty$. If

$$\int_0^\pi \left| \frac{f(x_0 + x) + f(x_0 - x)}{2} - L \right| \frac{dx}{x} < \infty$$

then $S_n(f, x_0) \to L$

**Proof:** BBT, pg 681